1. Introduction

Most flows encountered in engineering practice and in nature are turbulent. Some are

- Boundary layer on an aircraft wing
- The atmospheric boundary layer over the earth's surface
- Smoke from a chimney
- Water in a river or waterfall
- Water flows below the surface of oceans
- Most combustion processes

As implied by the above examples, an essential feature of turbulent flows is that the fluid velocity field varies significantly and irregularly both in space and time.

What is turbulence? A state of continuous instability!

It is hard to give a precise definition of turbulence, however, nearly everyone would agree with some characteristics of turbulent flows as

Irregularity, randomness: Turbulent flow fields exhibit a high degree of apparent *randomness* and *disorder*. However, close inspection often reveals the presence of ordered embedded flow structures (*coherent structures*). One cannot, however, say that a turbulent flow is "completely random".

> This makes a deterministic approach to turbulence problems impossible; instead, one relies on statistical methods.

Diffusivity: Diffusivity of turbulence causes rapid mixing and increased rates of momentum, heat, and mass transfer. Thus, advected tracers are rapidly mixed by turbulent flow.

> This is the single most important feature as far as applications are concerned.

Large Reynolds numbers: Turbulent flows always occur at high Reynolds numbers (a *nonlinearity parameter*). Turbulence often originates as an instability of laminar flows if the Reynolds number becomes too large.

In unstable flows small perturbations grow spontaneously and frequently equilibrate as finite amplitude disturbances. On further exceeding the stability criteria, the new state can become unstable to more complicated disturbances, and the flow eventually reaches a *chaotic state*.

Three-dimensional vorticity fluctuations: Turbulence is *rotational* and three dimensional. Turbulence is characterized by high levels of *fluctuating vorticity*. The random vorticity fluctuations could not maintain themselves if the velocity fluctuations were two dimensional, since an important *vorticity-maintenance mechanism* known as *vortex stretching* is absent in two-dimensional flows.

Random waves on the surface of oceans are not in turbulent motion since they are essentially irrotational. Random waves essentially *nondissipative*, though they often are *dispersive*.

Dissipation: Turbulent flows are always *dissipative* (have a high rate of viscous energy dissipation). Viscous shear stresses perform *deformation work* which increases the internal energy of the fluid at the expense of *kinetic energy of the turbulence*.

Turbulent flows therefore require a continuous supply of energy to make up for the *viscous losses*. If no energy is supplied, turbulence decays rapidly.

Continuum: Turbulence is a *continuum phenomenon*, governed by the equations of fluid mechanics. Even the smallest scales occurring in a turbulent flow are ordinarily far larger than any molecular length scale.

Turbulent flow has a very complex structure, involving broad range of space and time scales.

Turbulent flows are flows: Turbulence is not a feature of fluids but of fluid flows. Most of the dynamics of turbulence is the same in all fluids, if the Reynolds number of the turbulence is large enough. Since every flow is different, it follows that every turbulent flow is different, even tough all turbulent flows have many characteristics in common.

2. Length scales in turbulent flows

In turbulent flows a wide range of length scales exists, bounded from above by the dimensions of the flow field and bounded from below by the diffusive action of molecular viscosity (***separation of scales)***; the division of a turbulent motion into (interacting) motions on various length scales is useful because the different scales play rather different role in the dynamics of the motion. This is often expressed by talking of "eddies of different sizes".

An "eddy" eludes precise definition, but it is conceived to be a turbulent motion, localized within a region of size $\mathbf{l}$, that is at least moderately coherent over this region. The region occupied by a large eddy can also contain smaller eddies. Eddies of size $\mathbf{l}$ have a characteristic velocity $u(\mathbf{l})$ and timescale $t(\mathbf{l}) \equiv \mathbf{l}/u(\mathbf{l})$.

162. "Typical eddy" in a turbulent boundary layer. Oil fog is illuminated by a sheet of laser light to show the lower two-thirds of a turbulent boundary layer in side view. The vortex-ring structure just below and to the right of center, which resembles a sliced mushroom leaning left, is an example of what Falco has called a "typical eddy." It scales on wall variables (figure 161) rather than on the boundary-layer thickness. *Photograph by R. E. Falco*

The main physical process that spreads the motion over a wide range of wavelengths (length scales) is ***vortex stretching*** (the largest eddies interact and extract energy from the mean flow by the vortex stretching process). The turbulence gains energy if the vortex elements are primarily oriented in a direction in which the ***mean velocity gradients*** can stretch them. Most importantly, wavelengths that are not too small compared to the mean-flow width interact most strongly with the mean flow. Consequently, the larger-scale turbulent motion carries most of the energy and is mainly responsible for the enhanced diffusivity and attending stresses.

The large eddies are dominated by inertia effects and viscous effects are negligible. The large eddies are therefore effectively inviscid and angular momentum is conserved during vortex stretching. This causes the rotation rate to increase and the radius of their cross-sections to decrease, as shown in Figure 2. Thus the process creates motions at smaller transverse length scales and also at smaller time scales.

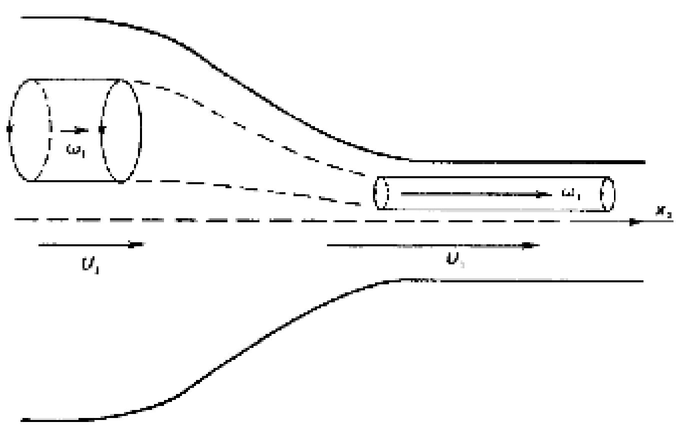

Figure 2. Vortex stretching in a wind-tunnel contraction. As the flow speeds up from left to right, the vorticity component w_1 is amplified because angular momentum has to be conserved [3].

The behavior of the *small-scale motions*, on the other hand, is determined almost entirely by ***the rate at which they receive energy from large scales, and by the viscosity***. Smaller eddies are themselves stretched strongly by larger eddies and more weakly by the mean flow. In this way the kinetic energy is handed down from large eddies to progressively smaller and smaller eddies in what is termed the ***energy cascade***. It is natural to ask what the ***characteristics of the small-scale motions*** are.

166. Turbulent water jet. Laser-induced fluorescence shows the concentration of jet fluid in the plane of symmetry of an axisymmetric jet of water directed downward into water. The Reynolds number is approximately 2300. The spatial resolution is adequate to resolve the Kolmogorov scale in the downstream half of the photograph. *Dimotakis, Lye & Papantoniou 1981*

2.1. Multiple Scales in Laminar Boundary Layers

For steady flow of an incompressible fluid with constant viscosity, the Navier-Stokes equations are

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{r}\frac{\partial p}{\partial x_i} + v\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1}$$

By introducing some characteristic scaling parameters and defining several nondimensional variables

$$\tilde{u}_i = \frac{u_i}{U}, \; \tilde{x}_i = \frac{x_i}{L}, \; \tilde{p} = \frac{p - p_\infty}{p_0 - p_\infty} \tag{2}$$

we can nondimensionalize the equations of motion as (the tilde decoration is omitted for simplicity)

$$u_j \frac{\partial u_i}{\partial x_j} = -[Eu]\frac{\partial p}{\partial x_i} + \left[\frac{1}{\mathrm{Re}}\right]\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{3}$$

where the nondimensional parameters are $[Eu] = (p_0 - p_\infty)/rU^2$, $[\mathrm{Re}] = UL/v$. The Reynolds number is the ratio of the inertia terms U^2/L to viscous terms vU/L^2. Hence, at large Reynolds numbers the viscous terms should become negligible. However, boundary conditions (no-slip BCs) or initial conditions may make it impossible to neglect viscous terms everywhere in the flow field. The viscous terms can survive at high Reynolds numbers only by choosing a new length scale $\mathbf{l}$ such that the viscous terms are of the same order magnitude as the inertia terms. Formally,

$$U^2/L \sim uU/\mathbf{l}^2 \tag{4}$$

The viscous length scale $\mathbf{l}$ (a ***transverse*** length scale, which represents width of the boundary layer) is thus related to the scale L of the flow field as

$$\frac{\mathbf{l}}{L} \sim \left(\frac{u}{UL}\right)^{1/2} = \frac{1}{\mathrm{Re}^{1/2}} \tag{5}$$

The boundary layer thickness may be considerably smaller than the length scale L of the flow field (convective or longitudinal length scale) in which the boundary layer develops. The wide separation between the diffusive (lateral) length scale across the flow and a convective (longitudinal) length scale along the flow in shear flows lead to very attractive simplifying approximations in the equations of motion.

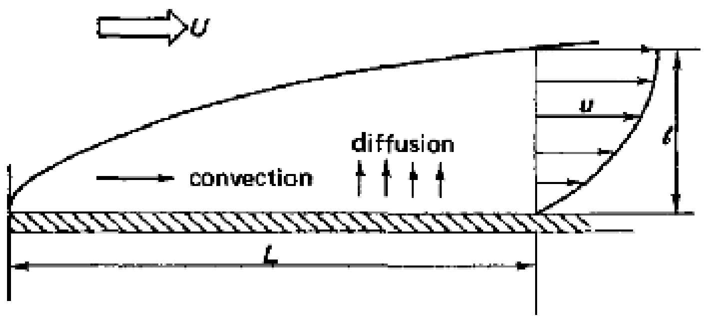

Figure 4. length scales, diffusion, and convention in a laminar boundary layer over a flat plate. [3]

2.2. Turbulent boundary layers

The relevant length and velocity scales in a turbulent boundary layer are illustrated in Figure 5. The turbulent eddies transfer momentum deficit away from the surface. With characteristic velocity fluctuations of order u, the boundary-layer thickness ℓ presumably increases roughly as $d\ell/dt \sim u$. The time interval elapsed between the origin of the boundary layer and downstream position L is of order L/U (convective time scale), so we may estimate $\ell \sim ut \sim uL/U$. In effect, we are equating the turbulent diffusion time scale ℓ/u to the convective time scale L/U. Thus, we can write the scale relations for turbulent boundary layers as

$$\ell/L \sim u/U \tag{6}$$

$$\ell/u \sim L/U \tag{7}$$

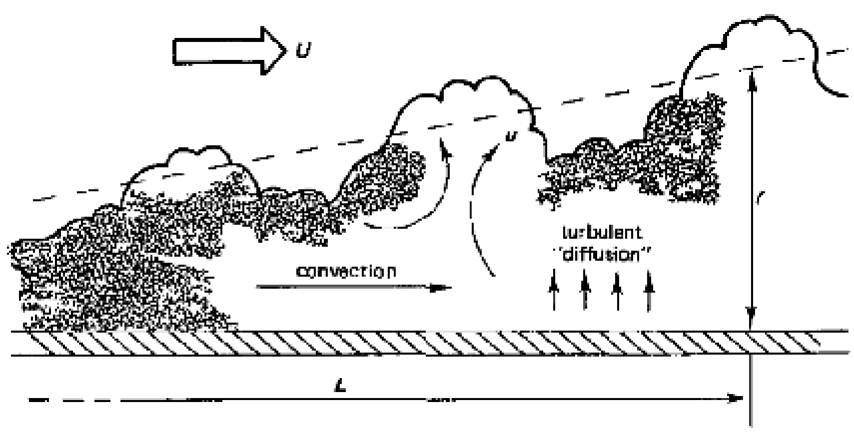

Figure 5. Length and velocity scales in a turbulent boundary layer. The time passed since the fluid at L passed the origin of the boundary layer is of order L/U [3].

The relation between the time scales, (7), rephrases that in a situation with an imposed external flow the turbulence must have a time scale commensurate with the time scale of the flow. However, this assumption conflicts with ***eddy-viscosity concepts***. Fortunately, not all of the turbulence has such a large time scale: the small eddies in turbulence have very short time scales, which tend to make them statistically independent of the mean flow.

2.3. The energy Cascade

The first concept in Richardson's view of the energy cascade is that the turbulence can be considered to be composed of ***eddies*** of different sizes. Richardson's notion is that the large eddies are unstable and break up, transferring their energy to somewhat smaller eddies. These smaller eddies undergo a similar break-up process, and transfer their energy to yet smaller eddies. This energy cascade – in which energy is transferred to successively smaller and smaller eddies – continues until the Reynolds number is sufficiently small that the eddy motion is stable, and molecular viscosity is effective in dissipating the kinetic energy and smoothing out velocity fluctuations; the viscous terms prevent the generation of infinitely small scales of motion by dissipating small-scale energy into heat.

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).

174. **Turbulent wake of a cylinder.** *A sheet of laser light* slices through the wake of a circular cylinder at a Reynolds number of 1770. Oil fog shows the instantaneous flow pattern, covering 40 diameters centered 50 diameters downstream. *Photograph by R. F. Falco*

The idea of the energy cascade is that kinetic energy enters the turbulence (through the production mechanism) at the largest scales of motion. This energy is then transferred (by ***<u>inviscid processes</u>***) to smaller and smaller scales until, at the smallest scales the energy is dissipated by viscous action.

One reason that this picture is of importance is that ***<u>it places dissipation at the end of a sequence of process</u>***. The rate of dissipation e is determined, therefore, ***<u>by the first process in the sequence, which is the transfer of energy from the largest eddies.</u>*** These eddies have energy of order u_0^2 and timescale $t_0 = \mathbf{l}_0 / u_0$, so the rate of transfer of energy can be supposed to scale as $u_0^2 / t_0 = u_0^3 / \mathbf{l}_0$. Consequently, consistent with the experimental observations in free shear flows, this picture of the cascade indicates that e scales $u_0^3 / \mathbf{l}_0$, independent of v

Several fundamental questions remained unanswered.

- Ø What is the size of the smallest eddies that are responsible for dissipating the energy?
- Ø As $\mathbf{l}$ decreases, do the characteristic velocity and timescales $u(\mathbf{l})$ and $t(\mathbf{l})$ increase, decrease, or remain the same?

These questions and more are answered by the theory advanced by Kolmogorov which is stated in the form of three hypotheses. A consequence of theory is that both the velocity and timescales $u(\mathbf{l})$ and $t(\mathbf{l})$ decreases as $\mathbf{l}$ decreases.

2.5. The Kolmogorov Hypotheses (Small Scales in turbulence)

The first hypothesis concerns the isotropy of the small-scale motions. In general, the large eddies are anisotropic and are affected by the boundary conditions of the flow. Kolmogorov argued that the directional biases of the large scales are lost in the chaotic scale-reduction process, by which energy is transferred to successively smaller and smaller eddies.

Kolmogorov's hypothesis of local isotropy: At sufficiently high Reynolds number, the small-scale turbulent motions $\mathbf{l} << \mathbf{l}_0$ are statistically isotropic.

It is useful to introduce a lengthscale $\mathbf{l}_{EI}$ (with $\mathbf{l}_{EI} \approx \frac{1}{6}\mathbf{l}_0$, say) as the demarcation between the anisotropic large eddies $(\mathbf{l} > \mathbf{l}_{EI})$ and the isotropic small eddies $(\mathbf{l} < \mathbf{l}_{EI})$.

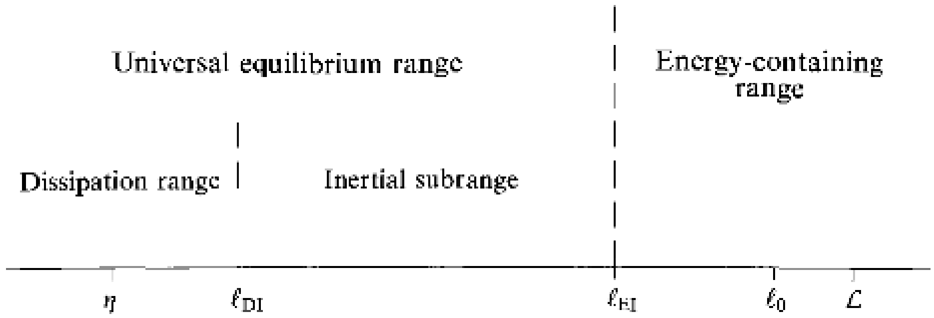

Figure 7. Eddy sizes (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges [2].

Just as the directional information of the large scales is lost as the energy passes down the cascade, Kolmogorov argued that all information about the geometry of the large eddies – determined by the mean flow and boundary conditions – is also lost. As a consequence, ***<u>the statistics of the small-scale motions are in a sense universal</u>***, similar in every high-Reynolds number turbulent flow.

- On what parameters does this statistically universal state depend?

In the energy cascade $(\mathbf{l} < \mathbf{l}_{EI})$ the two dominant processes are the transfer of energy to successively smaller scales, and viscous dissipations. *A plausible hypothesis, then, is that the important parameters are the rate at which the small scales receive energy from the large scales (which we denote by T_{EI}) and the kinematic viscosity v*. The dissipation rate e is determined by the energy transfer rate T_{EI}, so that these two rates are nearly equal, i.e., $e \approx T_{EI}$. Consequently, the hypothesis that the statistically universal state of the small scales is determined by v and the rate of energy transfer from the large scales T_{EI} can be stated as:

Kolmogorov's first similarity hypothesis: In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions $(\mathbf{l} < \mathbf{l}_{EI})$ have a universal from that is uniquely determined by v and e.

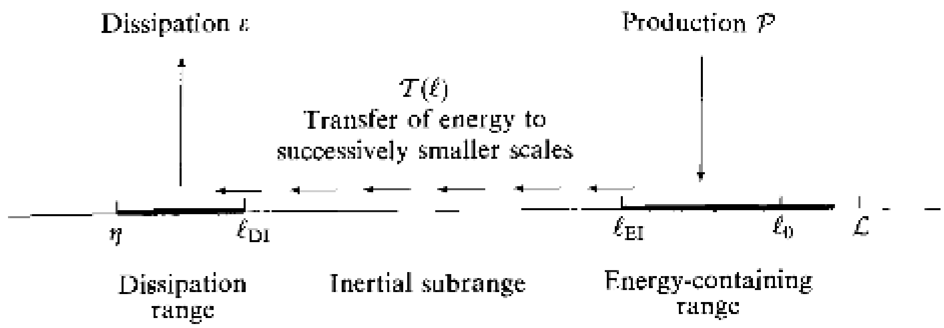

Figure 8. A schematic diagram of the energy cascade at very high Reynolds number [2]

The size range $\mathbf{l} < \mathbf{l}_{EI}$ is referred to as the ***universal equilibrium range***. In this range, the timescales $\mathbf{l}/u(\mathbf{l})$ are small compared with $\mathbf{l}_0/u_0$, so that the small eddies can adapt quickly to maintain a dynamic equilibrium with the energy-transfer rate T_{EI} imposed by the large eddies.

This discussion suggests that the parameters governing the small-scale motion include at least the dissipation rate per unit mass $e\left(m^2 \sec^{-3}\right)$ and the kinematic viscosity $v\left(m^2 \sec^{-1}\right)$. With these parameters, one can form length, time, and velocity scales as follows

$$h \equiv \left(v^3/e\right)^{1/4},\ t_h \equiv \left(v/e\right)^{1/2},\ u_h \equiv \left(ve\right)^{1/4} \tag{7}$$

These scales are referred as the ***Kolmogorov microscales*** of length, time, and velocity.

Two identities stemming from these definitions clearly indicate that the Kolmogorov scales characterizes the <u>***very smallest, dissipative eddies***</u>. First, the Reynolds number based on the Kolmogorov scales is unit, i.e., $hu_h/v = 1$, which illustrates that the small-scale motion is quite viscous and that the viscous dissipation adjusts itself to the energy supply by adjusting length scales. Second, the dissipation rate is given by

$$e = v\left(u_h / h\right)^2 = v / t_h^2 \tag{8}$$

Showing that $\left(u_h / h\right) = 1 / t_h$ provides a consistent characterization of the velocity gradients of the dissipative eddies.

The ratios of the smallest to the largest scales are readily determined from the definitions of the Kolmogorov scales and from the scaling $e \sim u_0^3 / \mathbf{l}_0$. The results are

$$h / \mathbf{l}_0 \sim \mathrm{Re}^{-3/4} \tag{9.1}$$

$$u_h / u_0 \sim \mathrm{Re}^{-1/4} \tag{9.2}$$

$$t_h / t_0 \sim \mathrm{Re}^{-3/4} \tag{9.3}$$

Evidently, at high Reynolds numbers, the velocity scales and timescales of the smallest eddies $\left(u_h, t_h\right)$ are – as previously supposed – small compared with those of the largest eddies $\left(u_0, t_0\right)$. Notice that $\mathrm{Re} = \mathrm{Re}_T = k^{1/2}\mathbf{l} / v$ is the usual turbulence Reynolds number.

Inevitably the ratio $h / \mathbf{l}_0$ decreases with increasing Re. As a consequence, at sufficiently high Reynolds number, there is a range scale $\mathbf{l}$ that are very small compared with $\mathbf{l}_0$, and yet very large compared with h, i.e., $\mathbf{l}_0 >> \mathbf{l} >> h$. Since eddies in this range are much bigger than the dissipative eddies, it may be supposed that their Reynolds number $u(\mathbf{l})\mathbf{l}/v$ is large, and consequently that their motion is little affected by viscosity. Hence, following from this and from the first similarity hypothesis, we have (approximately stated):

> ***Kolmogorov's second similarity hypothesis***: In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale $\mathbf{l}$ in range $\mathbf{l}_0 >> \mathbf{l} >> h$ have a universal from that is uniquely determined by e and independent of v.

Lengthscales, velocity scales, and timescales cannot be formed from e alone. However, given an eddy size $\mathbf{l}$ (in the inertial subrange), characteristic velocity scales and timescales for the eddy are those formed from e and $\mathbf{l}$:

$$u(\mathbf{l}) = (e\mathbf{l})^{1/3} = u_h(\mathbf{l}/h)^{1/3} \sim u_0(\mathbf{l}/\mathbf{l}_0)^{1/3} \tag{10}$$

$$t(\mathbf{l}) = (\mathbf{l}^2/e)^{1/3} = t_h(\mathbf{l}/h)^{2/3} \sim t_0(\mathbf{l}/\mathbf{l}_0)^{2/3} \tag{11}$$

A consequence, then, of the second similarity hypothesis is that (in the inertial subrange) the velocity scales and timescales $u(\mathbf{l})$ and $t(\mathbf{l})$ decrease as $\mathbf{l}$ decreases.

In the conception of the energy cascade, a quantity of central importance – denoted by $T(\mathbf{l})$ – is the rate at which energy is transferred from eddies larger than $\mathbf{l}$ to those smaller than $\mathbf{l}$. If this transfer process is accomplished primarily by eddies of size comparable to $\mathbf{l}$, then $T(\mathbf{l})$ can be expected to be of order $u(\mathbf{l})^2/t(\mathbf{l})$. The identity

$$u(\mathbf{l})^2/t(\mathbf{l}) = e \tag{12}$$

Stemming from Equations (10) and (11), is particularly revealing, therefore, since it suggest that $T(\mathbf{l})$ is independent of $\mathbf{l}$ (for $\mathbf{l}$ in the inertial subrange). Hence we have

$$T_{EI} \equiv T(\mathbf{l}_{EI}) = T(\mathbf{l}) = T_{DI} \equiv T(\mathbf{l}_{DI}) = e \qquad \text{for } (\mathbf{l}_{DI} < \mathbf{l} < \mathbf{l}_{EI}) \tag{13}$$

That is the rate of energy transfer from the large scales, T_{EI}, determines the constant rate of energy transfer through the inertial subrange $T(\mathbf{l})$; hence the rate at which energy leaves the inertial subrange and enters the dissipation range T_{DI}; hence the dissipation rate.

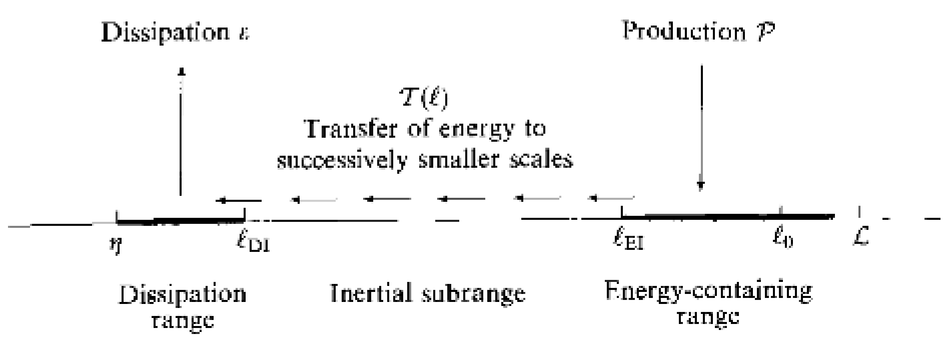

Figure 8. A schematic diagram of the energy cascade at very high Reynolds number [2]

<u>It remains to be determined how the turbulent kinetic energy is distributed among the eddies of different sizes</u>.

2.6 The energy Spectrum

Since turbulence contains a continuous spectrum of scales, it is often convenient to cast our analysis in terms of the ***spectral distribution*** of energy. If k denotes wavenumber and $E(k)dk$ is the ***turbulence kinetic energy*** contained between wavenumber k and $k + dk$, we can say

$$k \equiv \frac{1}{2}\langle u_i' u_i' \rangle == \int_0^\infty E(k)dk \tag{14}$$

Note that k is half the trace of the autocorrelation tensor, $\Re_{ij}$, defined

$$\Re_{ij}(X, t; t') \equiv \langle u_i'(X, t) u_j'(X, t + t') \rangle \tag{15}$$

Correspondingly, the energy spectral density or energy spectrum function, $E(k)$, is the Fourier transform of half the trace of $\Re_{ij}$. In general, we regard a spectral representation as a decomposition into wavenumbers (k) or, equivalently, wavelengths $(2p/k)$. Here we think of the reciprocal of k as the eddy size.

In general, $E(k)$ is a function of $v, e, \mathbf{l}_0, k$ and the mean strain rate, S. We needn't consider k as it can be expressed in terms of e, $\mathbf{l}_0$. As part of his universal equilibrium theory, explained above, Kolmogorov also made the hypothesis that for very large Reynolds number, there is range of eddy sized between the largest and smallest for which the cascade process is independent of the statistics of the energy-containing eddies (so that S and $\mathbf{l}_0$ can be ignored) and of the direct effects of molecular viscosity (so that v can be ignored). The idea is that a range of wavenumbers exists in which the energy transferred by inertial effects dominates, wherefore $E(k)$ depends only upon e, and k. On dimensional grounds, he thus concluded that

$$E(k) = C_K e^{2/3} k^{-5/3}, \quad \frac{1}{\mathbf{l}_0} << k << \frac{1}{h} \tag{16}$$

where C_K is the Kolmogorov constant. Because inertial transfer of energy dominates, Kolmogorov identified this range of wavenumbers as the inertial subrange. The existence of the inertial subrange has been verified by many experiments and numerical simulations, although many years passed before definitive date were available to confirm its existence. Figure 9 shows typical energy spectrum for a turbulent flow.

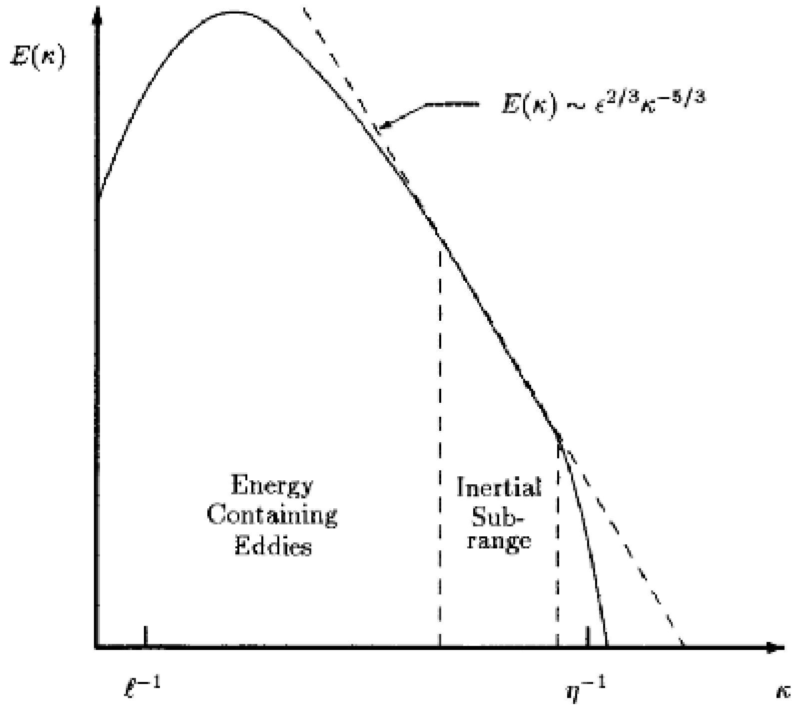

Figure 9. Energy spectrum for a turbulent flow – Log-log scales [2]

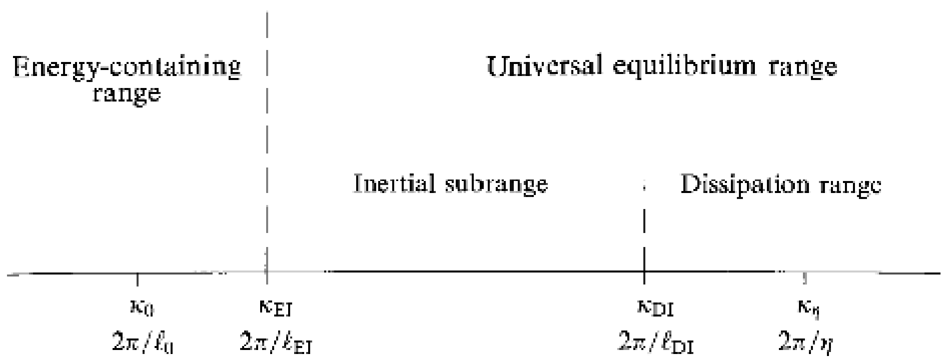

Figure 9. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges [2]

3. References

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