Integrals

(a) \[ \int \sin^3(u) \cos^2(u) du \]

Note: 3 = odd power. Single a \( \sin(u) \) out, use \( \sin^2(u) = 1 - \cos^2(u) \) then sub \( v = \cos(u) \).

(b) \[ \int \frac{e^x}{1 + e^{2x}} dx \]

\( e^{2x} = (e^x)^2 \), use sub \( u = e^x \)

(c) \[ \int y^2 \sqrt{1 + y^3} dy \]

\( u = 1 + y^3 \) since \( \frac{1}{2} du = y^2 dy \), which appears already.

(d) \[ \int_1^\infty \frac{\ln(x)}{x^{101}} dx \]

Improper so \( \int_1^\infty \frac{\ln(x)}{x^{101}} dx = \lim_{a \to \infty} \int_1^a \frac{\ln(x)}{x^{101}} dx \). Then IBP (with \( u = \ln(x) \), \( dv = x^{-101} dx \)). You need L’Hospital for the limit.) Answer: \( \frac{1}{101} \) (Alternative (harder) solution \( u = \ln(x), x = e^u \), then one IBP)

(e) \[ \int \frac{x}{\sqrt{1 - x^4}} dx \]

First \( u = x^2, \frac{1}{2} du = x dx \) to simplify, then trigonometric substitution \( u = \sin(\theta) \). Remember to go back: \( \theta \to u \to x \).

(f) \[ \int \frac{1}{x^2 \sqrt{16 - x^2}} dx \]

Trigonometric Substitution \( x = 4 \cos(x) \) (better than \( u = 4 \sin(x) \) for this problem since you don’t have to use an antiderivative that was not discussed in the course; \( u = 4 \sin(x) \) works too if you know \( \int \frac{1}{\sin^2(y)} dy \) BUT don’t learn it.)

(g) \[ \int \cos(\sqrt{x}) dx \]

Start with \( u = \sqrt{x} \). Bit tricky but you should only have \( u \)'s in the new integral. Then IBP.
\[
\int \frac{\cos(t)}{\sqrt{\sin^2(t) + 1}} \, dt
\]

\(u = \sin(t), \) since \(du = \cos(t) \, dt\) already appears and then bottom becomes \(\sqrt{u^2 + 1}\) for which \(u = \tan(\theta)\). Here, you end up with \(\int \sec(\theta) \, d\theta = \ln|\sec(\theta) + \tan(\theta)| + C.\) **do NOT memorize this, it’s tricky to get, will NOT be tested; focus on the ideas here.**

**Remark:** there was a typo in the original, it should have been \(dt,\) not \(dx.\)

### Series: Converges or diverges?

1. \(\sum_{n=1}^{\infty} \frac{2 \ln n}{n^b}\)
   - **converges** by the Comparison Test, comparing with \(\sum_{n=1}^{\infty} \frac{2n}{n^b} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{b-1}}.\) To see why we can compare with the latter check that \(\ln x < x,\) (hint: do monotonicity analysis on \(f(x) = \ln x - x\)).
   - **alternative solution:** use the Integral Test (check the decreasing property); the corresponding integral is almost the same as (d) above.

2. \(\sum_{n=1}^{\infty} (-1)^n \frac{n^3 + 1}{n^3 - 7}\)
   - **diverges** by the Divergence Test, since for even \(n\) \(a_n = \frac{n^3 + 1}{n^3 - 7} \to 1,\) while for odd \(n\) \(a_n = (-1) \frac{n^3 + 1}{n^3 - 7} \to -1\) so \(a_n\) doesn’t have a limit.

3. \(\sum_{n=1}^{\infty} \frac{7}{n5^n}\)
   - **converges** by the Comparison Test, since \(\frac{7}{n5^n} \leq \frac{1}{n}\) and the series \(\sum_{n=1}^{\infty} \frac{7}{n5^n} = 7 \sum_{n=1}^{\infty} \frac{1}{n5^n} = 7 \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n\) which is a converging geometric series. Alternatively, one can use the Ratio Test (limit = \(\frac{1}{5} < 1\))

4. \(\sum_{n=1}^{\infty} \frac{2n^2}{9n^2 - 7}\)
   - **diverges** by the Divergence Test, since \(a_n = \frac{2n^2}{9n^2 - 7} \to \frac{2}{9} \neq 0\)

5. \(\sum_{n=1}^{\infty} \frac{1}{4 + \sqrt{n^3}}\)
   - **diverges** by the Limit Comparison Test; notice that for large \(n\) the fraction behaves like \(\frac{1}{\sqrt{n^3}},\) whose corresponding series diverges (p-series with \(p < 1\)). Since we cannot use the Comparison Test (check that the inequality is in the unhelpful direction), the problem calls for the Limit Comparison Test.
   - Check that \(\lim_{n \to \infty} \frac{\frac{1}{4 + \sqrt{n^3}}}{\frac{1}{\sqrt{n^3}}} = 1 > 0\) hence by the Limit Comparison test our series also diverges.

6. \(\sum_{n=1}^{\infty} (-1)^n \frac{n^{32n}}{n!}\)
   - **converges** by the Ratio Test (limit is 0)
7. 
\[ \sum_{n=1}^{\infty} \frac{10 + 9^n}{5 + 8^n} \]

\textbf{diverges} by the Divergence Test, since \( a_n = \frac{10 + 9^n}{5 + 8^n} \to \infty \) (which you can see either by using L’Hospital’s rule or by dividing both numerator by 9^n and the denominator by 8^n)

8. 
\[ \sum_{n=1}^{\infty} \frac{(-1)^n \cos n}{n^5} \]

\textbf{converges}; we will show that this series is \textit{absolutely convergent}, thus is convergent. Consider the series of the absolute values which becomes \( \sum_{n=1}^{\infty} \frac{\lvert \cos n \rvert}{n^5} \). Now, by the Comparison Test (can now use since my series has non-negative terms) since \( 0 \leq \lvert \cos n \rvert \leq 1 \) the series of the absolute values converges by comparison to a converging p-series (\( p = 5 \)). We have shown that the series converges absolutely and hence converges.