Sequential Innovation and Patent Policy

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February 2017†

Abstract

I examine the impact of patent policy—characterised by patent length and strength—on R&D investment dynamics and the number of competitors in the context of sequential innovation. Overly protective policies can decrease the pace of innovation through two mechanisms: they delay firms' investments toward the end of the patent's life, and decrease the number of competitors in the market. I study the policy that maximises innovative activity, and how this policy depends on market characteristics. Patent length and strength complement each other in the sense that one tool is effective at encouraging innovation in markets where the other is not.

JEL: D43, L40, L51, O31, O32, O34

Keywords: Patent policy, sequential innovation, R&D dynamics, replacement effect, patent length, patent strength

*I am indebted to Michael Whinston and Wojciech Olszewski for their advice and support. I also thank Nancy Gallini, Joshua Gans, Jorge Lemus, Igor Letina, Fernando Luco, Guillermo Marshall, Alessandro Pavan, Bruno Strulovici, Ralph Winter and seminar participants at Bates and White, ESEM 2013, 2013 FTC Microeconomics Conference, IESE, IIOC 2015, LAMES 2013, Mannheim University, Northwestern University, TOI 2013, University of British Columbia and Washington University. I gratefully acknowledge the financial support of Northwestern's Center of Economic Theory. Álvaro Parra, Sauder School of Business, University of British Columbia, 2053 Main Mall, Vancouver, BC, V6T1Z2, Canada. alvaro.parra@sauder.ubc.ca

†*First draft:* November 2012

1 Introduction

When new innovations cannibalise the incumbents' patented innovations, the incumbents have less incentives to invest in R&D than firms who do not hold patents. This effect was called the replacement effect by Arrow (1962), and has since played a central role in understanding firms' incentives to innovate. While patent policy is what causes the replacement effect, how patent policy shapes the dynamics of the replacement effect has been ignored by the literature. Consider for instance an incumbent's incentives to innovate throughout a patent's life. When the patent is recently issued, the replacement effect is at its highest level, giving the incumbent little incentives to innovate because of how a new innovation would cannibalise the value of its recently awarded patent. In contrast, when the patent is about to expire, the replacement effect disappears, causing the incumbent's innovation incentives to peak. Patents of different length will thus induce different innovation patterns, implying that patent policy plays an important role in determining both the *magnitude* and *timing* of R&D investments.

In this article, I study how patent policy—through its impact on the replacement effect—shapes firms' incentives to innovate and its consequences for optimal patent design in the context of a quality-ladder model (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Aghion *et al.*, 2001). Innovations may come from a leader trying to prolong its lead or from followers aiming to become the new market leader. I show that the value of possessing a patent, the extent of the replacement effect, and firms' investment decision are *non-stationary* and *endogenously* determined by patent policy. A patent is represented by a two-dimensional policy determining how long a leader will be able to exclude others from using its technology—patent *length*—and how enforceable its patent will be against future innovations—patent *strength*. Following Lemley and Shapiro (2005) and Farrell and Shapiro (2008), I treat patent strength as probabilistic, capturing both the uncertainty that exists when a replaced leader tries to enforce its patent against a new innovation and the leniency of courts towards new innovators. When a follower develops a new innovation, the replaced leader files an infringement lawsuit against the innovating follower. The patent authority—for example, a U.S. federal court—may decide, with certain probability, to uphold the claim or to declare it invalid.[1] In the former case, a compulsory license fee, equal to the damages caused

[1]In a study on the validity of litigated patents, Allison and Lemley (1998) find that in 46% of cases that go to litigation, the suing patent is found invalid. In their annual patent litigation re-

by the commercialization of the new innovation, must be paid by the infringing firm before the firm can commercialise the new invention and obtain economic profits.

Beyond the technical contribution of studying the non-stationarity induced by finite patent protection and advancing the literature a step towards more realistic models of patents, there are three novel insights derived from incorporating dynamic incentives into the analysis. First, although patents are necessary to incentivise firms to develop new innovations, I show that longer patent protection intensifies the replacement effect inducing market *leaders* to *delay* their investments towards the end of the patent's life. Furthermore, when patent protection is strong, *followers* internalise the replacement effect—through the expected license fees they pay in the case they obtain a breakthrough—leading them to also delay investments when longer protection is offered. The extent of this internalization can be substantial, under strong patent systems Arrow's result *reverses* and followers have less incentives to innovate than market leaders at every moment of the patent's life. Together, these results imply that overly protective policies *decrease* the market's innovation pace, implying that the cost of, for instance, offering longer patent protection lies beyond extending the deadweight loss associated with the lack of competition due to the existence of a patent.

In order to explore the policy consequences of this delay in R&D investments, I examine the combination of length and strength that maximises the speed of innovative activity in a given market and explore how this policy depends on market characteristics.[2] The second main finding is that, from the perspective of a policymaker, patent length and strength *complement* each other in the sense that one tool effectively encourages innovation in circumstances where the other tool does not. The optimal levels of patent length and strength varies with the market's R&D productivity. In particular, among markets in which innovations take longer to produce or are costlier to develop, such as the pharmaceutical sector, patent length is a more effective tool for promoting innovation: long but weak patents

port, PricewaterhouseCoopers (2015) documents that 35% of infringement claims were successful in U.S. federal courts in 2014, and that the success rate varies across sectors.

[2]It is important to observe that even within the WTO's TRIPS agreement, which fixes the maximum patent length to 20 years, there is room for policy changes that affect the effective length of patent protection. For instance, the prosecution time—the time lapse between the filing of a patent application and its approval—is discounted from the 20 years of protection. Thus, a policy that aims to speed up the prosecution process can be effectively interpreted as an increase in patent length.

maximise innovative activity.[3] In contrast, markets in which innovations are less costly to produce or are more frequently generated, such as the software industry, patent strength is a more effective tool for promoting innovation: short but strong patents maximise innovative activity.[4] These results shed new light on industry-dependent patent policy literature, which has mainly focused on how to adapt patent policy to better balance enhanced R&D incentives with the deadweight loss associated with patent protection (Gilbert and Shapiro, 1990; Klemperer, 1990; Denicolò, 1999; Scotchmer, 1999; Cornelli and Schankerman, 1999; Hopenhayn and Mitchell, 2001; Acemoglu and Akcigit, 2012). By showing that protective policies do not necessarily lead to higher R&D incentives and that this non-monotonicity varies across industries, a policy contingent on market characteristics can be more effective at inducing faster technological progress at a lower social cost.

Patent policy also plays an important role in determining the number of firms competing in an innovative industry. The last main contribution of this paper is to endogenise the number of followers competing, finding that overly protective policies can disincentivise innovation by discouraging market entry. *Stronger* patents have an immediate effect on entry by decreasing the followers' innovation rents via higher expected license fees. Perhaps surprisingly, *longer* patent protection may also discourage entry depending on the strength of the patent system. In a weak system, longer protection increases innovation rents and encourages entry. In a strong system, in contrast, longer protection encourages entry up to a point. Because the replacement effect permeates to followers under strong patents, longer patent protection delays followers' investments, delaying their reward from innovating; i.e., longer patents can decrease the followers' benefit from participating in the market, inducing exit. Overly protective policies, therefore, not only delay investments, but also reduce the number of firms investing in R&D.

The paper proceeds as follows. The next subsection contextualises the paper relative to the literature. Section 2 introduces the baseline model and Section 3 establishes the equilibrium's existence, uniqueness, finds its analytic solution, and performs basic comparative statics. Section 4 presents a comparative dynamics analysis of the firms' investment choices with respect to changes in patent policy. Section 5 studies the policy that maximises the innovative speed of a market and

[3]In a study on the rewards necessary to induce the development of a new drug, Dubois *et al.* (2015) find that, at the mean market size, an additional \$1.8 billion in revenue is required.

[4]Consistent with this result, in an empirical study on the impact of the extension of patent rights within the software industry, Bessen and Hunt (2007) find that R&D expenditure (relative to sales) declined between 1987 and 1996.

how this policy changes across markets. Section 6 extends the previous analysis by endogenizing the number of firms competing in the market. Finally, Section 7 briefly discusses some extensions of the baseline model, and Section 8 concludes. All proofs are omitted from the main text and can be found in Appendix A.

1.1 Literature Review

Despite its importance, the set of incentives that a patent system provides is not yet fully understood. Single-innovation models (Schumpeter, 1942; Nordhaus, 1969; Kamien and Schwartz, 1974; Loury, 1979; Lee and Wilde, 1980; Reinganum, 1982; Gilbert and Shapiro, 1990; Klemperer, 1990; Denicolò, 1999) incorporate the idea that longer and more tightly enforced patents increase the returns to successful innovation.[5] As a consequence, these models predict that protective policies induce higher incentives to invest in R&D and the only cost of such policy is the efficiency loss due to lack of competition in the product market. As argued by Hall (2007) and Boldrin and Levine (2013), however, the claims that protective policies lead to a higher pace of innovation have weak empirical support; recent evidence by Qian (2007) and Lerner (2009) suggest that protective patents encourage R&D only up to a point, becoming detrimental to innovation when they are too protective.

Due to the mathematical complexities that arise in the context of sequential innovation, early work in this area focuses on the study of models of a sequence of two innovations. These theories recognise that patent protection can hinder innovation by creating a tension between the incentives given to develop first-generation technologies and to develop innovations that build upon (or complement) a first-generation technology (Scotchmer and Green, 1990; Scotchmer, 1991; Green and Scotchmer, 1995; Denicolò, 2000; Denicolò and Zanchettin, 2002; Bessen and Maskin, 2009). While insightful, these models are unable to explain how patent policy resolves this tension in a fully sequential context, where innovations both *build upon* previous technologies and *enable* future inventions.

In the context of an infinite sequence of innovation, recent papers have studied the problem of patent (or in some cases, mechanism) design. In order to solve the game analytically, however, these papers focus on environments that deliver stationary investments and payoffs, shutting down the dynamic incentives studied here. Stationarity has been attained by assuming an exogenous arrival of innovations (O'Donoghue *et al.*, 1998; Hopenhayn *et al.*, 2006; Hopenhayn and Mitchell,

[5] See Krasteva (2014) for an exception.

2013); by restricting the policy space to patents of infinite length (O'Donoghue, 1998; Bessen and Maskin, 2009; Denicolò and Zanchettin, 2012; Acemoglu and Cao, 2015); to patents that terminate stochastically in a Poison fashion (Acemoglu and Akcigit, 2012; Kiedaisch, 2015); and by restricting the agents performing R&D only to potential followers (Riis and Shi, 2012) or to only market leaders (Mookherjee and Ray, 1991; Horowitz and Lai, 1996). This paper builds upon previous literature by studying a tractable non-stationary model of sequential innovation in which patents expire at a known date; where innovations may come from the market leader or from followers; and where the value of patents, firms' investment rates, and the numbers of competitors in the industry are endogenously determined.

In environments where innovation dates are deterministically chosen, earlier work has recognised that protective policies may be detrimental to innovation. In contexts where innovations are only generated by market leaders, Mookherjee and Ray (1991) and Horowitz and Lai (1996) respectively study the role of diffusion and imitation on the leader's R&D incentives. In the former article, the leader's R&D decision is driven by preemption motives (a la Gilbert and Newbery (1982)) and not by the replacement effect.[6] In the latter work, due to the lack of competition from followers, the leader only innovates when its active patent expires. Consequently, either no patent protection or an infinitely long patent lead to no innovation in the market. Their result, however, is not robust to environments in which followers can invest to leapfrog the market leader. Similarly, in contexts where only followers innovate, Koo and Wright (2010) recognise that followers may have incentives to delay their innovations in order to bargain for lower license fees.

This article also contributes to the literature on optimal patent design. Previous works has studied environments where more protective policies lead to a higher pace of innovation. Their main focus is to find the policy that balances R&D incentives with the cost (deadweight loss) associated with patent protection. With this trade-off in mind, and in the context of a single innovation, Gilbert and Shapiro (1990) and Klemperer (1990) study the optimal patent length and breadth as a function of market characteristics. Scotchmer (1999) and Cornelli and Schankerman (1999) study how patent renewal systems could lead firms to self-select into the right length, and Hopenhayn and Mitchell (2001) study conditions under which

[6]To highlight the underlying difference, suppose a leader that faces no competition. In the aforementioned paper no innovation would occur, regardless of characteristics of patent policy; innovation is driven to preempt competition. Here, the leader will still have incentives to delay when longer patent protection is offered.

firms self-select into the right combination of length and breadth. Denicolò (2000) study the trade-off of offering longer patents in the context of two innovations. In stationary models of an infinite sequence of innovations, Hopenhayn *et al.* (2006) study how a buyout scheme can implement the optimal policy when firms have private information about the innovation's characteristics. Acemoglu and Akcigit (2012) identify that the aforementioned trade-off changes with the technology gap that exists between leaders and followers. They show that a policy contingent on this gap is more effective to promote innovation at a lower social cost. This article contributes to this literature by showing that protective policies do not necessarily lead to higher R&D incentives and that this non-monotonicity varies across industries, adding a new relevant dimension to consider in the design of patent policy.

This article also aims to deepen our understanding about how patents and other institutional change shape incentives and market structure in innovative industries. Along these lines, this work complements Segal and Whinston (2007), who study how antitrust regulation in the post-innovation market affects firms' innovation outcomes; Marshall and Parra (2016, 2017), who study the tradeoffs that arise when mergers in an innovative industry are allowed. Finally, Katzwer (2015) simulates the market equilibrium of a model with similar features to the ones presented in this paper.

2 A Model of Sequential Innovation

Consider a continuous-time economy characterised by an infinitely-long ladder of innovations. Firms compete investing in R&D to (stochastically) achieve an innovation—protected by a patent—and temporarily reach the technological lead in the market. The firm with the leading technology is called the *leader* and is denoted by l. All other firms, the *followers*, only have access to obsolete technologies and are denoted by f. The leader invests in R&D in order to extend its lead in the market, whereas followers invest in R&D to *leapfrog* the current leader in the technological ladder and become the new market leader. Payoffs are discounted at a rate $r > 0$.

Denote by t the time that has passed since the *last* innovation, i.e., $t = 0$ represents the arrival of a new innovation. Let v_t represent the leader's value of possessing a patent that has been active for t years. The value of v_t is endogenously determined and depends on the underlying patent system, the profit flow that the leader receives while holding the patent, and the R&D decisions of every firm in the market. A *patent system* consists of a *statutory length* $T \in \mathbb{R}^+$, denoting the amount of time that a leader will be able to exclude others from commercializing its *current* technology, and of a *patent strength* $b \in [0, 1]$, denoting the probability that a *new* innovation will be considered to effectively infringe on the leader's patent. I assume that all innovations are patentable. While a leader's infringement of its own patent has no active consequences, followers must pay a compulsory license fee to be able to profit from any innovation that infringes on an active patent.[7] The license fee is assumed equal to the damages that the leader suffers from the commercialization of the new product; i.e., for an innovation that occurs at instant t, license fee are equal to v_t. For all of the participants in this market, the tuple (T, b) is considered common knowledge and exogenously given.[8,9]

While a patent is active, the leader receives a monopoly flow of profits $\pi > 0$. When the patent expires, at $t = T$, competition in the product market drives the leader's profit flow to zero and the patent loses its value, i.e., $v_T = 0$. As soon as an innovation occurs, the innovating firm patents its new technology, gaining the right to exclude others from using it, in exchange for making this new technology known to the public. As a consequence of this release of information, any innovation produced by a follower will build upon the latest technology, leapfrogging the current leader. For ease in exposition, I assume that the patent of obsolete technologies that have not yet expired are too costly to enforce. Thus, obsolete technologies are imitated, driving the profits of a *replaced* leader to zero. In concrete terms, the conjunction of these assumptions implies that there is, at most, a

[7]In a case of an infringement, the leader may also choose to forbid the utilization of the new innovation. In order to show that overly protective policies slow the pace of innovation down, I use the best case for patents by assuming compulsory licensing (Tandon, 1982).

[8]To better illustrate how a patent system works, consider a patent that grants no strength ($b = 0$). Under such system the leader is able to preclude imitation of its *current* technology for T years. The leader, however, has no protection against innovations that *advance* through the technology ladder; i.e., no license fees can be collected from any innovation that *improves upon* the leader's technology.

[9]There may be concern about the possibility of using patent strength as a policy tool. Although, for tractability purposes, I treat b as a continuous policy parameter, there are certainly episodes where the strength of patents has changed. Jaffe and Lerner (2011) argue that the creation of the U.S. States Court of Appeals of the Federal Circuit significantly increased the number of infringement claims found to be valid. The Federal Trade Commission (2003) proposed a number of changes to promote innovation. Among the suggestions was weakening patent protection by lowering the requirements to prove the invalidity of an active patent. In particular, the Commission proposed changing the law from "clear and convincing evidence" to "the preponderance of evidence".

one-step lead between the technology leader and its competitors.[10]

In order to develop an innovation, firms invest in R&D. These investments lead to a stochastic arrival of innovations, which is an increasing function of the firms' investments. At every t, each firm $k \in \{l, f\}$ chooses an R&D investment flow $x_{k,t} \geq 0$. The instantaneous cost flow of this investment is given by the cost function $c(x)$. I assume that the cost is increasing in x, differentiable, strictly convex, $c'(0) = 0$ and $\lim_{x\to\infty} c'(x) = \infty$ (in order to obtain an analytical solution, below I will focus on the case in which $c(x) = x^2/2$).

Firm k's investment induces an arrival of innovations described by a non-homogeneous Poisson process. The arrival rate of the leader at instant t is $\lambda x_{l,t}$ with $\lambda > 0$, whereas the arrival rate of the followers is given by $\mu x_{f,t}$ with $\mu > 0$. The parameters λ and μ represent different levels of R&D productivity that firms may have. For instance, with $\lambda > \mu$ we can represent a situation in which the leader has an advantage in building upon its own technology.[11] The Poisson processes are independent among firms and generate a stochastic process that is memoryless but potentially non-stationary. The waiting time between two innovations is described by an exponential distribution with the probability of observing an innovation by instant t equal to $1 - \exp(-z_{0,t})$ where $z_{\tau,t} = \int_\tau^t (\lambda x_{l,s} + \mu x_{f,s})\, ds$ is a measure of the accumulated investments from instant τ to instant t.

Timing of the game The timing of the game, depicted in Figure 1, is as follows. When an innovation arrives, the time index t is reset to zero. From that time and onward, and while the leader's status lasts, the patent holder receives the monopoly profit flow π. The followers, on the other hand, obtain zero profit as they only have access to obsolete technologies. In order to simplify exposition and obtain analytical results, it is assumed that at each instant in time t the leader faces a different follower. Each follower is assumed to invest in R&D only once in the game, maximizing its instantaneous payoff. Thus, the investment $x_{f,t}$

[10]The assumption that old patents are costly to enforce also implies that two consecutive innovations by a leader do not increase the stream of profits π, i.e., the only benefits that a leader derives from an innovation are extending the clock of its patent protection and, in equilibrium, discouraging followers from investing. Section 7 explores the scenario in which consecutive innovations by the leader not only extends the patent clock but also increases the profit flow. I show that the main intuitions derived in the simplified model are still present there.

[11]In the context of the quadratic cost assumption, λ and μ are also measures of how costly it is to innovate for each type of firm. For instance, assume that the leader's productivity is $\tilde{\lambda}$ and its cost function is $\tilde{c}(x) = \rho c(x)$ for some positive ρ. Then, after redefining $\lambda = \tilde{\lambda}/\sqrt{\rho}$, we could proceed with the original (λ, $c(x)$) formulation with the reinterpretation that higher cost of innovation ρ will lead to lower productivity for the leader λ.

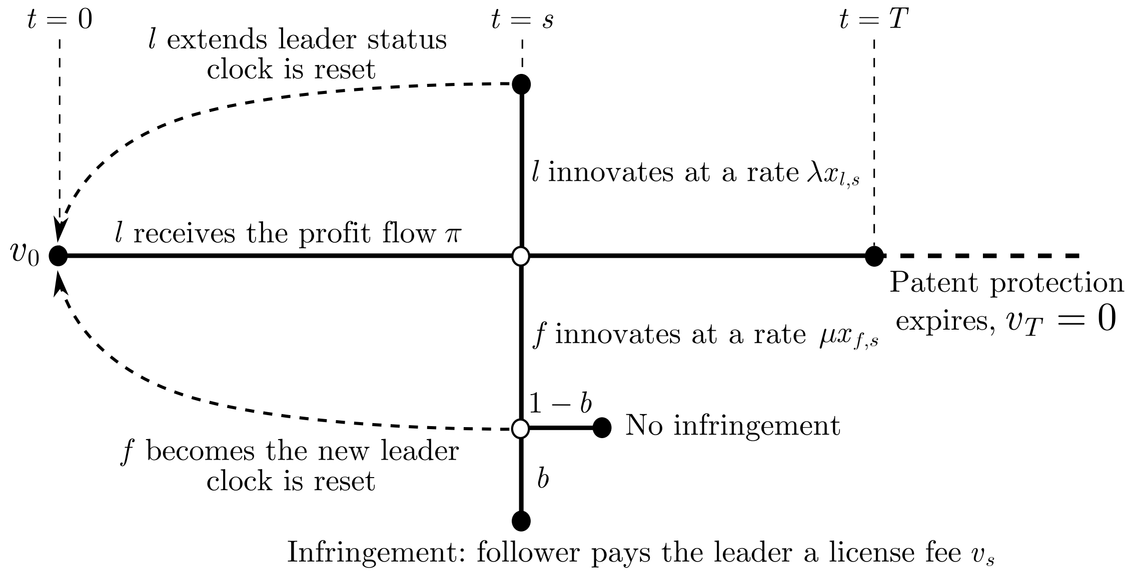

Figure 1: Timing of the game

represents how the R&D of the different followers evolves through time. At every t, both the leader and the investing follower choose their investments simultaneously, determining the arrival rate of innovation for both firms. Section 6 studies a model in which the number of followers is endogenously determined and these followers play throughout the game. I show that key intuitions and results are robust to this simplified treatment of followers.

When an innovation occurs, the succeeding firm becomes the new leader, and its technology renders the currently patented technology obsolete. If the innovating firm is a follower, with exogenous probability b, the leader is able to enforce the patent, and the follower's innovation is considered to infringe on the leader's patent. In that case, the follower must pay a compulsory license fee of v_t, equal to the damages caused to the leader due to the commercialization of the new innovation. If no innovation has occurred within the statutory length of the patent, the patent holder loses its leader status and becomes one of the many followers of the game. Consequently, no license fees can be charged for innovations that occur after T and the continuation value of a patent is zero; i.e., $v_T = 0$.

Model interpretation The model admits a wide variety of applications commonly studied in the literature. A natural interpretation is to understand each breakthrough as a process (cost-saving) innovation in the context of a homogeneous-good market under price competition. In that context, only the firm with the lowest marginal cost of production obtains positive profits.[12] Similarly, the model can also be interpreted as firms competing through the quality of their products. For example, firms compete in price and the consumers' willingness to pay is equal to the product's quality. Then, the leader's profit is a function of the quality gap between its product and that of the followers (see O'Donoghue *et al.* (1998)).

The model also accommodates the traditional (Schumpeterian) creative-destruction interpretation in which each innovation completely replaces the old technology rendering the previous one obsolete—e.g., the microprocessor industry. Finally, the profit flow π can be interpreted as coming from the direct commercialization of the innovation, or from licensing the technology to a downstream market.

Payoffs and strategies Given any sequence of investments by the followers $\{x_{f,t}\}_{t=0}^{T}$, from the perspective of time s, the leader's value of possessing a patent that has been active for s years is:

$$v_s = \max_{\{x_{l,t}\}_{t=s}^{T}} \int_s^T \left(\pi + \lambda x_{l,t} v_0 + \mu x_{f,t} b v_t - c\left(x_{l,t}\right)\right) e^{-z_{s,t}} e^{-r(t-s)} dt. \tag{1}$$

That is, with probability $\exp(-z_{s,t})$, no innovation has occurred between instant s and t, and the patent is still active at t. At that instant t, the leader receives the flow payoff π and pays the flow cost of its investment $c(x_{l,t})$. The R&D investment results in an innovation at a rate of $\lambda x_{l,t}$, obtaining the benefit of a brand new patent v_0. On the other hand, the follower investing at instant t may succeed at a rate of $\mu x_{f,t}$, in which case it may infringe on the current patent with probability b, and have to pay the leader a compulsory license fee of v_t. All of these payoffs are discounted by $\exp(-r(t-s))$.

On the other hand, at each instant t, a new follower decides how much to invest by maximizing its instantaneous flow payoff. At every t during which a patent is active, the flow payoff is given by:

$$\mu x_{f,t} \left(\left(1-b\right) v_0 + b(v_0 - v_t)\right) - c\left(x_{f,t}\right).$$

This is the follower's reward from a new innovation v_0, minus the expected license fee bv_t, adjusted by the arrival rate induced by its investment $\mu x_{f,t}$, minus the cost

[12] For illustration purposes, consider the isoelastic demand $q = a/p$. Suppose that each innovation decreases the marginal cost of production by a factor of $\beta \in (0,1)$; so that, after n innovations the marginal costs $c_n = \beta c_{n-1}$. Then, due to price competition and for any n, the market leader profits are equal to $\pi = (p-c)q = (c_n - \beta c_n)a/c_n = (1-\beta)a$.

of its investment $c(x_{f,t})$. Thus, the investment rate of the follower at instant t is implicitly given by:

$$c'(x^*_{f,t}) = \mu(v_0 - bv_t). \tag{2}$$

Similarly, when no patent is active and no license fee can be charged for an innovation, the followers' investments become constant and are implicitly given by $c'(x^*_{f,t}) = \mu v_0$.

In this model, t is the only state variable of the dynamic game. I study the Markov Perfect equilibria of this game by restricting attention to strategies that are a mapping from the time since the last innovation occurred, t, to an R&D intensity.

3 The Leader's Problem

In this section, I solve the leader's problem by using optimal control techniques. I start by assuming that the value of a new innovation is known and equal to $\hat{v}$.[13] Next, I apply the Principle of Optimality to derive the Hamilton-Jacobi-Bellman (HJB) equation, which provides necessary and sufficient conditions for a maximum. Maximizing the HJB equation, I find the leader's optimal investment rule, which is used to solve for the value of possessing a patent at t. The previous solution is implicitly defined in terms of the conjectured value $\hat{v}$. I show that there is a unique value of $\hat{v}$ that is consistent with the solution; i.e., there is a unique value $\hat{v}$ such that $v_0 = \hat{v}$.

Let $x_t = \lambda x_{l,t} + \mu x_{f,t}$, starting at an arbitrarily small time interval $[t, t+dt)$; the leader's value of having a patent for t years must satisfy the Principle of Optimality:

$$v_t = \max_{x_{l,t}} \left\{ (\pi + \lambda x_{l,t}\hat{v} + \mu x_{f,t} b v_t - c(x_{l,t}))dt + e^{-rdt}\mathbb{E}\left[v_{t+dt}|x_t\right] \right\}.$$

That is, evaluated at the optimal strategy, the value of having a patent at t is equal to the payoff flow at that instant in time, plus the discounted expected continuation value of the patent.

For sufficiently small dt, the discount factor $\exp(-rdt)$ is equal to $1 - rdt$. On the other hand, the expected continuation value of the patent $\mathbb{E}[v_{t+dt}|x_t]$ is equal to the probability of not having an innovation today $1 - x_t dt$, multiplied by the value

[13] To be clear, from this point on, v_0 represents the value of an active patent that was just issued, and $\hat{v}$ represents the value of the next innovation; i.e., a patent that has not yet been issued. In equilibrium, $v_0 = \hat{v}$.

of a patent tomorrow $v_{t+dt} = v_t + v_t' dt$, plus the probability that an innovation occurs $x_t dt$ times the continuation value of the *current* patent after an innovation, which is zero. Using the previous expressions in the equation derived from the Principle of Optimality and neglecting terms of order dt^2, I obtain the following HJB equation:

$$rv_t = \max_{x_{l,t}} \left\{ \pi + \lambda x_{l,t} \left(\hat{v} - v_t \right) - \mu x_{f,t} \left(1 - b \right) v_t - c \left(x_{l,t} \right) + v_t' \right\}. \tag{3}$$

Condition (3) is necessary and sufficient for a solution to be a maximum. The leader's optimal R&D intensity is determined by its first-order condition:

$$c' \left(x_{l,t}^* \right) = \lambda \left(\hat{v} - v_t \right). \tag{4}$$

Equations (2) and (4) are very informative about the firms' R&D investment dynamics. They state that at any instant t, the firms' marginal benefit of their R&D investments is a function of the *incremental value* that the firms obtain from innovating. For the leader, this is the expected profits from a new patent $\hat{v}$, minus the expected profit loss from giving up the currently active patent v_t. For the follower investing at instant t, the incremental value corresponds to the profits from a new patent, minus the expected license fee bv_t that the follower has to pay in order to commercialise its innovation.

Proposition 1 (R&D over time) *At the beginning of a patent race ($t = 0$), leaders do not invest in R&D whereas followers invest at a positive rate whenever $b < 1$. As the patent approaches its expiration date, both types of firms perform increasing investments over time. When firms are equally productive ($\lambda = \mu$), the leader's and followers' investments converge at the end of the patent life.*

When the value of an active patent declines with the proximity of its expiration date, both types of firms perform increasing investments over time (see Figure 2(a)). At the beginning of a patent race, and as long as $b < 1$, followers invest at a higher rate than the leader. The leader starts performing zero R&D at $t = 0$ as $v_0 = \hat{v}$ in equilibrium, whereas the followers' investments start at $c'(x_{f,0}^*) = \mu(1 - b)\hat{v}$. Investments reach their maximum at $t = T$, when patent protection expires and the value of the patent becomes zero. At this point, the leader invests at an implicitly given rate of $c'(x_{l,T}^*) = \lambda \hat{v}$ and, from then on, the followers invest at a rate of $c'(x_{f,t}^*) = \mu \hat{v}$.

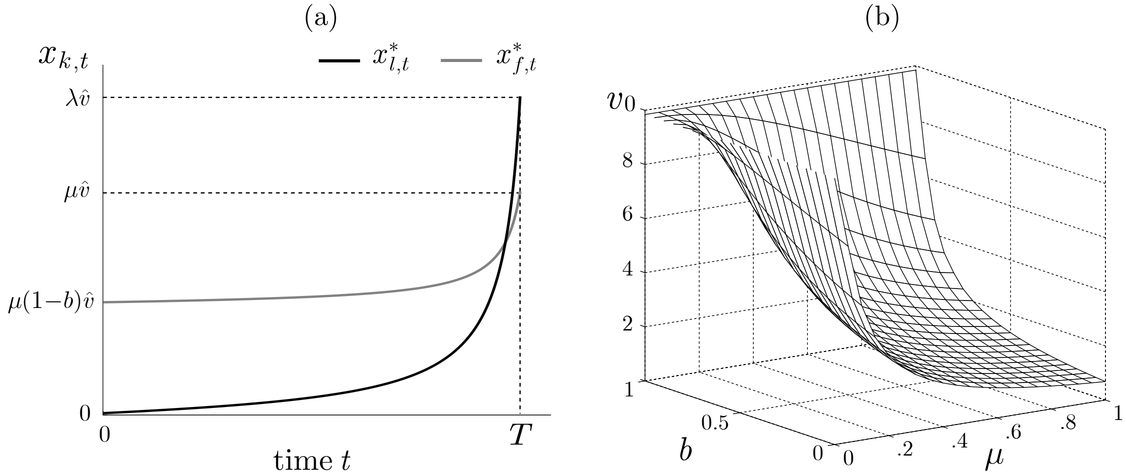

Figure 2: (a) R&D through time when $\lambda > \mu$. (b) Value of a new patent under different patent strength and followers' productivity.

Notes: (a) Depiction of the quadratic cost case. (b) Parameters's values are $r = 5\%$, $T = 20$, $\pi = 1/2$, and $\lambda = 1$.

To obtain an analytic solution, I assume a quadratic R&D cost $c(x) = x^2/2$. Substituting the leader's and the followers' investments into (3) and using the cost assumption, the following ordinary differential equation is obtained:[14]

$$-v_t' = av_t^2 - \theta v_t + \pi + \frac{1}{2}(\lambda\hat{v})^2 \tag{5}$$

where $a = \lambda^2/2 + \mu^2 b(1-b)$ and $\theta = r + \lambda^2\hat{v} + \mu^2(1-b)\hat{v}$ are positive constants. This is a separable Riccati differential equation, which has a unique solution satisfying the boundary condition that a patent has no value at its expiration date $v_T = 0$. The solution to equation (5) is given by:

$$v_t = \frac{(2\pi + (\lambda\hat{v})^2)(e^{\phi(T-t)} - 1)}{\theta(e^{\phi(T-t)} - 1) + \phi(e^{\phi(T-t)} + 1)} \tag{6}$$

where $\phi = (\theta^2 - 2a(2\pi + (\lambda\hat{v})^2))^{1/2}$ (see Online Appendix C for details).

Equation (6) shows that the value of a patent v_t depends on the conjectured value $\hat{v}$ and is a decreasing function of t. In order to have a well-defined solution, it is necessary to show that a fixed point to $v_0 = \hat{v}$ exists. The next proposition establishes the existence and uniqueness of such a fixed point.

[14] This specification of equation (5) assumes that firms' investments are strictly positive for $t > 0$ as will be the case in equilibrium. However, the conjectured value $\hat{v}$ may be sufficiently low that the leader or both firms may choose not to perform R&D for some t. Online Appendix C provides details on this matter.

Proposition 2 (Existence and uniqueness) *There is a unique* $\hat{v} > 0$ *such that* $v_0 = \hat{v}$. *The value of a patent* v_t *decreases with* t *and is given by equation (6) evaluated at the fixed-point* $\hat{v}$. *In equilibrium, firms' R&D investments are given by* $x^*_{l,t} = \lambda(v_0 - v_t)$ *and* $x^*_{f,t} = \mu(v_0 - bv_t)$.

When innovation is sequential, the value of a patent is endogenously determined by details of patent policy, the parameters of the model, and the firms' investment decisions. In equilibrium, an exogenous change in a parameter of the model will have two, often opposing, effects. On the one hand, there is a direct effect on the patent race taking place at the moment of the change. These types of effects can often be captured in single-innovation models. On the other hand, there is an indirect effect on the patent races that take place in the future, which is captured through changes in the fixed-point and affecting the value of a patent's v_t at every point of the patent life. By construction, these latter effects can only be identified by sequential innovation models in which the value of innovations are endogenously determined. These two effects will play an important role in understanding the impact of patent policy in the firms' investment dynamics. Despite these two forces, the next proposition shows that the main comparative statics of the model work as expected.

Proposition 3 (Value of a patent) *The equilibrium value of patent* v_t *increases with:* i) *an increase in the profit flow* π; ii) *a decrease in the interest rate* r; iii) *an increase in the statutory length of patents'* T; iv) *an increase in the leader's productivity* λ; v) *a decrease in the followers' productivity* μ *under low levels of patent strength* b *and has no effect when patents are strongest (i.e,* $b = 1$*); and* vi) *an increase in patents strength* b *for all* t *when* $b \leq 1/2$ *and, for* $b > 1/2$*, there exists* $\hat{t} < T$ *such that* v_t *increases in* b *whenever* $t \geq \hat{t}$.

Claims in Proposition 3 are quite intuitive. If the discounted flow of monopoly profits is higher, if the leader is relatively more productive, or if patents are more protective, the equilibrium value of a patent increases. Numerical results suggest that the conditions in vi), for stronger protection to increase the value of a patent, are not necessary and an increase in strength would always lead to an increase in the value of a patent (i.e., $\hat{t} = 0$; see Figure 2(b) for an example).[15] Under full patent strength ($b = 1$), an increase in the followers' productivity has no effect

[15] Despite the direct effect of b in v_t being always positive. The indirect effect that the fixed-point has in v_t is hard to sign when $b > 1/2$.

on the value of a patent as, when a follower innovates, the leader is able to fully capture—through license fees—the reminder value of its patent v_t.

It is interesting to observe that an increase in the overall R&D productivity within an industry—i.e., a proportional and simultaneous increase in both λ and μ—may have different effects on the value of a patent depending on the strength of the patent system. The strength of a patent affects the relative weight given to the increase in productivity of each type of firm. Under a strong patent system, the value of a new patent increases with the overall R&D productivity in the market. Strong patent protection dissipates the effect of an increase in competition, due to higher follower productivity, through the expected license fees that the leader gets. When patent protection is weak, in contrast, the value of a patent may actually decrease, as the effect of an increase in competition can dominate the increase in the leader's productivity. In practical terms, this implies that unless a market leader has strong protection against future innovations, it will not generally benefit from a policy that facilitates innovation at an industry-wide level; i.e., market leaders may have incentives to lobby against such measures.

To conclude this section, I connect the model to the literature on growth through innovations (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Aghion *et al.*, 2001). In these models, patents do not expire ($T = \infty$), and, as a consequence, investments are constant through time, and market leaders perform no R&D.

Proposition 4 (Stationarity) *In the limit, when patent protection is infinitely long ($T = \infty$), the firms' investments become constant. The leader performs no R&D, whereas followers' investments are $x_f = \mu(1-b)v_\infty$, where v_∞ is the value of a patent, which is independent of t and equal to:*

$$v_\infty = \frac{1}{2\mu^2 (1-b)^2} \left(\sqrt{r^2 + 4\mu^2 \pi (1-b)^2} - r \right) \tag{7}$$

when $b < 1$, and equal to $v_\infty = \pi / r$ when $b = 1$.

When patent protection is infinitely long, incentives become stationary and the leader performs no R&D. For the leader, this is because a new innovation merely replaces the currently active patent with one of the same value. Since the protection of a patent never expires, the leader faces the same incentives at any two moments in time, and the value of an active patent remains constant over time.

Similarly, the followers' incentives become stationary, as the license fees they have to pay in the case of an infringement do not decrease over time.

4 Patent Policy and R&D Dynamics

The purpose of this section is to study how the different tools granted by patent policy—patent length and strength—affect firms' R&D dynamic incentives. In particular, I investigate the asymmetries that may exist between leaders and followers at the moment of reacting to a policy change and how the two tools interact at the moment of incentivizing firms.

Proposition 5 (Patent length and leader R&D) *An increase in patent length T delays the leader's investments; i.e., decreases the leader's R&D at the beginning of the patent's life, but increases it towards the end.*

Proposition 5 explores how a change in patent length affects the leader's R&D investment throughout the patent life. As Arrow (1962) identified, at any instant t, the leader's investment is a function of the incremental value of an innovation. For the leader, the incremental value is equal to $v_0 - v_t$, corresponding to the value of a new patent minus the cannibalised benefits of the old patent. An increase in patent length increases the value of an active patent v_t for all $t < T$ (Proposition 3). Consequently, the equilibrium effect of an increase in patent length will depend on how the magnitude of the increase in v_t changes throughout the patent life t. The driving force of Proposition 5 is that the increase in v_t becomes larger the closer the active patent is to its expiration date, decreasing the leader's value of an innovation and inducing the leader to decrease its R&D investments at that instant t. The delay effect follows from observing that the leader's investment at the end of its patent life, $x_{l,T+dT} = \lambda v_0$, must be higher as the value of a new patent is an increasing function of patent length T (see Figure 3(a)).

The intuition of why the leader delays its investments follows from observing that the *effective* duration of a patent generally differs from its *statutory* length. When longer patent protection is offered, the probability of actually reaching and benefiting from the patent extension is higher when the patent is close to its expiration date T. This implies that the *effective* gain due to the increase in duration is larger the closer the patent is to its expiration date, reducing the incremental value of an innovation $v_0 - v_t$, decreasing investments at any instant $t < T$. The

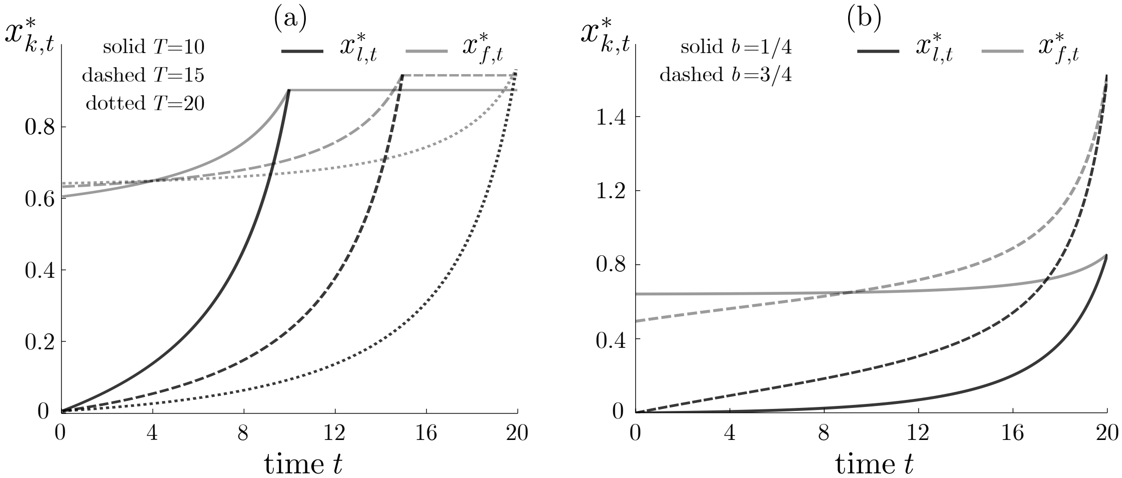

Figure 3: R&D investments under different patent: (a) length. (b) strength.
Note: Parameters' values are $r = 5\%$, $\pi = 1/2$, $\lambda = \mu = 2/5$, and, when fixed, $T = 20$ and $b = 1/3$.

net effect of a change in patent length on the leader's total investment in R&D is hard to quantify and is explored further in Section 5. We can say, however, as a consequence of Proposition 4, that the total effect must be non-monotonic in T, as both $T \in \{0, \infty\}$ induce leaders to perform no R&D.

Proposition 6 (Patent length and follower R&D) *The effect of an increase in patent length T on followers' investments depends on the strength of the patent system. When patents offer no protection against future innovation ($b = 0$), followers' investments increase in T. When patents strength is maximal ($b = 1$), followers delay their investments.*

The effect that an increase in patent length has on the followers' R&D investments depends on the strength of the patent system. This can be seen by rewriting the followers' investments at t as:

$$x_{f,t} = \mu \left[(1 - b)v_0 + b(v_0 - v_t)\right].$$

The total effect of an increase in patent length on the followers' investments is a convex combination of the impact it has on the value of a new patent v_0 and on the incremental value of developing an innovation, $v_0 - v_t$. On the one hand, longer patent protection increases the value of an innovation, v_0, incentivizing followers to invest in R&D. On the other hand, stronger patent protection leads followers to internalise the replacement effect, leading followers to delay their investments.

At the limit, when $b = 1$, followers fully internalise the replacement effect, delaying investments as much as the leader. Figure 3(a) suggests that the followers' internalization of the replacement effect is quite strong, dominating the increase in value of a new patent even with low levels of patent strength. In terms of policy design, explored further in Section 5, these observations imply that a system providing long and strong protection will discourage innovation, decreasing the economy's innovation pace. For instance, at the limit—as can be seen in Proposition 4—under the most protective patent system ($T = \infty$ and $b = 1$), no R&D would be performed.

The next proposition connects the sequential model with traditional single-innovation models by highlighting what drives the delay effect in Proposition 5.

Proposition 7 (Grandfathering) *If an increase in the statutory length T does not apply to currently active patents, but does apply to all subsequent innovations, then the leader and followers will increase their R&D intensity in the patent race in which the change in policy takes place.*[16]

It is interesting to observe the contrast in incentives that exist between sequential and single-innovation models. In the latter, more protective patents—modeled as an increase in the value of achieving the next innovation—induce all firms to invest more in R&D. This effect is present in sequential models and can be isolated by looking at the effects on current R&D investments when a change in policy has been grandfathered until the next innovation arrives. When an increase in patent length only affects future innovations; i.e., it does not apply to patents that are currently active (which in the model translates to an increase in v_0 but not in v_t), the effect in R&D coincides with that predicted by single-innovation models: larger rewards lead to higher innovation rates. This observation tells us that is precisely the sequential structure of the model, and its implied endogeneity of v_t, that leads to changes in policy to affect the replacement value of an active patent, inducing the delay in investments.

Proposition 8 (Patent strength and R&D) *An increase in patent strength b that increases the value of a new patent* i) *delays the followers' investments when $b \geq 1/2$, and* ii) *increases the leader's R&D towards the end of the patent's life.*

[16] To be clear, when the policy change has been grandfathered, R&D will increase only in the first race; in all subsequent races, R&D will present the dynamics described in Proposition 5.

Stronger patents have a direct negative effect on the followers' incremental value of an innovation due to higher expected license fees paid in the case of achieving a breakthrough. This leads to a decrease in the followers' investment rates at the beginning of the patent's life.[17] As the patent expiration date approaches, expected licenses fees decrease to zero, fading the effect of an increase in strength away. In particular, at $t = T$, the effect of an increase in b in the value of an active patent v_T is zero. Hence, the market leader and followers increase their R&D investment towards the end of the patent's life. These effects are shown in Figure 3(b), which depicts firms' investment dynamics for different levels of strength b.

The combination of Propositions 5 and 8 provides clear and testable empirical predictions. First, the probability that a leader innovates upon its own technology increases as the patent expiration date approaches. Second, followers are relatively more likely to innovate at the beginning of the patent's life, but this difference converges, and may even reverse, as the patent expiration date approaches. In addition, the proportion of innovations generated by followers depends on the strength of the patent. In markets with strong patent protection, the innovation patterns of the followers would tend to mimic those of the leader; whereas, under weak patent protection—or in markets in which infringements are harder to determine—followers' innovations will be more prevalent.

5 Patent Policy and the Pace of Innovation

In this section, I study different policies in terms of their capacity to generate higher innovation rates. In particular, I study the policy that maximises the pace of innovation and how this policy varies across markets according to the market's R&D productivity. There are many reasons to follow this methodology, as opposed to maximizing total welfare.[18] First, because one of the main findings of the previous section is the delay induced by protective policies, focusing on the pace of

[17] Once again, despite the direct effect of an increase in b always decreases $x_{f,0}$, the indirect effect (fixed-point change) makes this comparative static hard to sign. More generally, it can be proven that: if π is larger than a very mild lower bound, for each b there exists $\hat{T}$ such that $T \geq \hat{T}$ implies that an increase in b decreases $x_{f,0}$.

[18] To introduce welfare into the analysis entails an extra layer of complexity that goes beyond the scope of this paper. Although total welfare is affected by both the pace of innovation and the deadweight loss induced by patent protection. This loss depends on factors that are outside the model such as: the demand elasticity, how firms compete (prices, qualities), and the nature of innovations (cost saving, quality improvements, new products). It is intuitive to see that adding the deadweight loss into the analysis will result in shorter prescribed patents.

innovation allows us to quantify the extent of this delay and enables us to obtain a measure that can be directly compared across markets. Also, by focusing on the pace we can better understand how patent length and strength incentivise firms to perform R&D and, in particular, capture the interaction between these tools at the moment of incentivizing firms to engage in R&D. Finally, because, in principle, the pace of innovation is easier to measure than total welfare, much of the applied work and policy discussions focus on the former rather than the latter.[19]

Start by decomposing the followers' R&D productivity into two factors $\mu = \lambda\alpha$. The factor λ is now common among firms and represents the *market's R&D productivity*.[20] On the other hand, the factor α captures the relative productivity of the followers with respect to the leader. I study the policy (T^*, b^*) that maximises the pace of innovation as a function of the market's R&D productivity λ. For simplicity, from now on I refer (T^*, b^*) as the optimal policy, with the understanding that I mean the policy that maximises the pace of innovation.

To define our measure of innovative activity, I leverage from the property that innovations follow a non-homogeneous exponential distribution. In particular, I study the policy that minimises the market's expected waiting time between innovations, which is given by:[21]

$$\mathbb{E}[t] = \int_0^\infty x_t t e^{-z_{0,t}} dt. \tag{8}$$

Proposition 9 (Long patents discourage R&D) *The optimal policy* (T^*, b^*) *consists of a finite patent length.*

When innovation is sequential, longer patents promote R&D investments with diminishing returns and, at some point, they become detrimental to innovation (see Figure 4(a) for an example). The intuition here is quite simple: when patent protection is short, rewards are too low to induce high innovation rates and, in the limit ($T = 0$), no R&D is performed. On the other hand, although longer patents increase investments after patents expire, they also delay the leader's investments and possibly those of the followers (depending on patent strength). Under infinitely

[19] Article 1, Section 8, Clause 8 of the U.S. Constitution explicitly defines the goal of intellectual property as "to promote the progress of science" instead of to maximise society's welfare.

[20] As mentioned in footnote 11, λ is also a measure of how costly it is to produce an innovation.

[21] For the purpose of illustration, if investments were constant through time and equal to λ (i.e., $x_t = \lambda$ for all t), the distribution of successes will follow an exponential distribution with an arrival rate equal to λ. In that case we have $\mathbb{E}[t] = \lambda^{-1}$, and the expected waiting time between innovations corresponds exactly to the inverse of the market's R&D productivity.

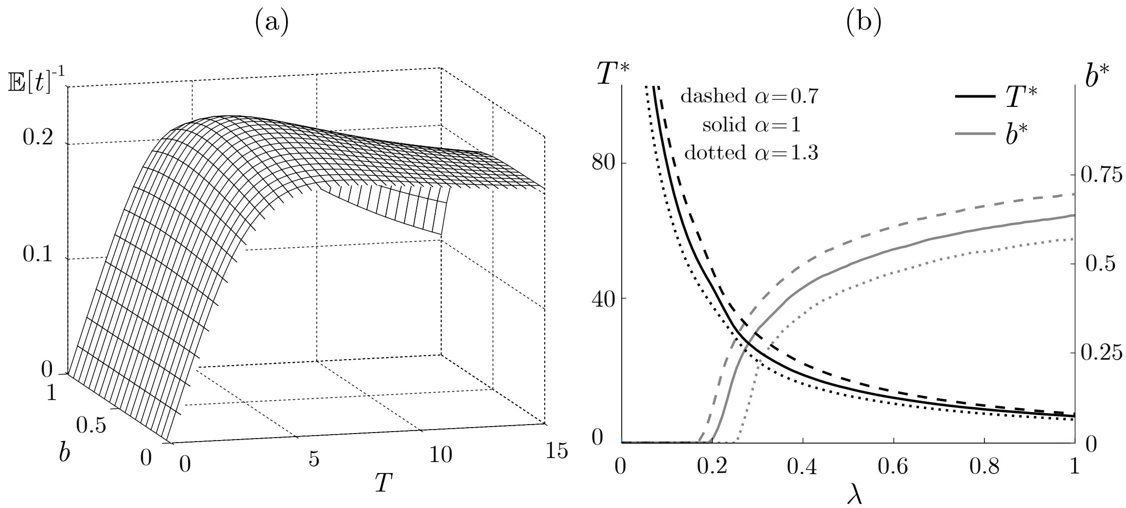

Figure 4: (a) Optimal policy has an interior solution. (b) Optimal policy as a function of the market's R&D productivity λ.

Note: Parameters' values are $r = 5\%$, $\pi = 1/20$. Figure (a) also uses $\lambda = \mu = \alpha = 1$.

long patents, the increase in R&D after the patent expires becomes irrelevant, and the leader delays its investments perpetually (Proposition 4), performing no R&D, and thus increasing the waiting time between innovations.

Proposition 9 builds upon a literature that has shown different mechanisms under which long patent protection may be detrimental to innovation. In the context of a single innovation, Gallini (1992) shows that longer patents encourage entry by counterfeiters, thus limiting the ability to reward innovators and becoming ineffective after some point. Horowitz and Lai (1996) study an environment in which innovations are only generated by a market leader who waits until the patent expires to introduce its new innovation. They find that infinitely long patents induce no innovation, as the leader delays the innovation introduction forever. Proposition 9 extends these results by showing that the discouragement effect of a patent still exists in environments in which followers can also perform R&D and breakthroughs occur stochastically. Finally, Bessen and Maskin (2009) show that when innovations are sequential and *complementary*—i.e., a new innovation increases the value of the whole sequence of innovations—patent protection may be detrimental to innovation as it makes it difficult to appropriate the rents created from a new innovation, diminishing the overall incentives to perform R&D. Proposition 9 extends their result to a scenario in which innovations do not add additional value, as innovations completely cannibalise previous ones.

Unfortunately, despite having a unique equilibrium with closed form solutions for the value of a patent v_t and firms R&D investments x_t, the integral (8) does not have a closed form solution when $b > 0$. This, added to changes in policy also leading to a new equilibrium fixed point, makes the analytical computation of (T^*, b^*) unattainable. I, therefore, use numerical methods to compute (8) exactly. As shown by Figure 4(a), equation (8) is smooth on the model's parameters and its maximization delivers a unique solution.

Result 10 ((T^*, b^*) across markets) *An increase in the market's R&D productivity λ decreases the optimal length T^* and increases the optimal strength b^*.*[22]

Result 10 characterises how the optimal policy changes across different markets according to the market's R&D productivity; see Figure 4(b) and Table 1. From the perspective of a policymaker, the result states that patent length and strength are *complementary* in the sense that one tool is more effective at providing incentives when the other tool decreases its effectiveness. It tells us that long but weak patents are more effective in markets where innovations are costly to produce or are harder to achieve, and that short but strong patents are more effective in markets where innovations occur frequently or are not too costly to produce.

To understand the intuition behind this result, compare the incentives present in markets that innovate quickly, such as the software industry, with those that are present in markets that innovate more slowly, such as the pharmaceutical sector. In markets in which innovations are frequent or are not too costly to generate (high λ), patent length is an ineffective tool for encouraging innovation, as the *expected* duration of a patent changes little when longer protection is offered. For instance, to increase patent length from twenty to twenty-one years in an industry in which the current expected arrival time between innovations is three years does very little to increase the value of an innovation, as the patent is likely to become obsolete long before the patent's expiration date becomes *binding*.[23] Moreover, longer patent protection will induce the leader to delay its investment, thereby slowing down the pace of innovation. In this context, patent length plays little role in providing incentives to R&D and, therefore, only strength can be used to reward innovation. Strong patents, on the other hand, allows the social planer to provide—while the patent is active—a stream of profits for innovators, and to benefit from higher investment rates towards the end of the patent's life and after

[22]The term Result is used to highlight that the proof of the statement is numerical.

[23]The term binding is used in the sense that there is a high probability that no innovation would occur before the patent expires, making a change in T relevant to incentivise R&D.

Table 1: Optimal patent under different λ and α, and a quantification of the delay in innovation pace $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ induced by implementing an inefficient policy.

					$T = 10$		$T = 20$	
	λ	T^*	b^*	$\mathbb{E}[t]^*$	$b = 1/3$	$b = 2/3$	$b = 1/3$	$b = 2/3$
	1/3	28.3	.42	24.5	70.1%	69.4%	7.1%	6.5%
$\alpha = .7$	1/2	18.5	.55	13.5	30.3%	27.5%	1.3%	1.1%
	2/3	13.7	.62	9.2	11.5%	7.2%	6.8%	8.3%
	1/3	23.7	.35	16.8	35.3%	34.9%	1.1%	2.6%
$\alpha = 1$	1/2	15.7	.49	9.9	11.1%	9.6%	1.7%	4.7%
	2/3	11.8	.56	6.9	3.2%	1.3%	6.8%	10.7%
	1/3	21.2	.27	12.7	19.5%	19.7%	0.1%	3.8%
$\alpha = 1.3$	1/2	14.0	.42	7.7	4.3%	4.2%	2.2%	7.2%
	2/3	10.6	.50	5.5	0.8%	1.0%	5.6%	11.1%

Note: Parameters used: $r = 5\%$ and $\pi = 1/20$. (T^*, b^*) represent the optimal length and strength. $\mathbb{E}[t]^*$ is the minimal waiting time between innovations, and $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ quantifies (in percentage points) the delay of implementing an inefficient policy.

the patent expires. Hence, short but strong patents are optimal.

In contrast, in markets where innovations occur less frequently or are costly to generate (low λ), the statutory length of patents T is binding for a larger range of values, making T a useful tool for increasing the value of an innovation and promoting R&D. However, since longer patents induce leaders to delay their investments, the followers' innovation is crucial to speed up innovative activity. Thus, weak patents have to be offered in order to induce followers to perform R&D in the early stages of the patent life and to increase the market's pace of innovation.

Result 11 ((T^*, b^*) and followers' productivity) *Markets where followers are more productive, the optimal patent is both shorter and weaker.*

Table 1 shows the optimal patent policy under different levels of market's R&D productivity λ and different relative productivity of the followers α (also see Figure 4(b)). It shows that when the relative productivity of the followers increases, followers' R&D efforts become more predominant, and the optimal patent is characterised by lower levels of strength. In addition, more productive followers decrease the expected waiting time between innovations $\mathbb{E}[t]^*$, making the length of patent less binding and, as a consequence, the optimal patent length is also shorter. Table 1 also quantifies, in percentage points, the delay in innovation pace induced by implementing an inefficient policy ($\mathbb{E}[t]/\mathbb{E}[t]^* - 1$). The cost of an inefficient policy can be substantial. It can easily decrease the market's innovation pace by 10%. As can be observed, the cost of implementing an inefficient length tends to be one order of magnitude larger than the cost of implementing an inefficient strength. Also, patents that are shorter than the optimal length seem to harm the innovation pace of the economy more than patents that are too long.

These results directly contribute to the literature on the optimal patent design, which has argued that a policy contingent on market characteristics is more appropriate to incentivise innovators at a lower social cost. Previous works, however, has focused on environments in which more protective policies lead to higher innovation pace, aiming to find the policy that balances R&D incentives with the cost (deadweight loss) associated with patent protection. This work adds a new layer to the discussion by showing that protective policies do not necessarily lead to higher R&D incentives and that this non-monotonicity varies across industries. The results presented here naturally opens the question on whether, given the dynamic incentives induced by patent policy, there is a self-enforced mechanism under which firms self-select into the right policy inducing faster technological progress at a lower social cost. This question is regarded as future research.[24]

6 Endogenous Market Structure

This section extends the previous analysis by allowing an endogenous number of followers to compete throughout the race. The main objective is to show that previous results are robust to, the previously unaccounted, long-term strategic interaction among firms and to explore the role that patent policy plays in determining market structure.

Formally, I extend the previous model to allow for a market leader and n endogenously determined symmetric followers. At the beginning of each race (at $t = 0$), followers decide whether to enter the R&D race by paying an entry cost K. Let w_t denote the value of being a follower that is competing against a leader at instant t. Followers will enter the race as long as $w_0 > K$. Since the followers' value of participating in this market will be decreasing in the number of competitors, in

[24] Hopenhayn and Mitchell (2001) identify a sufficient single-crossing condition on (what in the context of this model would correspond to) λ, under which patent length and breadth substitute for one another in the optimal mechanism. This condition does not hold here precisely due to the non-monotonicity in incentives induced by finite patents.

equilibrium we will have $w_0 = K$. When an innovation occurs, the non-successful firms have to repay the entry cost K in order to participate in the next race, representing the cost of adjusting their labs to be able to develop the next technology. To keep notation simple, and because changes in the relative productivity of followers α will be internalised by the number of followers in the market, throughout this section I assume that leaders and followers are equally productive; i.e., $\mu = \lambda$.

Competition after patent protection expires When patent protection expires ($t > T$), the (no-longer) patented technology is imitated and the leader's profits are cannibalised to zero. The market, thus, becomes a stationary race with $n + 1$ symmetric firms competing to achieve the next innovation. The value for firm i to be competing in this scenario is:

$$\begin{aligned} q &= \max_{x_i} \int_0^\infty \left(\lambda x_i v_0 + \lambda x_{-i}(w_0 - K) - (x_i)^2/2\right) e^{-(r+\lambda(x_i+x_{-i}))t} ds \\ &= \max_{x_i} \frac{2\lambda x_i v_0 - (x_i)^2}{2(r + \lambda(x_i + x_{-i}))} \end{aligned} \tag{9}$$

where $x_{-i} = \sum_{j \neq i} x_j$ is the sum of the innovation rates of all other firms in the market, and $w_0 = K$ was used to derive the second equality. Maximizing equation (9), I find the value of competing in this race q and the optimal strategy x_i^*, which are given by:

$$q = \frac{r + \lambda^2(n+1)v_0 - \rho}{\lambda^2(2n+1)}; \quad x_i^* = \frac{\lambda^2 n v_0 + \rho - r}{\lambda^2(2n+1)}$$

where $\rho = ((r + \lambda^2 n v_0)^2 + 2\lambda^2 r v_0)^{1/2}$.

Competition under patent protection Let $q_t = q \cdot \exp(-z_{t,T} - r(T-t))$ represent the expected-discounted continuation value q at time t. Redefine the license fees paid for an infringement at instant t by $\ell_t = v_t - q_t$. The leader's valuation for its active patent at instant t, v_t, is given by:

$$\max_{\{x_{l,s}\}_{s=t}^T} \int_t^T \left(\pi + \lambda x_{l,s} v_0 + \lambda n x_{f,s}\left(b\ell_s + w_0 - K\right) - (x_{l,s})^2/2\right) e^{-r(s-t)} e^{-z_{t,s}} ds + q_t.$$

There are four key differences with respect to the payoff described in equation (1): (i) The leader now faces n followers. (ii) Since license fees correspond to the damages caused by the commercialization of a new innovation, the payment ℓ_t discounts from v_t the continuation value q_t as the loss of q occurs regardless of whether or not a patent is in place. (iii) When replaced by an innovating follower, the leader receives the value of becoming a follower $w_0 - K$ plus the expected license fees $b\ell_t$. (iv) The value of an active patent takes into account that, when patent protection expires, the leader obtains the continuation payoff q.

Similarly, the value that a follower derives from competing in this market at instant t, w_t, is:

$$\max_{\{x_{f,s}\}_t^T} \int_t^T \left(\lambda x_{f,s} \left(v_0 - b\ell_s\right) + \lambda x_{-f,s}(w_0 - K) - (x_{f,s})^2/2\right) e^{-r(s-t)} e^{-z_{t,s}} ds + q_t \tag{10}$$

where $x_{-f,t} = x_{l,t} + (n-1)x_{f,t}$ is the R&D of all other firms in the market. At every instant t, a follower pays the costs of its R&D, receives the expected revenues of an innovation $v_0 - b\ell_t$ plus the expected revenue $w_0 - K$ if other firms innovate, and the continuation value after the patent expires q_t.

Following the optimal control techniques from Section 3 and using that $w_0 = K$ in equilibrium, I obtain the following necessary and sufficient conditions for a maximum:

$$\begin{aligned} rv_t &= \max_{x_{l,t}} \left\{ v_t' + \pi + \lambda x_{l,t}(v_0 - v_t) - \lambda n x_{f,t}(bq_t + (1-b)v_t) - (x_{l,t}^2)/2 \right\} \\ rw_t &= \max_{x_{f,t}} \left\{ w_t' + \lambda x_{f,t} \left(v_0 - w_t - b\ell_t\right) - \lambda x_{-f,t} w_t - (x_{f,t}^2)/2 \right\}. \end{aligned} \tag{11}$$

Taking first order conditions, the optimal investment rates for the leader and followers are $x_{l,t}^* = \lambda(v_0 - v_t)$ and $x_{f,t}^* = \max\{0, \lambda(v_0 - w_t - b\ell_t)\}$.

Proposition 12 (R&D dynamics) *At the beginning of a patent race ($t = 0$), leaders do not invest in R&D. For the followers, there exists $\hat{b} \in (0,1)$ such that: They invest at a positive rate whenever $b < \hat{b}$ and do not invest otherwise. For every $t \in (0,T)$ and when patents are sufficiently strong, followers' investments are lower than those of the leader. As an active patent approaches its expiration date, both types of firms perform increasing investments over time. When patent protection expires, the leader's and followers' investments converge.*

By looking at the leader's first order condition we can see that the dynamics described in Section 4 are replicated by this model; the leaders dynamic R&D incentives are driven by the value of a new innovation minus its replacement cost. It is interesting to observe that Arrow's result—that leaders have more incentives to innovate than followers—may reverse in this setting. Consider, for illustration

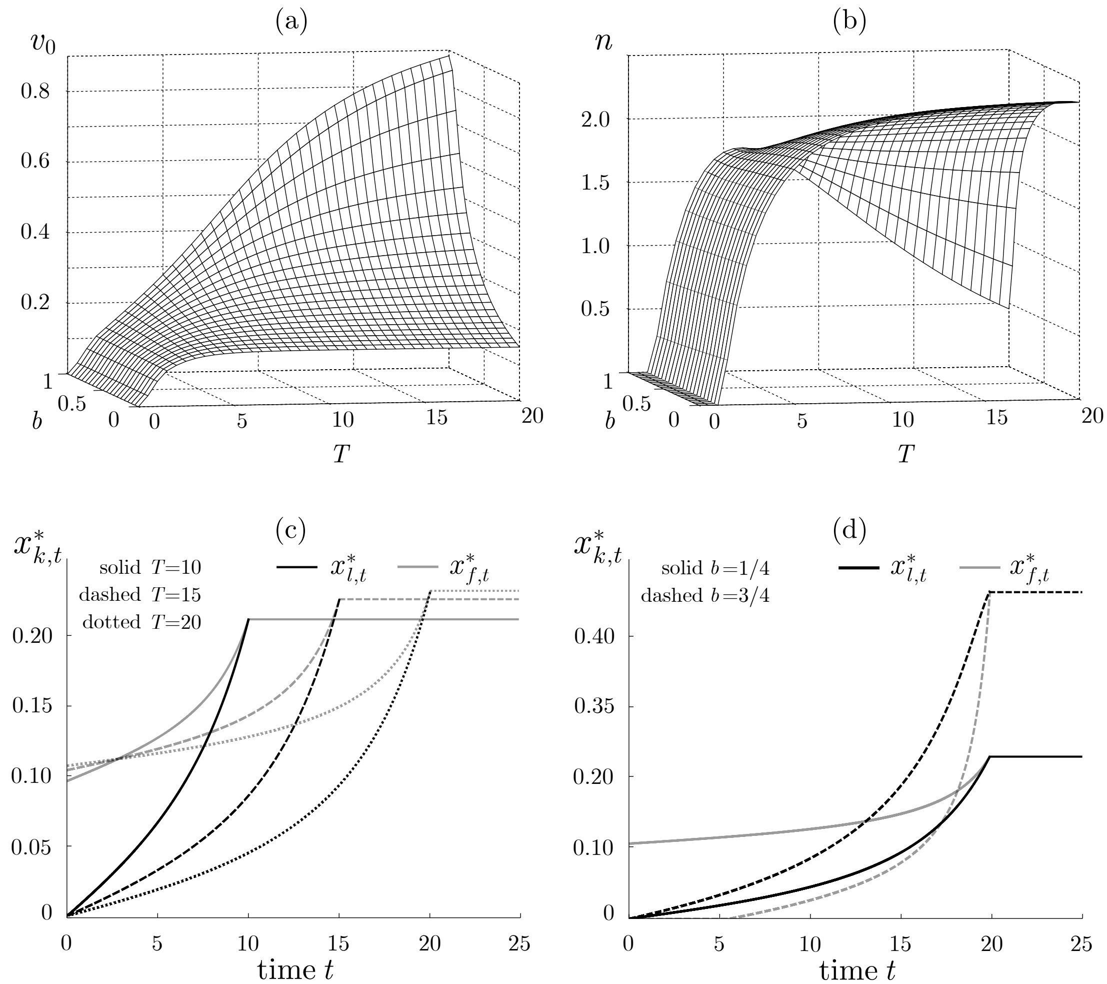

Figure 5: Endogenous market structure

Note: Parameters' values are $r = 5\%$, $\pi = 1/20$, $K = 1/30$, $\lambda = 1$, and, when fixed, $T = 20$ and $b = 1/3$. Value functions were approximated to the 4th decimal point.

purposes, the strongest patent ($b = 1$). At every t, a follower's investment is equal to $x^*_{f,t} = \max\{0, \lambda(v_0 - v_t + q_t - w_t)\}$. Equation (10) implies $w_t \geq q_t$; thus, followers invest at a lower rate than the leader and investments must be zero at the beginning of the patent's life (see Figure 5(d)). As before, strong patent protection makes followers internalise the cost of replacing a leader. In addition, when followers are long-lived, they also internalise the cost of replacing themselves in the race. Under sufficiently strong patents, this double replacement effect may induce followers to perform less R&D than the leader, reversing Arrow's prediction.

Replacing the optimal investments into equation (11), we derive the system of differential equations (17) described in Appendix B. Unfortunately, this system has no analytic solution, and the numeric method described in Appendix B is used to compute the equilibrium and perform comparative statics. Consistent with previous findings, a unique equilibrium was found for each set of parameters. Figure 5(a) shows that the main comparative statics for the value of a new patent remain unaltered: more protective policies lead to an increase in the value of a new patent. Figure 5(b) shows how the number of followers changes with patent policy. As expected, stronger patents decrease the number of followers competing in the market. Interestingly, the effect of patent length on the number of followers depends on the strength of the patent.

Result 13 (Patent policy and entry) *Patents that are too short induce no entry. An increase in patent length* i) *increases the number of competitors under weak patent protection, and* ii) *under strong protection, increases the number of competitors up to a point and then reduces the number of competitors.*

When patent protection is too short, no followers enter the market, as the value of participating is not high enough to compensate for the entry cost K. Under weak patent protection, longer patents induce more firms to enter the market. This also causes the value of a new patent, v_0, to not be very responsive to changes in patent length (see Figure 5(a)). In particular, when no forward protection is offered, we can see that most of the effect of increasing patent length is absorbed by the increase in the number of followers in the market, and the value of a patent increases only a small amount.

As patent protection becomes stronger, we find an additional countervailing effect of offering long patent protection: it not only delays the firms' investments, but also induces followers to exit the market (see Figure 5(b)). The exit of followers is induced by three effects that patent length has on the followers' value: (i) the followers' incremental rent from an innovation, $v_0 - b\ell_t$, start suffering the delay effect discussed in Proposition 5. This effect delays the expected arrival time of a breakthrough, decreasing the followers' value. (ii) The leader is able to charge license fees for a follower's innovation for a longer period of time. (iii) Longer patent protection delays the arrival of the continuation value of competing in a race with no patent protection q. The conjunction of these three effects makes the market less attractive to followers, decreasing the number of competitors. Notice that when patent protection is strong, the value of a new innovation is very responsive to an increase in patent length, which is consistent with the leader simultaneously benefiting from stronger protection and less competition.

Table 2: Optimal patent under different λ and a quantification of the delay in innovation pace $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ induced by implementing an inefficient policy.

					$T = 10$		$T = 20$	
λ	T^*	b^*	$\mathbb{E}[t]^*$	n^*	$b = 1/3$	$b = 2/3$	$b = 1/3$	$b = 2/3$
0.5	33.6	0	6.26	3.10	19.2%	23.8%	6.6%	21.4%
0.75	14.4	0	4.42	2.57	2.4%	8.6%	5.8%	18.5%
1.0	9	0.02	3.48	2.18	1.7%	8%	7%	17.8%
1.25	5.7	0.22	2.87	1.81	3.8%	9.7%	9.36%	19.2%
1.50	4.1	0.24	2.45	1.55	6.6%	11.8%	12.2%	19.6%
1.75	3.2	0.25	2.14	1.35	9.6%	14.1%	15.1%	23.6%

Note: Parameters used: $r = 5\%$ and $\pi = 1/20$. (T^*, b^*) represents the optimal length and strength. $\mathbb{E}[t]^*$ is the minimal waiting time between innovations, and $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ quantifies (in percentage points) the delay of implementing an inefficient policy.

Result 14 (Patent policy and R&D) *Longer patent protection delays the leader's investments and, when patents are also strong, they delay the followers' investments.*

Figures 5(c) and 5(d) show that the key comparative statics on the effect of patent policy in R&D investments remain unaltered. Longer patents delay the leader's investments and, depending on patent strength, may delay the followers' investments as well. Similarly, stronger patent protection discourages followers from performing R&D, but promotes that of the leader. As noted above, however, Arrow's conclusion that followers always perform more R&D than leaders does not always hold. Depending on the strength of the patent system, a single follower may perform more or less R&D than the leader throughout the patent's life. As before, when patents approach their expiration date, investments tend to converge.

Table 2 shows the optimal patent under different market's R&D productivity and quantifies the cost of implementing the incorrect policy ($\mathbb{E}[t]/\mathbb{E}[t]^* - 1$). Consistent with previous results, patent length and strength complement each other, with one tool being more effective in markets in which the other tool is not. It is interesting to contrast these results with those in Table 1. The cost of implementing the incorrect policy is still quite substantial, and the cost of implementing the incorrect strength is larger than the previous scenario. In addition, for markets with similar R&D productivity λ, the optimal patent prescribed in this scenario is weaker but longer. These results are consistent with the additional misincentive that patent strength has on the number of competitors in the market. The ability of patent strength to promote R&D depends heavily on how responsive the followers are to entry incentives. In industries in which the number of competitors is fixed, as in the baseline model, we can rely on stronger patent systems to promote R&D. In industries where followers respond to entry incentives, weaker policies are preferred.

7 Extensions

In this section I briefly discuss the robustness of previous results to extensions of the baseline model that, due to space limitations, are not fully developed here.

License fees: Bargaining The proposed framework can accommodate the study of incentives provided by different forms of license fees. In particular, we can explore the effects of allowing a bargaining process between the leader and an infringing follower to determine license fees *beyond* the profit loss v_t; i.e., $\ell_t = v_t + \beta(v_0 - v_t)$, where β can be interpreted as the Nash bargaining power of the leader or as the breadth of the patent. In this context, investments are given by $x_{l,t} = \lambda(v_0 - v_t)$ and $x_{l,t} = \mu(v_0 - b\ell_t)$. The delay effect that longer patents have on the firms' investments is still present. In addition, greater bargaining power of the leader increases the discouragement effect that patent strength has over the followers' investments. Interestingly, since expected license fees may be actually higher than the residual value of a patent at t—for instance, $b\ell_T = b\beta v_0 > 0 = v_T$—patents that are too strong may harm the leader. Stronger patents discourage followers' R&D, causing the leader's valuation for a patent to decrease as the leader prefers to be replaced by a follower and extract higher rents through license fees.

License fees: Undiscounted damages I have also examined the effects of computing the damages as the undiscounted sum of the stream of profit loss; i.e., $\ell_t = (T - t)\pi$ instead of v_t. Once again, this specification does not alter the incentives to delay induced by longer patent protection, nor the effect that stronger patent protection discourages followers' investments. It is interesting to observe that, depending on the R&D productivity in the market and on the strength and length of the patent, the expected license fee $b(T - t)\pi$ may be higher than v_0, inducing followers not to invest during the first years of the patent. Once again, this effect always fades away as the patent expiration date approaches and license fees decrease to zero.

Extending the leader's techonogical lead The delay effect on the leader's investment does not fade away once we allow the leader to not only extend its patent clock but also to extend its technological lead in the market. In particular, let's modify the baseline model by assuming that profits π_m are increasing in the number of consecutive innovations that a leader has achieved, m, and that the leader can extend the protection of its previous innovations with the arrival of a new innovation. Let $v_{m,t}$ be the value of possessing m consecutive patents with the latest innovation occurring t years ago. It can be shown that the functions $v_{m,t}$ are increasing in the number of consecutive innovations m. Then, equilibrium investments are given by:

$$x_{l,m,t} = \lambda(v_{m+1,0} - v_{m,t}), \quad x_{f,m,t} = \mu(v_{1,0} - bv_{m,t}).$$

Since $v_{m,t}$ is increasing in m, the followers get discouraged when facing a leader with a greater lead, as they have to pay higher expected license fees in the case of an innovation. In contrast, the leader experiences increased incentives to invest. For instance, at $t = 0$, we have $x_{l,m,0} = \lambda(v_{m+1,0} - v_{m,0})$, and investments are positive because the replacement effect of a new innovation does not completely cannibalise the value of the previous innovation. Similar to the effect of patent length on the investments of followers in the baseline model, an increase in T initially increases the leader's R&D investments, then decreases the leader's investments towards the middle of the patent's life, and then increases investments when the patent is about to expire. In other words, the delay incentive exists but becomes weaker for the leader. For the followers, in contrast, the incentive to delay under strong patent protection is intensified by a larger quality gap m. The sensitiveness of $v_{m,t}$ with respect to T increases with m, making followers delay their investments even more. It is not hard to see that Arrow's prediction that leaders invest less than followers may also reverse once we allow leaders to increase their lead.

8 Concluding Remarks

This paper studied how patent length and strength affect the innovation incentives of market leaders and followers. Longer patent protection delays the leader's investments and, depending on patent strength, may encourage or delay the followers' investments. In contrast, stronger patent protection delays the follower's investments, but encourages those of the leader towards the end of the patent's life. Policies that maximise the innovative activity in an industry must balance the effects that patent length and strength have on the leader's and followers' investments. It was shown that short and strong patents are preferable in markets where innovations occur often or are not too costly to produce. Long and weak patents are preferable in markets where innovations take longer or are costly to produce. The cost of implementing an incorrect policy can be substantial. Furthermore, this is greatest in a scenario in which patent protection is both too long and too strong.

Patent policy also affects the number of firms competing in the market. While stronger patents always discourage the entry of new firms, longer patent protection may encourage or discourage entry depending on level of patent strength. In this context, a protective policy not only delays the firms' investments, but also decreases the number of competitors in the market. As a consequence, the ability to use patent strength to encourage innovation heavily depends on the elasticity of firm-entry to market incentives. In markets where the number of firms is very elastic, it is preferable to have weaker patents, as policies that are too protective drive firms out of the market.

Important questions about how patent policy affects innovation in a sequential context remain. I believe that the model presented in this paper can serve as a building block to study many open questions about how patent policy can affect firms' decisions regarding adoption of new technologies, quality choice, and disclosure of new innovations. In addition, this framework can be used to study the relation that exists between patent policy and (endogenous) growth in the economy.

Appendix

A Omitted Proofs

A.1 Proof of Proposition 1

The only statement not proven in the text is that investments increase towards the patent expiration date. To see this observe that $v_T = 0$, so that equation (3) evaluated at T and using the first order condition (4) becomes

$$-v'_T = \pi + x^*_{l,T} c'(x^*_{l,T}) - c(x^*_{l,T}).$$

Convexity of $c(x)$ plus the assumption that $c'(0) = 0$ implies that $xc'(x) > c(x)$. Then, the right hand side of the expression above must be positive; i.e., $v'_T < 0$. By continuity, there exist $\hat{t} < T$ such that $v'_t < 0$ for $t \in [\hat{t}, T)$ and v_t converges to zero from above, implying the result.

A.2 Proof of Proposition 2

I start by proving the existence of a fixed-point. From Online Appendix C, we know that there is a unique solution to (5), so I can restrict attention to show that there is a fixed-point $v_0 = \hat{v}$ for a positive value of $\hat{v}$.[25] To do so, I start by reformulating the problem, defining a function $f(z) = v_0(z) - z$ where $v_0(z)$ denotes the dependence of the solution (6) on the conjectured value z. Then, showing the existence of the fixed-point is equivalent to show that exists $\hat{v} > 0$ such that $f(\hat{v}) = 0$.

I show existence by means of the intermediate value theorem. Observe that ϕ and θ go to ∞ at a rate of z, when z goes to infinity. Then, it is easy to check that

$$\begin{aligned} \lim_{z\to\infty} f(z) &= \lim_{z\to\infty} \frac{\left(\frac{2\pi}{z} - z\mu^2(1-b) - r\right)\left(1 - \frac{1}{e^{\phi T}}\right) - \phi\left(1 + \frac{1}{e^{\phi T}}\right)}{\frac{\theta}{z}\left(1 - \frac{1}{e^{\phi T}}\right) + \frac{\phi}{z}\left(1 + \frac{1}{e^{\phi T}}\right)} \\ &= -\infty. \end{aligned}$$

It remains to show that there is z such that $f(z) > 0$. The result follows from choosing $z = 0$. There, $f(0) = v_0(0) - 0$. Given the behavior of firms in an equilibrium, and since there is no benefit from developing a new innovation, we are in phase 0 (see Online Appendix C) throughout the patent's life, so $v_0(0) = (\pi/r)(1 - exp(-rT)) > 0$.

To prove uniqueness, I make use of the fact that $f(z)$ is continuous and show that at any fixed point $f'(\hat{v}) < 0$ so $f(z)$ can single-cross zero from above just

[25] There may be other fixed points such that $\hat{v} \leq 0$; however, those do not have an economic meaning and, consequently, are ignored.

once. Define the function

$$\psi_t = \frac{e^{2\phi(T-t)} - 2\phi\,(T-t)\,e^{\phi(T-t)} - 1}{\phi\left(e^{\phi(T-t)} - 1\right)^2}. \tag{12}$$

Section D of the Online Appendix shows that, for all $t < T$, the function ψ satisfies $\psi_t > 0$, $\psi_t' < 0$, and $\psi_T = 0$. This function will be used in several proofs.

Because it will be useful later on, I compute the derivative of $v_t(z) - z$ with respect to z, evaluated at $\hat{v}$

$$\frac{dv_t(\hat{v})}{dz} - 1 = -\frac{\lambda^2(\hat{v} - v_t)^2 + \mu^2(1-b)v_t^2 + 2\pi + \psi_t k v_t^2}{2\pi + (\lambda\hat{v})^2},$$

where $k = \mu^2(2\lambda^2+\mu^2)(1-b)^2\hat{v} + r(\lambda^2+\mu^2(1-b))$ is a positive constant. Therefore, the previous derivative is negative for all t. In particular, the derivative is negative at $t = 0$ which corresponds to $f'(\hat{v})$ and the result follows.

Finally, investments are increasing through time because the value of a patent decreases with t

$$\frac{dv_t}{dt} = -\frac{2\phi^2\left(\theta^2 - \phi^2\right)e^{\phi(T-t)}}{\lambda^2\left(\left(\theta+\phi\right)e^{\phi(T-t)} - \left(\theta-\phi\right)\right)^2} < 0$$

where $\theta^2 - \phi^2 = (\lambda^2 + 2\mu^2 b(1-b))(2\pi + (\lambda\hat{v})^2)$.

A.3 Proof of Proposition 3

Let $f(z,\alpha) = v_0(z,\alpha) - z$ be the construction presented in the proof of Proposition 2, where its dependence on a parameter $\alpha \in \{\pi, r, T, b, \lambda, \mu\}$ has been made explicit. By the implicit function theorem, there is a function $V(\alpha)$ implicitly defined by $f\left(V(\alpha),\alpha\right) = 0$ that describes the equilibrium value of having a new patent. Then, the comparative statics for how the value of a new patent, v_0, changes due to a change in parameter is given by

$$\begin{aligned} \frac{dV(\alpha)}{d\alpha} &= -\frac{\partial f\left(V(\alpha),\alpha\right)}{\partial\alpha} \Bigg/ \frac{\partial f\left(V(\alpha),\alpha\right)}{\partial z}. \\ &= \frac{\partial v_0(\hat{v},\alpha)}{\partial\alpha} \Bigg/ \left(1 - \frac{dv_0(\hat{v},\alpha)}{dz}\right) \end{aligned} \tag{13}$$

From the proof of Proposition 2, we know that the denominator of (13) is positive. Thus, it is sufficient to look at the sign of the partial derivative $\partial v_0(\hat{v},\alpha)/\partial\alpha$.
Comparative static with respect to π: v_0 increases with an increase in π as

$$\frac{\partial v_t}{\partial\pi} = \frac{v_t}{2\pi + (\lambda\hat{v})^2}\left(2 + \psi_t v_t(\lambda^2 + 2b(1-b)\mu^2)\right) > 0$$

where ψ_t is the function defined in equation (12).

Comparative static with respect to r: v_0 decreases with an increase in r as

$$\frac{\partial v_t}{\partial r} = -\frac{v_t^2(1+\psi_t\theta)}{2\pi + (\lambda\hat{v})^2} < 0.$$

Comparative static with respect to T: v_0 increases with an increase in T as

$$\frac{\partial v_t}{\partial T} = \frac{2\phi^2(2\pi + (\lambda\hat{v})^2)e^{\phi(T-t)}}{\left(\theta\left(e^{\phi(T-t)}-1\right)+\phi\left(e^{\phi(T-t)}+1\right)\right)^2} \tag{14}$$

is positive for all $t \leq T$. Moreover, it can be easily checked that this derivative increases with t.

Comparative static with respect to b: The derivative of v_t with respect b is

$$\frac{\partial v_t}{\partial b} = \frac{\mu^2 v_t^2}{2\pi + (\lambda\hat{v})^2}\left(\hat{v} + \psi_t\left[(2\lambda^2+\mu^2)(1-b)\hat{v}^2 + r\hat{v} + 2\pi(1-2b)\right]\right).$$

This derivative is zero at $t = T$. The condition $b \leq 1/2$ is sufficient for the term in square breakers to be positive and the derivative to be positive for all $t < T$. When $b > 1/2$ the term in square brackets may be negative (in fact is negative when $b = 1$). Observe however that ψ_t continuously goes to zero as t approaches T. Then, by intermediate value theorem and as $\hat{v} > 0$, there exists $\hat{t} < T$ such that the expression in round parenthesis is positive for all $t \geq t$.

Comparative static with respect to λ: The derivative of v_0 with respect λ is

$$\frac{\partial v_t}{\partial \lambda} = \frac{2\lambda v_t}{2\pi + (\lambda\hat{v})^2}\left(\hat{v}(\hat{v}-v_t) + \psi_t v_t\left[\pi - r\hat{v} - (\mu(1-b)\hat{v})^2\right]\right)$$

A sufficient condition for this derivative to be positive it that the term in square brackets to be non-negative. Solving the quadratic equation derived from setting the square bracket to zero we find that the condition hold whenever $\hat{v} \leq v_\infty$, where v_∞ is the value of a patent when $T = \infty$ defined in Proposition 4. Since $\hat{v}$ is increasing in T, the result follows.

Comparative static with respect to μ: The derivative of v_t with respect μ is

$$\frac{\partial v_t}{\partial \mu} = -\frac{2\mu(1-b)v_t^2}{2\pi + (\lambda\hat{v})^2}\left(\hat{v} + \psi_t\left[(\lambda^2+\mu^2)(1-b)\hat{v}^2 + r\hat{v} - 2b\pi\right]\right)$$

A sufficient condition for this derivative to be negative it that the term in square brackets to be non-negative. Observe that the square brackets is strictly positive when $b = 0$ and, by continuity, it is positive for low values of b. When $b = 1$, the derivative is zero.

A.4 Proof of Proposition 4

I start by showing that the limiting value of a patent is given by (7). Taking the limit of (6) when T goes to infinity, for every t, delivers

$$v_{\infty} = \lim_{T \to \infty} v_t = \frac{2\pi + (\lambda v_{\infty})^2}{\theta + \phi}.$$

Solving this expression for v_{∞} delivers a unique positive solution. Since v_{∞} does not depends on t, investments are constant through time. When $b < 1$ the solution is

$$v_{\infty} = \frac{1}{2\mu^2 (1-b)^2} \left(-r + \sqrt{r^2 + 4\pi\mu^2 (1-b)^2} \right)$$

and when $b = 1$ the solution is $v_{\infty} = \pi/r$.

A.5 Proof of Proposition 5 and 6

Formally, we want to show that there exists $\hat{t} > 0$ such that for all $t < \hat{t}$ the derivative

$$\frac{dx_{l,t}}{dT} = \lambda \left(\frac{d\hat{v}}{dT} \left(1 - \frac{dv_t}{d\hat{v}} \right) - \frac{\partial v_t}{\partial T} \right). \tag{15}$$

is negative. Making use of equation (13), we can readily check that $dx_{l,0}/dT = 0$. From the proof of Proposition 3, we know that $d\hat{v}/dT > 0$ and that $\partial v_t/\partial T > 0$ and increasing in t. Hence, a sufficient condition for the result to hold is to show that $dv_t/d\hat{v}$ increases with t around $t = 0$. The derivative of previous expression with respect to t at $t = 0$ is

$$\frac{d^2 v_0}{dv dt} = -\hat{v} \frac{2\mu(1-b)v_0' + k\left(2v_0'\psi_0 + v_0\psi_0'\right)}{\hat{v}^2\lambda^2 + 2\pi},$$

where k is the positive constant defined in the proof of Proposition 2. This derivative is positive as v_0' and ψ_0' are both negative, and the result follows. Finally, to show that the terminal investment increases, simply observe that $x_{l,T+dT} = \lambda\hat{v}$ which increases with T as proven by Proposition 3. Proposition 6 follows from the discussion in the text and previous results.

A.6 Proof of Proposition 7

The total derivative with respect to the patent length is given by equation (15). When the change in policy is grandfathered to the next innovation, there is no direct effect in the current race, i.e., $\partial v_t/\partial T = 0$, and the derivative becomes

$$\frac{dx_{l,t}}{dT} = \lambda \frac{d\hat{v}}{dT} \left(1 - \frac{dv_t}{d\hat{v}} \right).$$

From Proposition 3, we know that $d\hat{v}/dT > 0$. From the proof of Proposition 2 we know $1 - dv_t/d\hat{v} > 0$ and the results follows. Similar proof holds for the follower.

A.7 Proof of Proposition 8

Followers decrease R&D at the beginning of the patent's life as:

$$\frac{dx_{f,0}}{db} = \mu\left(-\hat{v} + (1-b)\frac{d\hat{v}}{db}\right) = -\frac{\mu\hat{v}\left(\psi_0\left(\mu^2\left(b\left(3-2b\right)-1\right)2\pi + \lambda^2 r\hat{v}\right)\hat{v} + 2\pi\right)}{\mu^2\left(1-b\right)\hat{v}^2 + k\psi_0\hat{v}^2 + 2\pi},$$

where k is the positive constant defined in the proof of Proposition 2. This derivative is positive whenever $b \geq 1/2$. Followers increase R&D at towards the end of the patent's life as $x_{f,T} = \mu\hat{v}$, which increases by assumption. Similar argument can be applied for the second claim.

A.8 Proof of Proposition 9

Observe that (8) can be written as

$$\mathbb{E}\left[t\right] = \int_0^T x_t t e^{-z_{0,t}} dt + e^{-z_{0,T}}\left(T + \frac{1}{(\lambda\alpha)^2\hat{v}}\right).$$

Taking the limit when $T \to 0$ shows that $\mathbb{E}\left[t\right] \to \infty$ as the value of a new innovation $\hat{v}$ converges to zero, precluding $T = 0$ to be optimal. I show $T^* < \infty$ by contradiction. Start by assuming that $T^* = \infty$. Using Proposition 4 we can compute $\mathbb{E}[t]$ which is equal to

$$\mathbb{E}\left[t\right] = \frac{2(1-b)}{\sqrt{r^2 + 4\mu^2\pi(1-b)^2} - r}. \tag{16}$$

Minimizing $\mathbb{E}\left[t\right]$ with respect to b we find that $b^* = 0$. Thus, the policy $(T^*, b^*) = (\infty, 0)$ is the only candidate for optimality if $T^* = \infty$ were to be optimal. When $b = 0$, the followers' investments are constant and equal to $x_{f,t} = \mu\hat{v}$ for all t. This scenario is the only case where $\mathbb{E}\left[t\right]$ can be solved analytically. For $(T^*, b^*) = (\infty, 0)$ to be a minimum, we need $\mathbb{E}\left[t\right]$ to converge to equation (16) from above, as T approaches infinity. Section E in the Online Appendix shows that $\mathbb{E}[t]$ converges from below, contradicting $T^* = \infty$ and proving the result.

A.9 Proof of Proposition 12

The proof that at $t = 0$ the leader does not invest, that followers may not invest depending on b, and that that followers may invest less than the leader for all t follows from the discussion in the text and the fact that functions are continuous in b. To show that investment are increasing towards the end of the patent life observe that equation (11) at $t = T$ becomes

$$-v'_T = \pi + \frac{\lambda^2}{2}(v_0 - q)^2 - (r + n\lambda^2(v_0 - q))q.$$
$$-w'_T = \frac{\lambda^2}{2}(v_0 - q)^2 - (r + n\lambda^2(v_0 - q))q.$$

Using the solution for q the previous expressions reduce to $v'_T = -\pi$ and $w'_T = 0$. Differentiating the firms investment rates with respect t

$$\frac{dx_{l,t}}{dt} = -v'_t \qquad \text{and} \qquad \frac{dx_{f,t}}{dt} = -(w'_t + b(v'_t - q'_t)).$$

Evaluating the derivatives at $t = T$ and using $q'_t = (r + n\lambda^2(v_0 - q))q > 0$ we obtain $x'_{l,T} = \pi > 0$ and $x'_{f,T} = \pi + q'_t$ implying, by continuity, that both investments increase towards the end of the patent life.

B Endogenous Market Structure

This section derives the system of ODEs describing how v_t and w_t evolve throughout t, and explains the numeric method used to compute the market equilibrium.

The Principle of Optimality Using the first order conditions $x^*_{l,t} = \lambda(v_0 - v_t)$ and $x^*_{f,t} = \max\{0, \lambda(v_0 - w_t - b\ell_t)\}$, I obtain the following system of differential equations when $x_{f,t} > 0$

$$\begin{aligned} -v'_t &= \alpha_1 v_t^2 + \alpha_2 v_t w_t - \alpha_{3,t} v_t + \alpha_{4,t} w_t + \alpha_{5,t} \\ -w'_t &= \alpha_6 w_t^2 + \frac{(\lambda b v_t)^2}{2} + \lambda^2(1 + bn)v_t w_t - \alpha_{7,t} w_t - \alpha_{8,t} v_t + \alpha_{9,t} \end{aligned} \tag{17}$$

where

$$\begin{array}{lll} \alpha_{0,t} = v_0 + bq_t & \alpha_{3,t} = r + (\lambda^2 + \alpha_2)v_0 + (1 - 2b)\alpha_{4,t} & \alpha_{8,t} = \lambda^2 b\alpha_{0,t} \\ \alpha_1 = \lambda^2/2 + b\alpha_2 & \alpha_{5,t} = \pi + (\lambda v_0)^2/2 - \alpha_{0,t}\alpha_{4,t} & \alpha_9 = (\lambda\alpha_{0,t})^2/2 \\ \alpha_2 = \lambda^2 n(1 - b) & \alpha_6 = \lambda^2(n - 1/2) & \\ \alpha_{4,t} = \lambda^2 nbq_t & \alpha_{7,t} = r + \lambda^2(v_0 + n\alpha_{0,t}) & \end{array}$$

When $x_{f,t} = 0$, the system becomes:

$$\begin{aligned} -v'_t &= \frac{(\lambda v_t)^2}{2} - (r + \lambda^2 v_0)v_t + \pi + \frac{(v_0)^2}{2} \\ -w'_t &= \lambda^2 v_t w_t - (r + \lambda^2 v_0)w_t \end{aligned}$$

Numerical Method The maximum value that a leader can obtain for an innovation is to receive the profit π forever. Thus, the value of being the leader is bounded above by π/r. The numeric method follows these steps:

1. Define V_p to be a partition of $[0, \pi/r]$. Each element of V_p will be tested as a candidate for v_0.
2. Fix $v \in V_p$. Start with $n = 0$ and define dn to be a small increase in n.
   (a) As a function of (v, n) compute the continuation value $q(v, n)$ using equation (9) in equilibrium.
   (b) Starting from $q(v, n)$, use the system of ODEs (17) backwards to compute the initial values of being a leader and a follower; i.e, set $v_T = w_T = q(v, n)$ and using (17) obtain $v_0(v, n)$ and $w_0(v, n)$.
   (c) If $w_0(v, n) > K$, increase n in dn and go back to (a). If $w_0(v, n) < K$ save results as a pair as $(v, n(v))$. Start step 2 with a different $v \in V_p$.[26]
3. Once all the pairs $(v, n(v))$ have been computed, the solution (v_0, n^*) corresponds to the pair $(v, n(v))$ where

$$v \in \underset{v \in V_p}{\operatorname{argmin}} \|v_0(v, n(v)) - v\|$$

[26]This steps assumes that $w_0(v, n)$ is monotonically decreasing in n, which seems to be the case.

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Online Appendix

Sequential Innovation and Patent Policy

by Álvaro Parra

Supplemental Material –Not for Publication

C Solving the Ordinary Differential Equation

This Appendix solves the differential equation that describes how the value of a patent evolves as its expiration date approaches. Depending on the conjectured value $\hat{v}$, competition during the life of the patent may go through one of three phases. Phase 0 occurs when the value $\hat{v}$ is low, i.e., when $\hat{v} < bv_t \leq v_t$. In this phase no firm will invest in R&D, as the cost of replacing the currently active patent is larger than its benefit. Phase 1 occurs when $bv_t \leq \hat{v} < v_t$, i.e., when only followers have incentives to perform R&D. Finally, phase 2 occurs when $\hat{v} \geq v_t$, in which case both firms will invest. In equilibrium, only phase 2 will be observed. However, for the purposes of proving the existence and uniqueness of the fixed-point (Proposition 2), the three phases have to be characterised. Let $v_{j,t}$ be the value of having an active patent in phase $j \in \{0, 1, 2\}$ at time t.[27]

Restate the differential equation (5), corresponding to phase 2, as

$$\frac{dv_{2,t}}{dt} + av_{2,t}^2 - \theta v_{2,t} + \hat{a} = 0$$

where

$$a = \frac{\lambda^2}{2} + \mu^2 b(1-b),\ \theta = r + (\lambda^2 + \mu^2(1-b))\hat{v},\ \text{and}\ \hat{a} = \pi + \frac{(\lambda\hat{v})^2}{2}.$$

This ODE is separable and of the form $dv/h(v) = dt$ where $h(v) = -(av^2 - \theta v + \hat{a})$. Separable ODEs have a unique non-singular solution that goes through its boundary condition, in this case $v_{2,t} = 0$.[28] To find the non-singular solution I integrate both sides to get

$$-\ln\left(\frac{\theta - 2v_{2,t}a + \sqrt{\theta^2 - 4a\hat{a}}}{\theta - 2v_{2,t}a - \sqrt{\theta^2 - 4a\hat{a}}}\right)\sqrt{\frac{1}{\theta^2 - 4a\hat{a}}} = \hat{C} + t$$

where $\hat{C}$ is a constant of integration. Define $\phi = (\theta^2 - 4a\hat{a})^{1/2}$ and solving for $v_{2,t}$,

[27] Phases 0, 1 and 2 correspond to situations in which there are zero, one, or two firms investing at a given instant in time.

[28] Singular solutions to (5) are found by setting $v_t' = 0$ and solving the quadratic equation. These solutions are disregarded, as they do not generically satisfy the boundary condition $v_T = 0$ and have no economic meaning.

we find

$$v_{2,t} = \frac{1}{2a}\left(\theta + \phi\frac{\left(1+e^{-\phi\left(\hat{C}+t\right)}\right)}{\left(1-e^{-\phi\left(\hat{C}+t\right)}\right)}\right), \tag{18}$$

which is the general solution to the ODE. To find the particular solution, we just make use of the boundary condition $v_{2,T} = 0$ to get

$$\hat{C} = -\frac{1}{\phi}\ln\left(\frac{\theta+\phi}{\theta-\phi}\right) - T. \tag{19}$$

Replacing back (19) in to (18) and rearranging terms, we obtain $v_{2,t}$ which corresponds to equation (6). Now, I make sure that $v_{2,t}$ is well defined for all positive conjectures of $\hat{v}$. This clearly is true in cases where $\hat{v}$ is such that $\phi > 0$. I have to check the cases under which ϕ is either imaginary or zero. For the former case, let $\phi = qi$ where i denotes the imaginary number, and q is the positive real coefficient of i. Rewrite $v_{2,t}$ as

$$v_{2,t} = \frac{2\pi + (\lambda\hat{v})^2}{\theta + q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i}.$$

Observe that Euler's identity implies[29]

$$q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i = \frac{q\sin\left(q\left(T-t\right)\right)}{1-\cos\left(q\left(T-t\right)\right)},$$

establishing that the value of a patent $v_{2,t}$ is real when ϕ is imaginary.

Finally, for the case when $\phi = 0$, let v° be the value of v such that $\phi(v^\circ) = 0$. When $\phi = 0$ the value of the patent at every t becomes $v_{2,t} = 0/0$. Then, I define $v_{2,t}$ to be the $\lim_{\hat{v}\to v^\circ} v_{2,t}$ which can be computed by applying L'Hôspital's rule to equation (6) and is equal to[30]

$$v_{2,t} = \frac{\left(2\pi + (\lambda\hat{v})^2\right)(T-t)}{\theta(T-t)+2},$$

showing that $v_{2,t}$ is well defined for any possible value of $\hat{v}$.

Similar steps can be followed to obtain $v_{1,t}$; however, two key differences apply. First, the optimal investment rate of the leader is zero. Second, because $v_{2,t}$ is decreasing in t (see proof in Section A.2), there exists $t_2 \leq T$ that determines the time in which phase 1 finishes and phase 2 starts; at that point the boundary condition $v_{1,t_2} = v_{2,t_2}$ must hold. Under those conditions, I find

$$v_{1,t} = \frac{v_{2,t_2}\left(\theta_1 + \phi_1 + (\phi_1 - \theta_1)\,e^{\phi_1(t_2-t)}\right) + 2\pi\left(e^{\phi_1(t_2-t)} - 1\right)}{\phi_1\left(1 + e^{\phi_1(t_2-t)}\right) + \left(\theta_1 - 2a_1 v_{2,t_2}\right)\left(e^{\phi_1(t_2-t)} - 1\right)}$$

[29] Euler's identity: $e^{i\psi} = \cos(\psi) + i\sin(\psi)$.

[30] In this case, left and right limit converge to the same point, so this is a well defined construction.

where $a_1 = \mu^2 b(1-b)$, $\theta_1 = r + \mu^2(1-b)\hat{v}$ and $\phi_1 = (\theta_1^2 - 4a_1\pi)^{1/2}$. Similar steps as those shown above can be followed to show that $v_{1,t}$ is well defined for any conjecture of $\hat{v}$. Finally, the value of $v_{0,t}$ is

$$v_{0,t} = \frac{\pi}{r}\left(1 - e^{-r(t_1-t)}\right) + v_{1,t_1}e^{-r(t_1-t)}$$

where $t_1 \leq T$ is the instant of time in which phase 1 starts. To conclude, t_1 and t_2 are found by solving $bv_{1,t_1} = \hat{v}$ and $v_{2,t_2} = \hat{v}$.

D Properties of the Function Psi

To study ψ_t for $t \in [0,T]$ it is useful to make a change in variable. Define the new variable $x = \phi(T-t)$ and, since ϕ is just a constant with respect to t, define $\hat{\psi}(x) = \phi\psi_t$ under the respective change in variable. The domain of this new function is $x \in [0, \phi T]$ and is equal to

$$\hat{\psi}(x) = \frac{e^{2x} - 2xe^x - 1}{(e^x-1)^2}.$$

To show $\psi' < 0$ is equivalent to show $\hat{\psi}' > 0$. I start showing this for $x \in (0, \phi T]$:

$$\hat{\psi}'(x) = \frac{2e^x}{(e^x-1)^3}\left(x\left(1+e^x\right) + 2\left(1-e^x\right)\right)$$

the terms outside the parenthesis are positive, I need to determine the sign of $h(x) \equiv x(1+e^x) + 2(1-e^x)$, which takes the value of 0 at $x=0$ and $h'(x) = xe^x - e^x + 1$, an expression that is always positive, thereby proving the result. To show that $\psi_T = 0$ is equivalent to showing $\hat{\psi}(0) = 0$. At that point we have that $\hat{\psi}$ is not well defined. To identify its limit, I apply L'Hôspital's rule (twice) and get:

$$\lim_{x\to 0} = \frac{-(x - e^x + 1)}{(2e^x - 1)} = \frac{0}{1} = 0.$$

The conjunction of these two results proves that $\hat{\psi}(x)$ is positive for all x which implies ψ_t is positive for all $t < T$.

E Omitted Details in Proposition 9

I start by proving a lemma and making computations that will be used in the proof. Recall that $b = 0$ is assumed throughout the proof.

Lemma 1 *For T sufficiently large, $\phi > r$.*

Proof: As ϕ continuously increases with $\hat{v}$, and $\hat{v}$ continuously increases with T, it is sufficient to show the result at $T = \infty$:

$$\begin{aligned}
\phi &= \left((r + \lambda^2(1+\alpha^2)v_\infty)^2 - \lambda^2\left(2\pi + \lambda^2 v_\infty^2\right)\right)^{1/2} \\
&= \left(\frac{r^2}{2}\left(1 + \sqrt{1 + \frac{4\pi(\alpha\lambda)^2}{r^2}}\right) + \frac{\pi(\alpha\lambda)^2}{2r^2}\right)^{1/2} \\
&> \left(r^2\frac{(1+\sqrt{1})}{2}\right)^{1/2} = r
\end{aligned}$$

where (7) was used in the second line. □

In the context of this proof, Lemma 1 implies that when T is large enough, the terms multiplied by $e^{T(r-\phi)}$ converge to zero.

Computations: We need to know $e^{-z_{0,t}}$ for $t \leq T$, where $z_{0,t} = \int_0^t x_t dt$. Start by integrating the value of an active patent with respect to time:

$$\int_0^t v_s ds = \frac{1}{\lambda^2}\left(t\left(\phi + \theta\right) - 2\log\left(\frac{\theta\left(e^{\phi T} - 1\right) + \phi\left(e^{\phi T} + 1\right)}{\theta\left(e^{\phi(T-t)} - 1\right) + \phi\left(e^{\phi(T-t)} + 1\right)}\right)\right).$$

Since $x_t = \lambda^2((1+\alpha^2)\hat{v} - v_t)$, we obtain

$$z_{0,t} = 2\log\left(\frac{\theta(e^{\phi T} - 1) + \phi(e^{\phi T} + 1)}{\theta(e^{\phi(T-t)} - 1) + \phi(e^{\phi(T-t)} + 1)}\right) - (r+\phi)t$$

Thus

$$e^{-z_{0,t}} = \begin{cases} \left(\frac{\theta\left(e^{\phi(T-t)} - 1\right) + \phi\left(e^{\phi(T-t)} + 1\right)}{\theta\left(e^{\phi T} - 1\right) + \phi\left(e^{\phi T} + 1\right)}\right)^2 e^{(\phi+r)t} & \text{if } t < T \\ \left(\frac{2\phi}{\theta\left(e^{\phi T} - 1\right) + \phi\left(e^{\phi T} + 1\right)}\right)^2 e^{(\phi+r)T} & \text{if } t = T \end{cases} \tag{20}$$

Proof: The proof proceed as follows: first we solve (8). Then, we compute its derivative. Since the integral converges as T goes to infinity, the derivative is the sum of terms that converge to zero at different rates. It is shown that the slowest term converging to zero is positive. Thus, the derivative is positive for T sufficiently large and the integral converges from below; i.e., $T = \infty$ can not be a minimum. Recall Equation (8)

$$\mathbb{E}\left[t\right] = \int_0^T (1+\alpha^2)\lambda^2 \hat{v} t e^{-z_{0,t}} dt - \int_0^T \lambda^2 v_t t e^{-z_{0,t}} dt + e^{-z_{0,T}}\left(T + \frac{1}{(\alpha\lambda)^2\hat{v}}\right). \tag{21}$$

Define $k_1 = \lambda^2(1+\alpha^2)\hat{v}/(\theta(e^{\phi T}-1)+\phi(e^{\phi T}+1))^2$; using (20), the first integral of (21) can be written as:

$$k_1\left((\theta+\phi)^2 e^{2\phi T}\int_0^T te^{(r-\phi)t}dt + (\theta-\phi)^2\int_0^T te^{(\phi+r)t}dt + 2(\phi^2-\theta^2)e^{\phi T}\int_0^T te^{rt}dt\right)$$

and using (20) and equation (6), the second integral of (21) can be written as:

$$k_2\left((\theta+\phi)\,e^{2\phi T}\int_0^T te^{(r-\phi)t}dt + (\theta-\phi)\int_0^T te^{(\phi+r)t}dt - 2\theta e^{\phi T}\int_0^T te^{rt}dt\right).$$

where $k_2 = (\theta^2-\phi^2)/(\theta(e^{\phi T}-1)+\phi(e^{\phi T}+1))^2$. Together, they imply that:

$$\begin{aligned}\int_0^T \lambda x_t te^{-z_{0,t}}dt = & \frac{(\phi-r)(\theta+\phi)^2 e^{2\phi T}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}\int_0^T te^{(r-\phi)t}dt \\ & - \frac{(r+\phi)(\theta-\phi)^2}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}\int_0^T te^{t(\phi+r)}dt + 2rk_2e^{\phi T}\int_0^T te^{rt}dt\end{aligned}$$

The generic solution to the three integrals above is given by:

$$\int_0^T te^{at}dt = \frac{1}{a^2}\left(e^{Ta}\left(Ta-1\right)+1\right).$$

With this information we solve (8) which is equal to:

$$\mathbb{E}[t] = \frac{(\theta+\phi)^2}{(\phi-r)}f_1(T) - \frac{(\theta-\phi)^2}{(r+\phi)}f_2(T) + \frac{2\left(\theta^2-\phi^2\right)}{r}f_3(T) + \frac{4\phi^2}{\lambda^2\hat{v}}f_4(T)$$

where

$$f_1\left(T\right) = \frac{e^{2T\phi}\left(e^{T(r-\phi)}\left(T(r-\phi)-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2} \quad f_2\left(T\right) = \frac{\left(e^{T(r+\phi)}\left(T\left(\phi+r\right)-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}$$

$$f_3\left(T\right) = \frac{e^{T\phi}\left(e^{rT}\left(rT-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi(e^{\phi T}+1)\right)^2} \quad f_4\left(T\right) = \frac{\left(1+T(\alpha\lambda)^2\hat{v}\right)e^{(\phi+r)T}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}.$$

As T approaches infinity, $f_1(T)$ converges to a positive constant. The other functions converge to zero. To know whether the integral increases when T approaches to infinity, we need to study its derivative. The derivative will be the sum of terms converging to zero at exponential rates. When T is large enough, only the slowest converging term is relevant. The derivatives with respect to T are:

$$\frac{df_1(T)}{dT} = (\phi-r)K + O\left(e^{-T\phi}\right) \quad \frac{df_2(T)}{dT} = 2\phi J - (\phi+r)K + O\left(Te^{T(r-2\phi)}\right)$$

$$\frac{df_3(T)}{dT} = \phi J - rK + O\left(e^{-T\phi}\right) \quad \frac{df_4(T)}{dT} = (\theta-\phi)J - \alpha^2\lambda^2\hat{v}K + O\left(Te^{T(r-2\phi)}\right)$$

where

$$J = \frac{(\theta+\phi)e^{T(2\phi+r)}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^3} \quad \text{and} \quad K = \frac{(\phi+\theta)\,(\phi-r)\,Te^{T(2\phi+r)}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^3}$$

are positive and converge to zero at a rate of $e^{T(r-\phi)}$ and $Te^{T(r-\phi)}$ respectively. The derivative of (8) with respect to T is given by

$$\frac{d\mathbb{E}[t]}{dT} = \frac{\partial\mathbb{E}[t]}{\partial T} + \frac{d\mathbb{E}[t]}{d\hat{v}}\frac{d\hat{v}}{dT}$$

where

$$\frac{\partial\mathbb{E}[t]}{\partial T} = \frac{2\phi^2(\theta-\phi)\left(2r(r+\phi)+(\alpha\lambda)^2\hat{v}(2r+\theta+\phi)\right)}{(\alpha\lambda)^2 r(r+\phi)\hat{v}}J + O\left(e^{-T\phi}\right)$$

$$\frac{d\mathbb{E}[t]}{d\hat{v}} \approx C + O\left(e^{-T(r-\phi)}\right) \quad \text{and} \quad \frac{d\hat{v}}{dT} \approx O\left(e^{-T\phi}\right),$$

where C is a positive constant. The K terms in $\partial\mathbb{E}[t]/\partial T$ cancel out. Equation (14) shows that $d\hat{v}/dT$ converges to zero at a rate of $e^{-\phi T}$. Using Proposition 4 it can be shown that $d\mathbb{E}[t]/d\hat{v}$ converges to a positive constant C. When T is large enough the terms of order $e^{-\phi T}$ (or smaller) are negligible. Thus the only relevant term is the positive constant accompanying J, and the derivative is always positive; i.e, $\mathbb{E}[t]$ converges to the limit from below, contradicting the conjecture that $(T^*, b^*) = (\infty, 0)$ is a minimum. ∎