

Entry Games under Private Information*

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Abstract

We study market entry decisions when firms have private information about their profitability. We generalize current models by allowing general forms of market competition and *heterogeneous* firms that *self-select* when entering the market. *Post-entry* profits depend on market structure, and on the identities and the private information of the entering firms. We introduce a notion of the firm's *strength* and show that an equilibrium where players' strategies are ranked by *strength*, or *herculean* equilibrium, always exists. Moreover, when profits are elastic enough with respect to the firm's private information, the *herculean* equilibrium is the unique equilibrium of the game.

Keywords: Entry, Oligopolistic markets, Private information

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1 Introduction

Understanding firms' entry decisions is essential in economics. Entry determines market structure, which directly affects prices and welfare. When deciding whether to enter a market, firms base their decisions on the market's observable characteristics, including the number of (potential) competitors and their technology, as well as their own private characteristics (e.g., marginal costs). Lacking a general theory that incorporates heterogeneous public and private firm characteristics, structural analysis has traditionally modeled entry decisions as a complete information game. It has been assumed that: (i) all information is known before entering (à la [Bresnahan and Reiss, 1991](#)) or, (ii) firms learn some private information only after entry has occurred (à la [Levin and Smith, 1994](#)). These assumptions turn market entry into a coordination problem which commonly leads to multiple equilibria ([Tamer, 2003](#)). Multiplicity weakens identification, precluding the possibility of performing robust counterfactual analysis ([Berry and Reiss, 2007](#)).

[Roberts and Sweeting \(2013, 2016\)](#) argue that, in addition to multiplicity, complete information entry models are restrictive and could produce biased counterfactuals. These models cannot account for firms *self-selecting* when entering the market, i.e., they are unable to account that more efficient firms are more likely to enter the market. Since the seminal work by [Seim \(2006\)](#), recent work have incorporated private information in firms' entry decisions. This literature introduces private information as an additive shocks to profits. When the private shock is interpreted as an entry or sunk cost, there can be no self-selection.

In this article, we examine the question of equilibrium uniqueness in a general class of entry games under *private information*. The model allows for diverse forms of *firm heterogeneity* and equilibrium outcomes in which firms may *self-select* when entering the market. To facilitate characterization of the firms' entry strategies, we propose a simple index, called *strength*, which summarizes a firm's capacity to endure competition. The use of *strength* in entry games is similar to the use of a Gittins index in multi-armed bandit games ([Gittins, 1979](#)) as it summarizes the game's relevant information, facilitating the search for equilibria. Unlike the Gittins index, which summarizes information in a single-agent context, *strength* summarizes information in a n -firm game using the public characteristics of every potential entrant. We show that the entry game always has an equilibrium where strategies are ordered according to *strength*, or *herculean* equilibrium. *Strength* reduces the computing power necessary to find equilibria and estimate entry models.

In particular, when there are n potential entrants, there are $n!$ possible rankings for firms' entry strategies; *strength* reduces this problem to the computation of n simple indexes.

Our main contribution is to identify a sufficient condition that guarantees the uniqueness of equilibrium in the entry game. The sufficient condition aids in the empirical identification of the model and broadens the scope of the applications in which market entry can be studied. The condition is easy to verify and requires that payoffs are stable under small changes in the firms's private information. We show that our sufficient condition is not too demanding as it also guarantees well-behaved comparative statics in the entry model. Similarly to monotone virtual valuations in mechanism design à la Myerson (1981), our sufficient condition establishes whether the entry game is sufficiently regular.

The proposed model allows for flexible forms of firm heterogeneity and rich strategic interactions. Firms may differ in their public characteristics, including those that are unobserved by the econometrician (as in Li *et al.*, 2016). To the best of our knowledge, this is the first model that can handle situations in which firms are not uniformly ranked in terms of their competitiveness. For example, our framework incorporates situations in which a subset of firms have larger (publicly observed) entry costs but are likely to have lower (privately observed) marginal cost than another set of competitors. *Private information* adds an extra layer of heterogeneity. Payoffs may not only depend on the opponents' actions but, conditional on entry, payoffs may also depend on the opponents' private information in *non-linear* ways. For example, if firms are privately informed about their marginal costs of production, facing a competitor with lower marginal costs will harm a firm's profitability. The magnitude of this effect will depend (non-linearly) on the firm's marginal costs, the elasticity of demand, the number of entrants and their degree of product substitutability. Our framework accommodates *post*-entry competition in price, quantity, quality, and follower-leader scenarios, among others.

One of the central topics in Industrial Organization is to understand the determinants of market structure and its consequences for welfare. Mankiw and Whinston (1986) studied entry in a model of homogeneous agents with complete information.¹ Levin and Smith (1994), studying an auction setting, examined the case in which firms are *ex-ante* homogeneous but learn their valuations only after entering the market; i.e, firms are *ex-ante* identical when taking their entry

¹In order to avoid confusion we use the expressions (hetero)homogeneous to refer to firms' *ex-ante* characteristics, whereas (a)symmetric is used to refer to firms' strategies.

decision, but heterogeneous *ex-post*. Brock and Durlauf (2001) is closer to our approach as they examine a model in which privately-informed agents choose a binary action. Our modeling shares the idea that both the *action* and the *type* of an agent affect the payoffs of other agents. Our analyses differ in that we examine a context in which entry decisions are *strategic substitutes* and in that we allow for *ex-ante* heterogeneous firms.

Bresnahan and Reiss (1990, 1991) and Berry (1992) developed the first models of market entry that explicitly accounted for the strategic interaction between *post*-entry market competition and firms' entry decisions. When firms are perfectly informed about market outcomes, the entry game often contains multiple equilibria. Berry (1992) shows that if the heterogeneity among the firms is limited, partial identification can be achieved. In contrast, our private-information model can handle any source of heterogeneity. Although Sweeting (2009) argued that multiplicity of equilibria may help with the model's identification in games without outside options, Tamer (2003) showed that, without further assumptions, multiple equilibria usually lead to set, rather than point, identification. Seim (2006) showed numerically that private information may solve the problem of multiplicity of equilibria. However, Berry and Tamer (2006) constructed examples of multiple equilibria under private information, raising the question of when uniqueness can be achieved. We contribute to this discussion by identifying sufficient conditions that guarantee equilibrium uniqueness in entry games under private information.

Models of perfect information are intended to understand long-run equilibrium outcomes, where *ex-post* regret is unlikely. Our framework is especially useful in settings where there is both substantial private information and substantial entry costs, so that *ex-post* regret is likely to arise in the medium term. Private information is particularly important in dynamic settings with information shocks over time as in Ericson and Pakes (1995). Grieco (2014) showed that, in practice, part of the observed heterogeneity comes from private information—marginal costs, contracts with suppliers, and managerial ability—that firms possess at the moment of making their entry decisions. Moreover, he showed that omitting private information from the model can lead to qualitatively different results. In the same spirit, Magnolfi and Roncoroni (2016), using the notion of Bayes Correlated Equilibrium (Bergemann and Morris, 2013, 2016), rejected the idea that their data is generated by a process of complete information. Our contribution is to provide a framework that allows for private information under rich forms of strategic interactions. The

framework allows for firm heterogeneity in both their observable characteristics as well as in their distribution of unobservable characteristics (private information).

The article is organized as follows. Section 2 introduces the model and Section 3 defines and discusses the notions of firm *strength* and *herculean* equilibrium. Section 4 studies the case with two potential entrants. We show that the existence of a *herculean* equilibrium is guaranteed and provide a sufficient condition for when the *herculean* equilibrium is the unique equilibrium of the game. Section 5 extends the analysis to cases with more than two firms in contexts that often arise in empirical applications. Section 6 discusses extensions of our model and Section 7 concludes.

2 A Model of Market Entry

Consider n firms simultaneously deciding on whether to enter a market. Each firm possesses private information about its profitability upon entering the market. The *post-entry* profits of firm i depend on every firm entry decision, i 's private information $v_i \sim F_i$ (scalar), and the private information of the entering firms. We assume that the draws of private information are independent across firms but not (necessarily) identically distributed. In particular, we assume that each firm i is endowed with a distribution function F_i that is atomless and continuously differentiable with full support on \mathbb{R} .²

Let $e_i \in \{0, 1\}$ be an indicator function that takes the value 1 if firm i enters the market. Denote by $e = (e_1, e_2, \dots, e_n)$ the vector of *ex-post* entry decisions, we also refer to e as the (realized) market structure. Define $E_i = \{e : e_i = 1\}$ be the set of all possible market structures in which firm i enters. For a given vector of entry decisions e , define $I(e) = \{i : e_i = 1\}$ to be the set of firms participating in market e . Similarly, define $I_i(e) = \{j \neq i : e_j = 1\}$ and $O_i(e) = \{j \neq i : e_j = 0\}$ to be the set of i 's competitors that are *in* and *out* of the market under structure e , respectively. Define $\mathbf{v} = (v_1, v_2, \dots, v_n)$ to be the vector containing the draws of private information (signals) of every firm. Similarly, v_{-i} represents the draws of private information of every firm except firm i ; and $v_e \equiv (v_k)_{k \in I(e)}$ is the vector of draws of every firm participating in market e .

With a slight abuse of notation, let $\pi_i(v_e)$ be a real valued function representing firm i 's profits of entering the market when the realized vector of private informa-

²All our results would go through if the support of F_i were any interval $[a, b]$ with $a < b$. The current formulation is used for consistency with the literature.

tion is \mathbf{v} and the realized market structure is e . To illustrate the workings of the notation observe that $\pi_i(v_i)$ represents firm i 's profit when its private information is v_i and i is the only firm entering the market. Similarly, $\pi_i(\mathbf{v}) = \pi_i(v_i, v_{-i})$ represents i 's profit when every potential firm enters the market and the vector of private information is given by \mathbf{v} . By adopting this notation we assume that the private information of non-entering firms is payoff irrelevant. In addition, we assume that a firm that does not enter receives zero profit, that the profit function is integrable with respect to v_e under any market structure e , and that the expectation of the profit function with respect to the distribution of private information is finite. Also, since we only require differentiability of $\pi_i(v_e)$ with respect to v_i , we use $\pi'_i(v_e)$ to denote such derivative. Conditional on i 's entry ($e_i = 1$), we assume that $\pi_i(v_e)$ satisfies the following properties.

(A1) Monotonicity: $\pi_i(v_e)$ is strictly increasing and differentiable in v_i ; i.e., $\pi'_i(v_e) > 0$ for all v_i and $v_{e \setminus i}$.³

Assumption A1 gives economic meaning to the private information, v_i . Upon entering the market, and regardless of the realized market structure e , firms' profits are increasing in v_i . In terms of traditional competition models, higher v_i can represent lower marginal cost of production, higher product quality, or higher managerial ability. A1 excludes, for instance, Bertrand competition under *homogeneous* goods, but not under *heterogeneous* goods. In the former case, firms may have profits that are only weakly increasing with respect to their own marginal costs and the profit function may not be differentiable everywhere.⁴

(A2) Competition: For each $j \in O_i(e)$, $\pi_i(v_e)$ is weakly decreasing in e_j . For each $j \in I_i(e)$, $\pi_i(v_e)$ is weakly decreasing in v_j .

Assumption A2 concerns how competition affects profitability. In general terms, it states that firm i 's profits decrease with competition, i.e., $\pi_i(v_e)$ decreases with entry and higher draws of private information by competitors. From the firm's perspective, A2 implies that firms' entry decisions are strategic substitutes and that the degree of substitutability increases with higher draws of private information from a rival. A2 is a minimal assumption on competition.

Finally, to ensure an interior solution we add a third assumption. With a slight

³For any $e \in E_i$, $e \setminus i$ denotes the vector e but with a zero in the i th position. Similarly, for any firm $j \in O_i(e)$, $e \cup j$ denotes the vector e but with a one in the j th position.

⁴Our results are still valid under a weaker version of monotonicity that includes homogeneous Bertrand competition. For brevity, clarity in exposition, and because it encompasses most applications, we choose to present the results under A1.

abuse of notation we define $\phi_i(v_e) = \prod_{j \in I_i(e)} f_j(v_j)$ to be joint density of the private information of i 's competitors under market structure e .

(A3) Interior: There exists values $\underline{v}_i < \bar{v}_i$ such that $\pi_i(\underline{v}_i) = 0$ and

$$\int_{\mathbb{R}^{n-1}} \pi_i(\bar{v}_i, v_{-i}) \phi_i(v_{-i}) d^{n-1}v_{-i} > 0.$$

where the integral is over each of the $n - 1$ dimensions of v_{-i} .

Assumption A3 concerns the entry problem. The value \underline{v}_i represents the minimal draw of private information required to incentivize a monopolist to enter the market. Jointly with A2, the first part of A3 implies that no firm would choose to enter the market, regardless of market structure, when its draw (or private information) is sufficiently low ($v_i < \underline{v}_i$). For this condition to hold, a monopolist should be able to obtain negative *post-entry* profits. Observe that, because entry costs are incorporated into $\pi(\cdot)$, *post-entry* profits can be negative even under the assumption of non-negative variable profits, as firms may always choose not to produce and obtain zero variable profits. The second part of A3 states that any firm may enter the market if its draw of private information is sufficiently high. In particular, there exists a value \bar{v}_i such that drawing $v_i > \bar{v}_i$ ensures entry regardless of whether every competitor decides to enter the market.

The timing of the game is as follows. Before making any entry decision, each firm privately observes v_i . After observing v_i and without observing v_{-i} , each firm independently and simultaneously decides whether to enter the market. After entry decisions are made, market structure e is realized and each firm entering the market gets a payoff $\pi_i(v_e)$. The tuple $(\pi_i, F_i)_{i=1}^n$ —which includes the number of potential entrants n —is commonly known by all the firms in the market.⁵

An entry strategy for firm i is a mapping from the firm's private information v_i to a probability of entering in the market $\tau_i : \mathbb{R} \rightarrow [0, 1]$. We assume that the strategy of player i is an integrable function with respect to its own type v_i . We study the Perfect Bayesian Equilibria of the entry game. Denote by $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ the vector of entry strategies. Given a strategy profile τ , the expected profits of firm i *after* drawing the private information v_i but *before* entry decisions are

⁵The model also accommodates partially informed firms, where firms first obtain a private signal about their market profitability and only after entry, do they become fully informed (see Section 6).

realized is

$$\Pi_i(v_i, \tau) = \tau_i(v_i) \left[\sum_{e \in E_i} \left\{ \int_{\mathbb{R}^{n-1}} \pi_i(v_e) \Pr[e|\tau_{-i}, v_{-i}] \phi_i(\mathbf{v}) d^{n-1}v_{-i} \right\} \right] \quad (1)$$

where $\Pr[e|\tau_{-i}, v_{-i}]$ is the probability of observing market structure e , given the vector of strategies τ_{-i} and the realization of private information v_{-i} . The integral is over each of the $n - 1$ dimensions of the private information of firm i 's competitors, v_{-i} . Conditional on i 's entry, which occurs with probability $\tau_i(v_i)$, the expected profits of firm i consist on the expected sum of profits that firm i would get under each feasible market structure, which is induced by the vector of strategies τ and the realization of private information \mathbf{v} , integrated over all possible realizations of the competitors' private information.

The model allows for general forms of firm heterogeneity. First, firms can differ in their distribution of private information F_i . For instance, firms may have publicly-known production technologies but have private contracts with (potentially different) suppliers. Or, firms could be located in different regions and thus their production is affected by local shocks. Perhaps more importantly, the model allows for firm heterogeneity in the profit function $\pi_i(v_e)$. As noted above, this formulation can accommodate firms with different entry costs. These can represent models in which firms make complementary investments or incur sunk entry costs, in which firms enter the market with different production capacities, or in which firms locate at various distances from some exogenously given market location. Also, the model allows for heterogeneity in the way firms compete *after* entry has occurred. In particular, the general form of the functions $\pi_i(v_e)$ accommodates situations in which firms compete with heterogeneous products and have different production capacities. The model can accommodate the existence of dominant firms, or even a predetermined order of play in the *post-entry* market, such as, competition à la Stackelberg.

The proposed formulation of $\pi_i(v_e)$ does have some restrictions on the nature of *post-entry* competition. First, $\pi_i(v_e)$ is a function rather than a correspondence, which imposes that either the *post-entry* game has a unique equilibrium or, under multiplicity of *post-entry* equilibria, the equilibria are payoff equivalent or that there is a market consensus about which equilibrium will be played. Another, subtler, restriction is the fact that $\pi_i(v_e)$ does not depend on the strategy profile τ . A natural interpretation for the model is that entering firms' private informa-

tion becomes public *after* entry occurs but *before* firms compete in the product market. As a consequence, firms carry no beliefs about their competitors' private information when playing in the *post-entry* game. Another interpretation is that no private information is revealed after entry and $\pi_i(v_e)$ is only observed at the end of the game. To see why the omission of τ in $\pi_i(v_e)$ is restrictive, consider the case in which v_e remains private in the *post-entry* game but the market structure e is observed. In such scenario firms will base their strategies in the *post-entry* game on their beliefs about the private information of their competitors. Through Bayesian updating, these beliefs would depend on the strategy profile τ and the observed market structure e , making it part of the *post-entry* profit function. Although important, the analysis of such models lie outside of the scope of this article.

3 Preliminaries

In this section we provide a general characterization of all equilibria in the entry game and establish the existence of equilibrium. It is shown that, without loss of generality, we can restrict attention to cutoff strategies. Then, using the cutoff structure of equilibria, we introduce two key definitions—firm *strength* and *herculean* equilibrium—for the main results in Sections 4 and 5. We provide intuitions on the nature of these constructions.

3.1 Characterization of Equilibria

Definition (Cutoff Strategy). A strategy $\tau_i(v_i)$ is called *cutoff* if there exists a threshold $x > 0$ such that

$$\tau_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases} .$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on whether its private information is above or below some given threshold. In any best response, there exists a draw of private information, v_i , that makes a firm indifferent between entering the market, or not. We break this indifference by assuming that firms enter. For a cutoff strategy, this means that a firm enters when its draw of private information is equal to its cutoff. Given a vector τ_{-i} , a best response is given by the strategy $\hat{\tau}_i$ that maximizes (1) at every value of

v_i . A (Bayesian Nash) equilibrium, is thus defined by a vector of strategies τ in which every firm best respond to each other. The next proposition establishes the existence of equilibrium and that, without loss of generality, we can restrict our analysis to cutoff strategies.

Proposition 0. *For any game $(\pi_i, F_i)_{i=1}^n$ there exists an equilibrium. For any vector τ_{-i} , firm i 's best response is a cutoff strategy. Therefore, every equilibrium of the game is in cutoff strategies.*

Existence follows from Brouwer's fixed-point theorem. The restriction to cutoff strategies is quite intuitive: regardless of which strategy competitors are playing, assumption A1 implies that firm i 's expected profit is strictly increasing in its private information. Because i 's expected profit is linear in its entry probability (see eq. (1)), i either prefers to enter with certainty when it is profitable to do so, or to stay out otherwise.

We abuse notation and denote a cutoff strategy by the cutoff itself. In particular, from now on $x_i \in \mathbb{R}$ represents the cutoff valuation under which, if $v_i \geq x_i$, the firm enters the market. We also simplify notation by writing $\pi_i(v_i, v_e)$ instead of $\pi_i(v_i, v_{e \setminus i})$ when the context leads to no ambiguity. Let n_e represent the number of firms entering under market structure e . For a vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \dots, x_n)$, define the function

$$H_i(\mathbf{x}) \equiv \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_i(e)}}^{\infty} \pi_i(x_i, v_e) \phi_i(v_e) d^{n_e-1} v_{e \setminus i} \right\} \quad (2)$$

where the integral is across each of the $n_e - 1$ dimensions of $v_{e \setminus i}$.⁶ The next Lemma characterizes all cutoff equilibria.

Lemma 1. *The vector \mathbf{x} of cutoff strategies constitutes an equilibrium if and only if $H_i(\mathbf{x}) = 0$ for each firm i .*

Lemma 1 characterizes every equilibrium of the entry game. The function $H_i(v_i, \mathbf{x}_{-i})$ represents firm i 's expected profit of entering the market when it draws the private information v_i and the opponents play the vector of cutoffs \mathbf{x}_{-i} . Firm i 's best response to \mathbf{x}_{-i} is defined by a cutoff x_i equal to the value of v_i that satisfies $H_i(v_i, \mathbf{x}_{-i}) = 0$. A profile of equilibrium cutoffs \mathbf{x} is, thus, constructed

⁶The following notation conventions are used throughout the article: $\sum \emptyset = 0$, $\prod \emptyset = 1$, and $\int_{\emptyset} a = a$.

by the collection of functions $H_j(\mathbf{x})$ evaluated at a point in which every firm j is indifferent between entering the market when drawing private information x_j .

Lemma 2. *The function H_i is differentiable, strictly increasing in x_i , and weakly increasing in x_j .*

Lemma 2 connects cutoff strategies with the nature of competition in the entry game. When a firm chooses a higher cutoff, it increases its expected payoff upon entry, but enters the market with lower probability. Similarly, when a competitor of firm i increases its entry cutoff, x_j , firm i faces less expected competition, leading to an increase in i 's expected payoff.

3.2 Strength and Herculean Equilibrium

In general, there might be multiple equilibria (see Section 4.1 for an example). The lemmas above provide little information about which firm plays which cutoff, and the number of equilibria in the entry game. The next definition, which ranks players according to a summary of the game fundamentals in their willingness to enter the market, is instrumental in characterizing the entry game further and establishing conditions for the uniqueness of an equilibrium.

Definition (Firm *Strength*). For a given game $(\pi_i, F_i)_{i=1}^n$, let $\sigma_i(s) \equiv H_i(s, \dots, s)$. The *strength* of player i is the unique number $s_i \in \mathbb{R}$ that solves $\sigma_i(s_i) = 0$; i.e.,

$$\sigma_i(s_i) = \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{\infty} \pi_i(s_i, v_e) \phi_i(v_e) d^{n_e-1} v_{e \setminus i} \right\} = 0.$$

We say that firm i is *stronger* than firm j if $s_i < s_j$.

Lemma 3. *$\sigma_i(s)$ is strictly increasing and single crosses zero.*

Lemma 3 shows that *strength* is well defined, assigning a unique scalar s_i to each firm i and, therefore, delivering a complete ranking of the firms. In words, *strength* ranks firms by finding the cutoff s_i that firm i would play under the assumption that all firms play the same cutoff s_i . Intuitively, *strength* ranks firms according to their ability to endure competition. Firm i being *stronger* than firm j ($s_i < s_j$) indicates that firm i , facing more competition than j , needs lower draws of private information than j to enter the market. This observation motivates our next definition.

Definition (*Herculean Equilibrium*). An equilibrium is called *herculean* if the equilibrium cutoffs are ordered by *strength*, with *stronger* players playing *lower* cutoffs.

Intuitively, because *stronger* firms are more able to endure competition, they should be more inclined to enter the market than weaker firms. Therefore, an equilibrium in which cutoffs are ordered by *strength* should naturally emerge in entry games. We show in Proposition 1 that this is indeed the case, and that, under certain conditions, the *herculean* equilibrium is the unique equilibrium of the game. The next definition and corollary will help us motivate and to convey intuitions behind our focus on *herculean* equilibria.

Definition (Homogeneous entry game). An entry game is *homogeneous* when every firm i has homogeneous CDFs: $F_i(v_i) = F(v_i)$ for all i and v_i ; and, homogeneous and anonymous profit functions: $\pi_i(v_i, v_{e \setminus i}) = \pi(v_i, v_{\text{perm}_i(e \setminus i)})$ for all i where $v_{\text{perm}_i(e)}$ is any permutation that fixes the realizations in v_e but changes the identities of the entering firms in e excluding the identity of firm i .

Homogeneous games correspond to the private-information analogue of complete information models à la Bresnahan and Reiss (1990, 1991). In the context of incomplete information, homogeneous-firms models have been studied by Brock and Durlauf (2001), Seim (2006) and Sweeting (2009), among others. It is important to observe that even if firms are *ex-ante* homogeneous, particular realizations of firms' private information can produce outcomes in which firms are *ex-post* heterogeneous. In homogeneous games the focus is usually restricted to *symmetric* equilibrium. The next corollary makes explicit the connection between *herculean* and symmetric equilibrium.

Corollary 1. *If firms are equally strong (i.e., $s_i = s$ for all i), there exists a unique herculean equilibrium. In this equilibrium every firm plays cutoffs equal to their strength ($x_i = s$ for all i). In particular, when the game is homogeneous, the (unique) herculean equilibrium and the symmetric equilibrium coincide.*⁷

Herculean equilibrium aims to extend the idea of symmetric equilibrium to games with heterogeneous agents; i.e., to construct an *asymmetric* analogue of the *symmetric* equilibrium. We do so by ranking firms in terms of their *strength*, which

⁷Notice that the result applies trivially to games with homogeneous firms, where firms have the same *strength* by construction, but also to games with heterogeneous firms when they simply happen to have the same *strength*.

summarize each firm's ability to endure competition by using the notion of symmetric strategies. In games with homogeneous firms, every firm is equally *strong*. Thus, the notions of *strength*, *herculean* equilibrium and *symmetric equilibrium* coincide. We show that *herculean* equilibria possess some desirable properties. Its existence is guaranteed, under certain conditions it is the only equilibrium of the game, and the cutoff order prescribed among firms is intuitive.

The uniqueness result in Corollary 1 follows from *strength* being uniquely defined. Because *strength* and the *herculean* cutoffs coincide when firms are equally *strong*, there is a unique *herculean*/symmetric equilibrium in the entry game. Note, however, that non-*herculean* equilibria may still exist.

4 Two Potential Entrants

To convey our intuition, we begin studying a market with two potential entrants. Section 5 extends the analysis to scenarios likely to arise in applied work under n potential competitors. We show that *herculean* equilibria always exist and establish a general sufficient condition for equilibrium uniqueness. Then we proceed to link our condition with models used in empirical applications. Finally, we study comparative statics of the entry game.

4.1 Herculean Equilibria: Existence and Uniqueness

Let $\Delta_i(v_i, v_j) \equiv \pi_i(v_i) - \pi_i(v_i, v_j)$ be firm i 's profit loss under private information v_i when firm j enters the market with private information v_j , i.e., the difference between monopoly and duopoly profits. In what follows, firms are ordered by *strength*, with firm 1 being the *strongest* firm in the game. Recall (from A3) that \underline{v}_i is the minimum value under which firm i enters the market.

Proposition 1. *In any entry game $(\pi_i, F_i)_{i=1}^2$, an herculean equilibrium always exists and is characterized by cutoffs $x_1 \leq x_2$ that solve $H_i(x_1, x_2) = 0$; i.e.,*

$$\pi_i(x_i)F_j(x_j) + \int_{x_j}^{\infty} \pi_i(x_i, y) dF_j(y) = 0.$$

Moreover, the entry game has a unique equilibrium if for every firm i , all $v_i > \underline{v}_i$, and all $v_j > \underline{v}_j$

$$\frac{\Delta_i(v_i, v_j)}{\pi_i'(v_i)} \frac{f_i(v_i)}{F_i(v_i)} < 1. \quad (3)$$

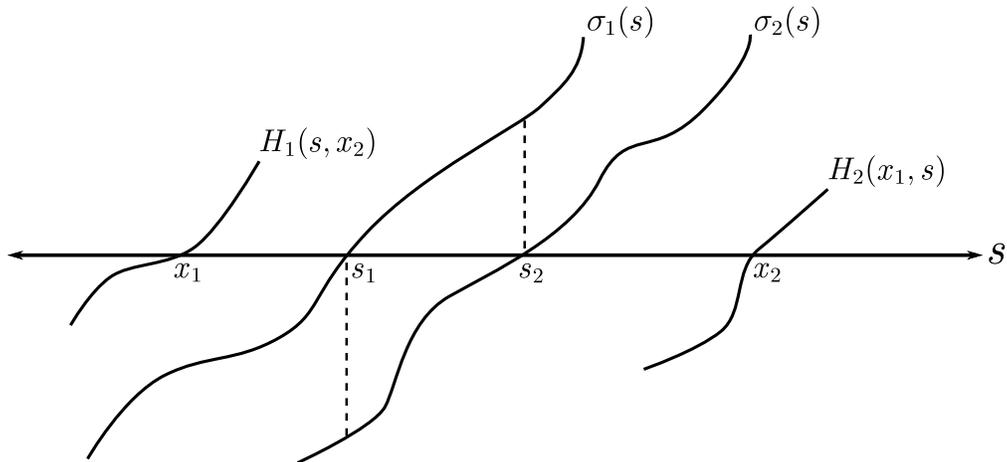


Figure 1: Strength and Herculean equilibrium

Proposition 1 guarantees the existence of *herculean* equilibria, confirming the intuition that this type of equilibria emerges naturally. Figure 1 depicts the construction of a *herculean* equilibrium. Consider the functions $\sigma_1(s)$ and $\sigma_2(s)$ defining the *strength* of each firm. Firm 1 is *stronger* than firm 2 as $s_1 < s_2$. By Lemma 3, $\sigma_i(s)$ is strictly increasing, crossing the horizontal axis once, at s_1 and s_2 respectively. At s_2 , $\sigma_2(s_2) = 0$ and firm 2 breaks even when both firms play the cutoff strategy s_2 ; i.e., when $x_i = s_2$ for all i . Since $\sigma_1(s_2) > 0$, however, firm 1 gets positive expected profits, so this cannot be an equilibrium. Because profits are increasing in the firm's private information, in equilibrium firm 1 plays a lower cutoff, so that $x_1 < s_2$. Observe that $x_1 < s_2$ implies that firm 2 is facing *more competition* than what it would face at its strength cutoff s_2 . Consequently, firm 2 responds by increasing its own cutoff above s_2 , so that $s_2 < x_2$. This establishes that when $x_1 < s_2$ we must have $x_2 > s_2$.

Similarly, at s_1 , $\sigma_1(s_1) = 0$ and firm 1 breaks even when both firms play the cutoff strategy s_1 , i.e., when $x_i = s_1$ for all i . Since $\sigma_2(s_1) < 0$, however, firm 2 gets negative expected profits, so this cannot be an equilibrium. Because profits are increasing in the firm's private information, in equilibrium firm 2 plays a higher cutoff, so that $x_2 > s_1$. Observe that $x_2 > s_1$ implies that firm 1 is facing *less competition* than what it would face at its *strength* cutoff s_1 . Consequently, firm 1 responds by reducing its own cutoff below s_1 , so that $x_1 < s_1$ whenever $x_2 > s_1$. Therefore, by the construction above and assumption A3, we must have a pair (x_1, x_2) such that $x_1 < s_1 < s_2 < x_2$; where the equilibrium thresholds are always further away from each other than the values of *strength*. Hence, there always exists

Table 1: Examples

		Firm 1	
		In	Out
Firm 2	In	$v_1 - \delta_1, v_2 - \delta_2$	$0, v_2$
	Out	$v_1, 0$	$0, 0$

Notes. (a) In examples 1, 2, and 3: $\delta_i = \delta$ for all i . (b) Profit ranking: $\delta_1 = 1/4$ and $\delta_2 = 1/6$.

an equilibrium where cutoffs are ordered by *strength*, i.e., an *herculean* equilibrium.

The argument above does not preclude the existence of multiple *herculean* equilibria, or of non-*herculean* equilibria. Another contribution of Proposition 1 is to provide a sufficient condition under which the *herculean* equilibrium is the unique equilibrium of the entry game, which is a stability condition. It guarantees that the gain in firm i 's profit induced by an increase in its own cutoff x_i cannot be overcome by any response by the competitor's cutoff x_j .⁸ Condition (3) uses only information with respect to each firm i , but it has to hold for every firm in the market. Also, it is sufficient to check the condition only where entry is feasible—where $v_k > \underline{v}_k$ for $k \in \{1, 2\}$ —as deviations for valuations below the minimal entry cutoff \underline{v}_k are always outside the equilibrium path.

To illustrate Proposition 1, consider the next three examples using the entry game described in Table 1 under the assumptions that firms are homogeneous ($F_i(v) = F(v)$ and $\delta_i = \delta$) and $\delta > 0$. These examples corresponds to type-independent extensive margin models (case B in Table 2) which have been the focus of most of the empirical literature. In this model, condition (3) becomes $f(v)/F(v) < \delta^{-1}$ and has to hold for every $v > 0 = \underline{v}_i$.

Example 1 (Multiplicity). Suppose that $F(v) = v^2$ with support in $[0, 1]$. By Corollary 1, homogeneous games possess a unique symmetric equilibrium, which is given by $x^s = (\sqrt{1 + 4\delta^2} - 1)/(2\delta) < 1$ for any δ . Notice that uniqueness is not guaranteed, as condition (3) is never satisfied for all $v > 0$.⁹ In particular, for $\delta > \sqrt{3}/4$, there are two asymmetric equilibria for $i = 1, 2$ given by

$$x_i = \left(1 + \sqrt{4\delta^2 - 3}\right) / 2\delta \quad x_{3-i} = \left(1 - \sqrt{4\delta^2 - 3}\right) / 2\delta.$$

⁸The implicit function theorem guarantees the local uniqueness of the vector \mathbf{x} defined by $\mathbf{H}(\mathbf{x}) \equiv (H_1(\mathbf{x}), H_2(\mathbf{x}), \dots, H_n(\mathbf{x})) = 0$. Gale and Nikaido (1965) show that, when the Jacobian of \mathbf{H} is a P-matrix, the vector \mathbf{x} becomes globally unique. Although the proof provided is constructive, it can be shown that condition (3) guarantees the Jacobian of \mathbf{H} to be a P-matrix.

⁹Since $f(v)/F(v) = 2/v$, for every δ there exists a value of $v > 0$ such that (3) does not hold.

Example 2 (Extreme-Value Distribution). Following Seim (2006), assume that v follows a standard type-I extreme value distribution. Then $f(v)/F(v) = e^{-v}$, which is decreasing in v . Since $f(0)/F(0) = 1$, condition (3) is satisfied for every $\delta < 1$, which implies that the symmetric equilibrium is the unique equilibrium for this set of parameters.

Example 3 (Normal Distribution). Berry and Tamer (2006) observe that, when $v_i \sim N(\mu, \sigma)$ and $\delta > \mu$, the entry game has multiple equilibria when $\sigma \rightarrow 0$ and a unique equilibrium when $\sigma \rightarrow \infty$. Let $\mu = 1$ and $\delta = 4$. Using that the reverse hazard rate for a normal distribution decreases in v , that $f(0)/F(0)$ decreases in σ , and condition (3), we can establish that the entry game has a unique equilibrium whenever $\sigma > 3.876$.

4.2 Uniqueness in Applications

To illustrate how our sufficient condition applies to commonly-used models in the literature, consider the following example under linear profit functions. When firm i is the sole entrant in the market its profits are $\pi_i(v_i) = v_i - K_i$ where K_i are the publicly known entry costs of firm i . If both firms enter, firm i 's profits become $\pi_i(v_i, v_j) = (1 - \gamma)v_i - \rho v_j - \delta - K_i$. The profit functions are increasing in firm i 's private signal v_i (assumption A1) and are weakly decreasing in j 's entry ($\delta \geq 0$ and $\gamma \in [0, 1]$) and in j 's private information ($\rho \geq 0$, as in A2). This parametrization captures the idea that entry by an opponent decreases firm profitability and that the magnitude of this loss depends on the traits of the competing firm.

Consider firm i 's profit loss when j enters the market: $\Delta_{i,j}(v_i, v_j) = \gamma v_i + \rho v_j + \delta$. Table 2 summarizes different entry models under various assumptions of market competition.¹⁰ Case A corresponds to scenarios in which firms do not interact. This may happen because firms are not direct market competitors, competitors are atomistics, or where firm i is the only potential entrant in the industry.

A type-independent extensive margin model (case B) assumes that j 's entry decreases i 's profit, but this decrease is *independent* of the characteristics of both firms. A large extent of the Industrial Organization literature, beginning with the seminal work of Bresnahan and Reiss (1991) and Berry (1992), has examined these effects in the context of a complete information model. Seim (2006) and Grieco (2014) are examples of case B models in the context of private information. The

¹⁰See the Online Appendix for a discussion on the micro-foundation of these cases.

Table 2: Different dimensions of competition in oligopolistic models

Case	Competition Model	$\Delta_{i,j}(v_i, v_j)$	Condition (3)	CDF
A	No Interaction	0	Always unique	No condition
B	Type-independent e.m.	δ	$\frac{f_i(v_i)}{F_i(v_i)} < \frac{1}{\delta}$	Bounded hazard rate
C	Type-dependent e.m.	γv_i	$\frac{v_i f_i(v_i)}{F_i(v_i)} < \frac{1}{\gamma}$	Concave*
D	Intensive margin	ρv_j	$\frac{v_i f_i(v_i)}{F_i(v_i)} < 1$	Concave*
E	Full Oligopoly	$\gamma v_i + \rho v_j + \delta$	$\frac{v_i f_i(v_i)}{F_i(v_i)} < 1$	Concave*

Notes: (a) e.m. stands for extensive margin. (b) The parameters of the model in each case are: A) $\gamma = \rho = \delta = 0$; B) $\gamma = \rho = 0$; $\delta > 0$; C) $\rho = \delta = 0$; $\gamma \in (0, 1)$; D) $\gamma = \delta = 0$, $\rho > 0$, and; E) $\delta, \rho > 0$, $\gamma \in (0, 1)$. (c) Conditions D and E make use of $\pi_i(v_i, v_j) = \max\{0, (1 - \gamma)v_i - \rho v_j - \delta\} - K_i$ as a firm may choose not to produce if it leads to negative variable profits. Thus, $\Delta_{i,j}(v_i, v_j)$ is always bounded above by v_i .

*When the support of F_i is \mathbb{R}_+ , concavity of the CDF is equivalent to $v f_i(v) < F_i(v)$ (cases C–E). When the support of F_i is \mathbb{R} , the conditions above are weaker than concavity.

proposed model also accommodates important aspects of competition that have not yet been fully explored in the literature.¹¹

A *type-dependent* extensive margin model (case C) incorporates the idea that j 's entry directly affects i 's profitability by taking away a share γ of the market. In contrast to the previous models, the profitability of i in its share of the market and, consequently, the magnitude of i 's profit loss, directly depends on i 's private information. This scenario may represent competition in differentiated products where a firm's private information corresponds to the quality or marginal costs of its product. Similarly, we could consider models in which the *intensive* margin effect (case D) is isolated. In these models, larger draws of v_j , which can be interpreted as facing a more competitive opponent, leads firm i to experience a larger profit loss when j enters the market.

More generally, our framework can account for competition models that incorporate both intensive and extensive margins effects (case E). It can also accommodate micro-founded models—such as price and quantity competition with general

¹¹Entry models in which private information is introduced as a linear additive shocks (case B) include Seim (2006); Aguirregabiria and Mira (2007); Bajari *et al.* (2007); Pakes *et al.* (2007); Pesendorfer and Schmidt-Dengler (2008); Sweeting (2009); Aradillas-Lopez (2010); Bajari *et al.* (2010); De Paula and Tang (2012); Vitorino (2012); Grieco (2014); Mazzeo *et al.* (2016).

demand forms—in which these effects can appear in non-linear ways. Finally, the model is flexible enough to capture heterogeneity among firms, incorporating the idea that firms may have different size, capacities, technologies, suppliers, locations, etc.

Table 2 also summarizes sufficient condition (3) for the different cases. Due to the linearity of the profit function, the sufficient condition translates to imposing structure to the distribution of private information. In type-independent extensive margin models (case B), for instance, it is sufficient that the reversed hazard rate of the distributions to be bounded by the inverse of the increase-in-competition effect, δ . When the reversed hazard rate $f_i(v_i)/F_i(v_i)$ is decreasing in v_i , as is the case with Type-I Extreme Value distributions, it is sufficient to check the condition at the entry lower bound \underline{v}_i . For cases C–E, a weak version of concavity of the CDF guarantees equilibrium uniqueness.

More generally, in micro-founded models of competition, condition (3) may be hard to verify. It requires computing closed-form solutions to both monopolistic and duopolistic outcomes. The next result provides a stronger sufficient condition that may be easier to check in applied work.¹²

Corollary 2. *Let $\tilde{\pi}_i(v_e) \geq 0$ be firms' i variable profits under market structure and realization of private information v_e . The entry game has a unique equilibrium if for every firm i and all $v_i > \underline{v}_i$:*

$$\varepsilon \equiv \frac{\tilde{\pi}'_i(v_i)}{\tilde{\pi}_i(v_i)} \bigg/ \frac{f_i(v_i)}{F_i(v_i)} > 1. \quad (4)$$

The corollary simply follows from observing that $\pi'_i(v_i) = \tilde{\pi}'_i(v_i)$ and $\Delta(v_i, v_j) = \tilde{\pi}_i(v_i) - \tilde{\pi}_i(v_i, v_j) \leq \tilde{\pi}_i(v_i)$ because variable profits are non-negative. The term ε corresponds to the elasticity of firm i 's monopolistic variable profits with respect its distribution of private information. Condition (4) tells us that uniqueness is achieved when monopoly variable profits are responsive to changes in private information; i.e., when a change in firm i 's expected profit induced by increasing x_i cannot be overcome by a change in best response x_j . Condition in (4) is stronger than (3) in the sense that there may exist models in which (3) is satisfied but none of the conditions in (4) hold. It is, however, easier to check as only requires information about each firm's monopolistic variable profits.

¹²See the Online Appendix for examples of micro-founded models and their conditions for uniqueness.

4.3 Comparative Statics

We now study comparative statics for the entry model. We begin by parameterizing the fundamentals of the model (π_i, F_i) . Suppose that, for each firm i , there are parameters (ω_i, θ_i) characterizing the distribution of private information $F_i(v_i) = F_i(v_i|\omega_i)$ and profits $\pi_i(v_e) = \pi_i(v_e, \theta_i)$, respectively. We assume that for every i , the family $(\omega_i)_{i=1}^n$ orders $F_i(v_i|\omega_i)$ in terms of first order stochastic dominance (FOSD). In particular, we assume $\omega' > \omega$ implies $F_i(v_i|\omega') \leq F_i(v_i|\omega)$ for every v_i . Similarly, the family of parameters $(\theta_i)_{i=1}^n$ order the firms' profits *uniformly*; i.e., $\theta' > \theta$ implies $\pi_i(v_e, \theta') > \pi_i(v_e, \theta)$ for every vector and market structure v_e . In words, under higher θ , firm i becomes uniformly more profitable. Examples of objects that can order firms profits uniformly are entry costs, fixed costs of production, or access to cheaper suppliers. Finally, using (2), define

$$\bar{H}_i(\mathbf{x}) \equiv \int_{x_i}^{\infty} H_i(s, x_{-i}) dF_i(s) \quad (5)$$

to be the *ex-ante* expected profit of firm i when the vector of cutoffs \mathbf{x} is played.

Proposition 2. *In equilibrium and under condition (3), an increase of ω_i or θ_i leads to: (i) A decrease in x_i and a increase in x_j . (ii) An increase in $\bar{H}_i(\mathbf{x})$ and a decrease in $\bar{H}_j(\mathbf{x})$.*

Proposition 2 is quite intuitive, it tells us that when firm i becomes more competitive—by having, for example, lower entry costs or systematically higher draws of private information—firm i is more likely to participate in the market and receives higher expected profits. In contrast, when an opponent becomes more competitive, a firm is less likely to enter the market and receives lower expected profits. Observe that the results in Proposition 2 are only guaranteed when the conditions for uniqueness in equation (3) hold. This suggests that models possessing multiple equilibria may be associated with ill-behaved comparative statics.

Some empirical studies of entry models under *complete* information deal with estimation problems due to multiplicity of equilibria using an equilibrium selection criteria. A common criteria used in practice is to assume that there exists an entry order among firms which is based on the firm's profitability. In particular, it is assumed that more profitable firms enter the market first (Berry, 1992; Jia, 2008). Because under incomplete information multiplicity of equilibria may still exist, a natural question to ask is whether this methodology can be extrapolated to this

context. More precisely, we examine whether firms' *ex-ante* expected profits—i.e., before firms observe their private information—relate to cutoff order and entry probabilities. *Ex-ante* expected profits are a good proxy for (*ex-post*) aggregated firm profitability, as the latter could be understood as the realization of multiple entry decisions in independently drawn markets. We show that the relative likelihood of entering a market, cutoff order, and firm profitability are related in the class models defined below.¹³

Definition (*Quasi-homogeneous in Distributions – QHD*). An entry game is *quasi-homogeneous in distribution* when firms are characterized by homogeneous and anonymous profit functions, a collection of parameters $(\omega_i)_{i=1}^n$ and CDFs $F_i(v_i) = F(v_i|\omega_i)$ such that ω_i orders F_i in terms of first order stochastic dominance

Definition (*Quasi-homogeneous in Profits – QHP*). An entry game is *quasi-homogeneous in Profits* when firms are described by homogeneous CDFs, a collection of parameters $(\theta_i)_{i=1}^n$, and $\pi_i(v_e) = \pi(v_e, \theta_i)$ where θ_i order i 's profits uniformly.

Quasi-homogeneous entry games corresponds to the private information analogue of a wide class of models studied in practice, including the analogue of [Berry \(1992\)](#) where firms can be *ex-ante* ranked by entry costs. Other examples of *quasi-homogeneous* games arise in situations where firms are ranked in terms of their production capacities, or where firms produce homogeneous goods but may have access to different production technologies. In practice, QHD games have been used in the empirical auction literature by [Athey et al. \(2011\)](#) where bidders are ordered in terms of hazard rates; and QHP models have been studied by [Vitorino \(2012\)](#) where profits are linear in the private information and v_i are identically distributed among firms.

Proposition 3. *Firms ex-ante profits (5) can be (inversely) ranked according to the order of equilibrium cutoffs when: (i) firms are homogeneous and play an asymmetric equilibrium; (ii) firms are quasi-homogeneous (in distributions or profits) and play herculean equilibria. In these situations, a lower entry cutoff also translates to higher probability of entering the market.*

In entry games where firms have identical distributions of private information, as in homogeneous entry games or in QHP games, entry cutoff ranking naturally

¹³The difference between quasi-homogeneous (QH) games and the order presented at the beginning of this section is that QH games order *across* instead of *within* a firm.

translates into a ranking of market-entry probabilities. Despite firms having different CDFs, this relation between cutoff and entry likelihood extends to QHD games whenever firms play *herculean* equilibria. In applied work, these connections are important, as they link a firm’s observed behavior (e.g., firm’s probability of entering a market) with equilibrium behavior and observed profitability.

In more general models of market entry, Proposition 3 does not necessarily hold. We can construct examples in which cutoff order, entry probability, and profit-ranking do not relate. To illustrate this, consider again the example in Table 1 corresponding to a type-independent extensive margin model (Table 2 case B). Consider the following set of assumptions: firms pay an entry cost $K = 1/2$, $\delta_1 = 1/4$, $\delta_2 = 1/6$, $v_1 \sim U[0, 1]$, and $v_2 \sim U[0, 4/5]$. In this scenario firms are not (*quasi*) homogeneous but are equally *strong* with $s_i = 4/7$ for $i \in \{1, 2\}$. Firm 1 is more profitable in expectation, because $F_1(v)$ FOSD $F_2(v)$. However, firm 1 suffers more losses when facing competition because $\delta_1 > \delta_2$. Also, it is not hard to verify that condition (4) holds for all $v_i > 1/2 \equiv \underline{v}_i$.¹⁴ Hence, the game has a unique equilibrium (the *herculean* equilibrium). By Corollary 1, firms playing cutoffs equal to their *strength*, i.e., $x_1 = x_2 = 4/7$, is an herculean equilibrium. Thus, firms are not (strictly) ordered in terms of their entry cutoffs. Observe that firm 1 is more likely to enter the market as $F_1(4/7) = 4/7 < 5/7 = F_2(4/7)$. Firm 1 also obtains higher expected profits as $\bar{H}_1(\mathbf{X})/\bar{H}_2(\mathbf{X}) = 225/64 > 1$. By continuity, one can construct similar examples with a slightly smaller (larger) δ_1 such that $x_1 < s_1 < s_2 < x_2$ ($x_1 > s_1 > s_2 > x_2$), while preserving equilibrium uniqueness and the previous rankings in terms of expected profit and probability of entering the market.

The example above shows that our framework can accommodate situations where the ranking of firm *profitability* (represented by v_i) is different than the ranking of firm *competitiveness* (represented by δ_i). In complete information models, this degree of firm heterogeneity would violate the assumption of the equilibrium selection criteria proposed by Berry (1992). To illustrate this, consider a perfect information version of the example in Table 1. Assume that monopoly profits are $\pi_1^M = 0.23$ and $\pi_2^M = 0.2$ and, using $\delta_1 = 1/4$ and $\delta_2 = 1/6$, duopoly profits are $\pi_1^D = -0.02$ and $\pi_2^D = 0.033$. This is a simple variation on the Bresnahan and Reiss (1991) model where δ_i is heterogeneous among firms.¹⁵ Because complete in-

¹⁴Simply observe that $f_i(v_i)/F_i(v_i) = v_i^{-1}$, $\Delta_i = \delta_i$, and $\pi'(v_i, v_e) = 1$ for all e . Then (4) for firm i becomes $v_i > \delta_i$ which holds for $v_i > \underline{v}_i = 1/2$.

¹⁵Ciliberto and Tamer (2009) propose an estimator to partially identify the parameters of the

formation models may have multiple equilibria, [Berry \(1992\)](#) proposed a *monotone* selection criteria where, at every step, the most profitable firm decides whether to enter the market. This criteria relies on the assumption that the ranking of firm profitability does not change across market structures. In the example above, this is not true. When there are no firms in the market, firm 1 has the largest monopolistic profits and the criteria prescribes entry. In the second step, firm 2 also enters the market because $\pi_2^D > 0$. This, however, cannot be an equilibrium because $\pi_1^D < 0$. Moreover, the unique equilibrium of the game is that only firm 2 enters, even though firm 1 would have higher monopoly profits. Although we assume that firms possess private information, one of our main contributions is to provide a framework and a methodology that can handle any type of firm heterogeneity, including situations in which the ranking of firm profitability changes across market structures.

5 Equilibria with More than Two Entrants

This section extends the ideas behind [Proposition 1](#) in contexts with more than two potential entrants. The first extension studies an environment with n homogeneous firms. The second extends results to a scenario in which firms belong to one of two groups, each group having an arbitrary number of potential entrants. Firms are identical *within* groups, but there are no restrictions on the degree heterogeneity in profit functions and distribution of private information that firms can have *across* groups.

The next definition extends firm's i profit loss after firm j enters the market under more general market structures. As before, this object will be important in defining the sufficient condition for uniqueness of equilibrium. For any firm i and fix a market structure e such that i participates (i.e., $e \in E_i$), for any firm j *not* participating in market e (i.e., $j \in O_i(e)$) define:

$$\Delta_{i,j}(v_i, v_j, v_e) \equiv \pi_i(v_i, v_{e \setminus i}) - \pi_i(v_i, v_j, v_{e \setminus i}) \quad (6)$$

to be firm i 's profit loss when firm j enters the market under initial market structure e , firm i 's private information is v_i , firm j 's private information is v_j , and the rest of the entering firms private information is represented by $v_{e \setminus i}$. Observe that, by

[Bresnahan and Reiss \(1991\)](#) model when δ_i varies by firm.

A2, this loss is non-negative for any v_i, v_j and v_e , i.e., $\Delta_{i,j}(v_i, v_j, v_e) \geq 0$.

5.1 Homogeneous Entry Games

We begin by extending the ideas behind Proposition 1 to n -firm homogeneous entry games which were formally defined in Section 3.2. We can drop the i sub-index from profits and distributions functions, and write i 's profit loss in market structure e as $\Delta(v_i, v_j, v_e)$, because under homogeneity profits functions are anonymous and identical across firms.

Proposition 1A. *In homogeneous entry games, there exists a unique herculean equilibrium characterized by the strength of the firms; i.e., $x_i = s$ for all i , where s is the unique number that solves*

$$\sigma(s) \equiv \sum_{r=0}^{n-1} \left\{ \binom{n-1}{r} F(s)^{n-1-r} \int_s^{\infty} \pi(s, \mathbf{y}_r) \phi_i(\mathbf{y}_r) d^r y \right\} = 0.$$

Moreover, if for every market structure $e \in E_i$, every opponent $j \in O_i(e)$, and draws of private information $v_k \geq \underline{v}_k$ for $k \in e \cup j$ the following holds

$$\frac{\Delta(v_i, v_j, v_e)}{\pi'(v_i, v_e)} \frac{f(v_i)}{F(v_i)} < 1, \quad (7)$$

the herculean (i.e., symmetric) equilibrium is the unique equilibrium of the game.

From Corollary 1 we know that homogeneous games have a unique *symmetric* equilibrium. Proposition 1A, thus, builds upon this result by providing a sufficient condition under which the symmetric equilibrium is the unique equilibrium of the entry game. Comparing conditions (7) and (3) two key differences arise: (i) condition (7) has to hold across more market structures e (not only monopoly-duopoly), and; (ii) it has to hold for every realization of v_e . This suggests that condition (7) is, in principle, more demanding than (3). In practice, however, this may not be the case. As shown in Proposition 4 in Section 6 below, for certain models of quantity competition, if condition (4) in Corollary 2 holds for the monopolist, then condition (7) holds for every possible market structure.

5.2 Two Groups of Firms

We now extend our model to a market characterized by two groups of firms $g \in \{1, 2\}$. Within each group firms are identical (homogeneous). Group g consists of $m_g \in \mathbb{N}$ firms described by the pair (π_g, F_g) . Let g_i be the group of firm i , define $\pi_i(v_e) \equiv \pi_{g_i}(v_i, v_{e \setminus i}^{g_i}, v_e^{3-g_i})$ where v_e^g represents the draws of private information of firms in group g that participate under market structure e . We assume that profits are *anonymous* to the private information of competitors within a group: i.e., $\pi_{g_i}(v_i, v_{e \setminus i}^{g_i}, v_e^{3-g_i}) = \pi_{g_i}(v_i, v_{\text{perm}_i^g(e \setminus i)}^{g_i}, v_{\text{perm}_i^g(e)}^{3-g_i})$ where $v_{\text{perm}_i^g(e)}$ is any permutation that fixes the group g and the realizations in v_e but changes the identities of the entering firms in e excluding the identity of firm i .

Because firms are homogeneous within a group, an *herculean* equilibrium prescribes that every firm within a group should play symmetric strategies. To formally characterize an equilibrium, fix a firm i , let $\varphi_i^g(v_e) = \prod_{j \in e \setminus i, g_j = g} f_g(v_j)$ be the probability that firms participating under the structure e that belong to group g but are not firm i , and draw the vector of private information v_e^g . Without loss of generality, we assume that firms in group 1 are stronger than those in group 2 ($s_1 \leq s_2$). For a pair of cutoff strategies $\mathbf{x} = (x_1, x_2)$ define $\mathbb{E}[\pi_{g_i}(v_i) | \mathbf{x}, r, k]$ as the expected profits of a firm in group g_i when it draws private information v_i , and r other firms of group g_i and k firms of group $3 - g_i$ have entered the market, that is

$$\mathbb{E}[\pi_{g_i}(v_i) | \mathbf{x}, r, k] = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} \pi_{g_i}(v_i, v_{e \setminus i}^{g_i}, v_e^{3-g_i}) \varphi_i^{3-g_i}(v_e) \varphi_i^{g_i}(v_{e \setminus i}) d^k v_e^{3-g_i} \right) d^r v_{e \setminus i}^{g_i}$$

where the integrals are over the r and k dimensions of $v_{e \setminus i}^{g_i}$ and $v_e^{3-g_i}$ respectively. Then, a pair of strategies $\mathbf{x} = (x_1, x_2)$ is an equilibrium if and only if, from the perspective of a firm in group g_i , \mathbf{x} satisfies:

$$\sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k] \right] \right\} = 0,$$

where, for notational ease, $i = g_i$ and $j = 3 - g_i$. To understand the equation above, fix a market structure e in which r firms of group i and k firms of group j participate in the market. Because there are m_j firms in group j , there exist m_j choose k possibilities to obtain a market structure with k competitors from group j , each occurring with probability $F_j(x_j)^{m_j-k}$. Similarly, because the analysis is from the perspective of a firm in group g_i , there exists $m_i - 1$ choose r possibilities to

observe r competitors from group i , each occurring with probability $F_i(x_i)^{m_i-1-k}$. The expression above is, thus, obtained by summing across every possible market structure e .

Proposition 1B. *An herculean equilibrium always exists. Moreover, the herculean equilibrium is the unique equilibrium of the game if for any market structure $e \in E_i$, any firm $j \notin e$, and any type $v_k \geq \underline{v}_k$ for $k \in e \cup j$ the following conditions holds:*

- If j belongs to i 's group (i.e., $g_j = g_i$):

$$\frac{\Delta_{i,j}(v_i, v_j, v_e)}{\pi'_i(v_i, v_e)} \frac{f_i(v_i)}{F_i(v_i)} < 1 \quad (8)$$

- If j does not belong to i 's group (i.e., $g_j \neq g_i$):

$$\frac{\Delta_{i,j}(v_i, v_j, v_e)}{\pi'_i(v_i, v_e)} \frac{f_i(v_i)}{F_i(v_i)} < \frac{1}{m_{O(e)}^{3-g_i}} \quad (9)$$

where $m_{O(e)}^{3-g_i}$ is the number of firms in group $3 - g_i$ that do not participate in market structure e .

Proposition 1B provides two conditions that need to be satisfied for uniqueness. Condition (8) is analogous to (7) in Proposition 1A, as it is required to hold among firms within the same group; i.e., among homogeneous firms. This condition guarantees that only group-symmetric strategies are played. Empirical applications usually restrict their analyzes to these type of strategies (see references below). Condition (8), thus, guarantees that this restriction is without loss. Recall that *herculean* equilibria are always group-symmetric equilibria. Firms belonging to the same group are equally strong and, therefore, play the same (*herculean*) strategy.

Condition (9), on the other hand, guarantees that the *herculean* equilibrium is the unique equilibrium of the game. Condition (9) has to hold for every firm that is not in the same group than firm i and is a bit more demanding than condition (8), as its right hand side (RHS) is a function of the number of firms not participating in market structure e . In other words, the larger the number of firms in group $3 - g_i$ participating in e , the more likely is that condition (9) holds. Notice that (9) has to hold for any $j \in O(e)$, thus, the smallest value for $m_{O(e)}^{3-g_i}$ is 1.

In applied work, models of two groups of potential entrants have been used by Athey *et al.* (2011) and Roberts and Sweeting (2013), who study the timberwood

industry, and distinguish between loggers and mills; and by [Krasnokutskaya and Seim \(2011\)](#), who study highway procurement, and divide firms between favored (small) and non-favored (large) firms. This type of group structure can arise in applications ranging from incumbents and entrants, high and low quality segments, local and international producers, discount retailers and traditional supermarkets, to legacy and low-cost airlines.

6 Discussion

In this section we discuss some results in greater depth, present some extensions, and relate our results to other topics outside the entry literature.

Uniqueness and Market Demand The sufficient condition for uniqueness (4) can be related to observable market objects. We can do this by interpreting private information as marginal cost. In particular, assume there exists a marginal cost function $c_i(v_i)$ that is decreasing and differentiable in v_i . Let $p_i^*(v_e)$ be firm's i equilibrium price under market structure and realization of private information v_e . Then, $\mu_i(v_e) \equiv (p_i^*(v_e) - c_i(v_i)) / c_i(v_i)$ represents i 's markup under market structure and private information v_e .

Lemma 4. *In games in which higher v_i leads to lower marginal costs $c_i(v_i)$, condition (4) is equivalent to*

$$\frac{-c_i'(v_i)}{c_i(v_i)} \bigg/ \frac{f_i(v_i)}{F_i(v_i)} > \mu_i(v_i). \quad (10)$$

This function identifies a unique equilibrium for entry games in which the elasticity of marginal costs with respect to the distribution of private information is larger than the mark-up. The mark-up connects uniqueness to demand elasticity. For example, markets with inelastic demand tend to have lower mark-ups, suggesting that those markets have a unique entry equilibrium. Similarly, if firms have a very elastic marginal cost function with respect to the private information v_i , the game is likely to have a unique equilibrium.

A Single Sufficient Condition The number of sufficient conditions to check increase exponentially in the number of potential entrants n . This is the case because, for each firm i , we have to check whether conditions hold for each feasible

market structure. The next proposition shows that, in quantity competition models, it is sufficient to check a single condition per firm. In particular, we show that when uniqueness condition (4) (or (10)) holds under monopoly, the conditions in markets with more competitors also hold.

Proposition 4. *In quantity competition games in which higher v_i leads to lower marginal costs $c_i(v_i)$ and where entry decreases markups, $\mu(v_e)$. If condition (4) (or equivalently condition (10)) holds for firm i at v_i , then condition (7) holds at $v_e = (v_i, v_{e \setminus i})$ for any realization of $v_{e \setminus i}$ and market structure $e \in E_i$.*

Consider a Cournot competition game with homogeneous goods and linear demand $P = 1 - Q$ where $Q = \sum_{i \in n} q_i$ and $c_i(v_i) : \mathbb{R} \rightarrow [0, 1]$ (see Online Appendix). As in Corollary 2, we can substitute $\Delta_{i,j}(x, y, v_e)$ for $\tilde{\pi}_i(x, v_e)$ in condition (7) and get a stronger sufficient condition. Then, it is not hard to verify

$$-\frac{(1 - c_i(v_i))}{2c'_i(v_i)} = \frac{\tilde{\pi}_i(v_i)}{\tilde{\pi}'_i(v_i)} \geq \frac{\tilde{\pi}_i(v_e)}{\tilde{\pi}'_i(v_e)} = -\frac{1}{2c'_i(v_i)} \left(\frac{1 + \sum_{j \in e \setminus i} c_j(v_j)}{n_e} - c_i(v_i) \right)$$

where $c'_i(v_i) < 0$ for all v_i . This inequality implies that checking for (4) (competition under monopoly) at the realization v_i is sufficient for (7) to hold at any $v_e = (v_i, v_{e \setminus i})$; i.e., (7) holds for any market structure e , independently of the realization of competitors' private information $v_{e \setminus i}$.

Selective Entry Recent structural analyses of market entry study scenarios in which firms are *ex-ante* homogeneous but become partially informed before their entry decision by receiving a signal correlated with their type. Since firms with high realizations are relatively more likely to enter the market, there is *selective entry*.¹⁶ At one extreme there are LS models (Levin and Smith, 1994) where signals are infinitely noisy; i.e., firms possess no private information before entry. At the other extreme are S models (Samuelson, 1985) in which firms become perfectly (and privately) informed about their type. The model presented in Section 2 belongs to the latter category. Below, we argue that both models with perfectly informative signals and models with a partially informative signals can be solved using the tools developed in this article. Notice, however, that these models may have different empirical predictions (Gentry and Li, 2014). For example, in a selective entry

¹⁶Selective entry models have been studied by Roberts and Sweeting (2013); Gentry and Li (2014); Bhattacharya *et al.* (2014); Sweeting and Bhattacharya (2015); Grieco (2014).

model with a partially informative signal, an entrant who receives a high signal will submit a “very low” bid after learning that his true value is low.¹⁷ This cannot happen in games with perfectly informative signals.

LS models, as argued by Roberts and Sweeting (2013, 2016) and Gentry and Li (2014), produce no selection in unobservables. These models can predict the number, but not the identity, of entrants from a particular group. The models also predict that there are no differences in unobservable (to other players) characteristics between entrants and non-entrants. These are the same predictions of models à la Bresnahan and Reiss (1991). Because entry in LS models occurs before private information is revealed, these types of entry games are *de facto* games of complete information. Li and Zheng (2009) and Krasnokutskaya and Seim (2011), modify the LS model by allowing entry costs to be private information before entry occurs, but realization of valuations to be learned after entry. This condition turns the entry stage into a private information game, which is embedded in our framework. However, because entry costs do not affect valuations, there is no selective entry: the distribution of valuations before and after firms’ entry decisions coincide.

We now show that under a simple informational assumption and a reinterpretation of the profit function, our results extend to general *selective entry* models. Let $F_i(v_i, \varepsilon_i)$ be a joint cumulative distribution of *signals* v_i and *types* ε with support in \mathbb{R}^2 . The distributions F_i are independent across firms. Before making their entry decisions, firms observe the signal v_i that partially informs firms about their type ε_i . Let $F_i(v_i) = \int_{-\infty}^{\infty} F_i(v_i, s) ds$ and let $F_i(\varepsilon_i|v_i) = F_i(v_i, \varepsilon_i)/F_i(v_i)$ be the CDF of ε_i conditional on v_i .

(A4) Affiliated Signals: For $v' > v$, $F_i(\varepsilon|v') < F_i(\varepsilon|v)$ for all i and ε .

Assumption A4 states that higher signals lead to higher expected types in terms of First Order Stochastic Dominance (FOSD) similarly to Marmer *et al.* (2013) and Gentry and Li (2014).

Let $\hat{\pi}_i(\varepsilon_e)$ be the profits of firm i when the market structure is e and the realization of private information for every firm in e is ε_e . Then, we can re-interpret $\pi_i(v_e)$ as

$$\pi_i(v_e) = \int_{-\infty}^{\infty} \hat{\pi}_i(\varepsilon_e) \prod_{k \in I(e)} f_k(\varepsilon_k|v_k) d^{m_e} \varepsilon_e$$

where the integral is across the n_e dimensions of ε_e . Given the properties of FOSD,

¹⁷Where a “very low” bid is in relation to the entry cost, the number of potential entrants and their distributions of private characteristics.

it is straightforward to see that if the profit function $\hat{\pi}_i(\varepsilon_e)$ satisfies analogous conditions to A1-A3, then $\pi_i(v_e)$ satisfies the assumptions of the model, and the results go through.

Observe that $\pi_i(v_e)$ is obtained by integrating $\hat{\pi}_i(\varepsilon_e)$ over all possible realizations of ε_e . This means that we could weaken the differentiability requirements of $\hat{\pi}_i$ and the everywhere (strictly) increasing assumption in ε_i , and still obtain A1 in π_i . This means that our results apply to selective entry models in which firms compete on prices with homogeneous goods and in second-price auctions. This case is studied further in [Espín-Sánchez *et al.* \(2018\)](#).

Dynamic Oligopoly Models with Entry We present our results in the context of static entry games. Our findings, however, extend to dynamic entry models á la [Ericson and Pakes \(1995\)](#) when firms possess private information when deciding whether to enter (e.g., [Aguirregabiria and Mira, 2007](#)). In a dynamic setting, it is sufficient to show that the entry value function (the sum of current profits and the discounted expected sum of future profits) satisfies assumptions A1-A3, when firms make their entry decisions. Although it is beyond the scope of this article to find conditions in the game fundamentals that guarantee that the value function satisfies these assumptions, we think it is an important avenue for future work.

Supermodular Games There are some similarities in our approach to the theory of supermodular games. Although, supermodularity requires players' actions to be strategic complements (see [Morris and Shin, 2003](#)), and while we treat actions by contrast as strategic substitutes, our uniqueness condition imposes a payoff structure that resembles supermodularity. To illustrate this, consider an entry game with two potential firms. The uniqueness condition requires that firm i 's profits from participating in the market are increasing in its private information, even when firm j *best-responds* to i 's entry. That is, once we account for the opponent's actions, higher draws of private information always leads to higher incentives to enter.

7 Concluding Remarks

In this article, we study entry games under private information. Our analysis generalizes previous models by allowing heterogeneous firms and rich strategic

interactions. The proposed model provides a framework that can be used in empirical work to incorporate firm selection. To characterize the game, we propose a simple firm index, called *strength*, that orders firms by competitiveness. This index greatly reduces the computational burden of calculating equilibria with heterogeneous firms. The equilibrium associated with this index (*herculean*) can be thought of as a focal equilibrium in games with heterogeneous firms. This equilibrium always exists and is unique under mild conditions that are easy to check.

The model presented here is general, encompassing entry models currently used in both the theoretical and the empirical literature. The tools we develop could also be useful in solving more general (non-entry) games. Although the research is beyond the scope of this article, the results could be extended to dynamic games with finite or infinite horizons. In that case, one would have to check that the value function in the Bellman equation is monotone on the firm's type, and that the other regularity conditions hold. We believe that this avenue of research has great potential.

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Appendix

A Omitted Proofs

Proof of Proposition 0. *Best responses are cutoff strategies:* Fix any firm i and vector of strategies τ . Because i 's profit is linear in τ_i , i 's best response is to participate with probability one whenever there is a positive payoff of doing so. Suppose i enters the market with certainty ($\tau_i(v_i) = 1$). A1 implies that profits are strictly increasing in v_i . By A3 $\Pi_i(\underline{v}_i, \tau) \leq 0$, and $\Pi_i(\bar{v}_i, \tau) > 0$. Thus, $\Pi_i(v_i, \tau)$ single crosses zero and i 's best response to τ_{-i} is the cutoff strategy defined by the value x_i that satisfies $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$.

Existence: We check the conditions of Brouwer's fixed-point theorem. Because F_i is atomless and has full support and $\pi_i(v_e)$ being continuous and differentiable in v_i , player i 's best response lies in the compact and convex set $[\underline{v}_i, \bar{v}_i]$. Thus the n -dimensional function of best responses is a continuous mapping from $\times_{i=1}^n [\underline{v}_i, \bar{v}_i]$ to itself and the conditions for the theorem are met. ■

Proof of Lemma 1. By the previous proof a cutoff strategy is defined as the value x_i satisfying $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$. Because in a cutoff equilibrium $\Pr[e|\tau, v_i]$ is either zero or one. Integrating (1) over payoff-irrelevant firms delivers (2). ■

Proof of Lemma 2. Start with the derivative of H_i with respect to i , then:

$$\frac{\partial H_i}{\partial x_i} = \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \int_{\{x_j\}_{j \in I_i(e)}}^{\infty} \pi'_i(x_i, v_e) \phi_i(v_e) d^{m_e-1} v_{e \setminus i} \right\} > 0. \quad (11)$$

For the derivative of H_i with respect to j , pick a market structure e such that $j \notin e$. Conditional on e , the derivative of H_i with respect to x_j is equal to:

$$f_j(x_j) \left(\prod_{k \in O_i(e) \setminus j} F_k(x_k) \right) \int_{\{x_k\}_{k \in I_i(e)}}^{\infty} \pi_i(x_i, v_e) \phi_i(v_e) d^{m_e-1} v_{e \setminus i} > 0.$$

Now take market structure e from above and, using Leibnitz differentiation, compute the derivative of H_i with respect to x_j conditional on market structure $e \cup j$; i.e., entry decisions by every firm remain the same as in e except that of firm j , which now enters:

$$-f_j(x_j) \left(\prod_{k \in O_i(e) \setminus j} F_k(x_k) \right) \int_{\{x_k\}_{k \in I_i(e)}}^{\infty} \pi_i(x_i, x_j, v_e) \phi_i(v_e) d^{m_e-1} v_{e \setminus i} < 0.$$

Observe that both expressions from above only differ in sign and in the profit function that is integrated over. Summing both equations delivers a positive expression where the integral is over $\Delta_{i,j}(x_i, x_j, v_e)$ which is the non-negative expression defined in (6).

Summing across every market structure we obtain:

$$\frac{\partial H_i}{\partial x_j} = f_j(x_j) \sum_{e \in E_i \setminus E_j} \left\{ \left(\prod_{k \in O_i(e) \setminus j} F_k(x_k) \right) \int_{(x_k)_{k \in I_i(e)}}^{\infty} \Delta_{i,j}(x_i, x_j, v_e) \phi_i(v_e) d^{n_e-1} v_{e \setminus i} \right\}. \quad (12)$$

Thus, the derivative is non-negative. \blacksquare

Proof of Lemma 3. We show that s_i exists and that $\sigma_i(s)$ single crosses zero.

Existence: Observe that A3 jointly with A2 imply $\sigma_i(\underline{v}_i) < 0$. We need to show that exist \hat{s} such that $\sigma_i(\hat{s}) > 0$, so that $\sigma_i(s_i) = 0$ exist by the Intermediate Value Theorem. Observe that, by Lemma 2, $\sigma_i(s) \geq H_i(x_i = s, -\infty_{-i})$.¹⁸ But, by A3, there exist \hat{s} such that $H_i(x_i = \hat{s}, -\infty_{-i}) > 0$. Thus, for the same \hat{s} , we have that $\sigma_i(\hat{s}) > 0$.

Uniqueness: By Lemma 2 and the chain rule we have that $\sigma'_i(s) > 0$. Thus, $\sigma_i(s)$ single crosses zero; i.e., there is a unique value s_i satisfying $\sigma_i(s_i) = 0$. \blacksquare

Proof of Corollary 1. By definition of *strength*, if $\sigma_i(s) = 0$ for every i , then $H_i(s, \dots, s) = 0$ for every i and $x_i = s$ for all i constitute an equilibrium. The equilibrium is the unique *herculean* equilibrium as, by Lemma 3, *strength* is uniquely defined. \blacksquare

Proof of Proposition 1. *Preliminaries:* Suppose first $s_1 = s_2 = s$. Then, by definition of strength, $x_1 = x_2 = s$ corresponds to a *herculean* equilibrium. Assume without loss of generality $s_1 < s_2$. Define $g(x)$ to be the function that solves $H_1(g(x), x) = 0$. Then, $g(x)$ corresponds to firm one's best response to firm two, when firm two plays the cutoff strategy x . By Lemma 7 in Appendix B, the value $g(x)$ exists and is unique; i.e., $g(x)$ is well defined.

Claim 1. $g(s_1) = s_1$, $g'(x) \leq 0$ and, under (3), $g'(x)$ is bounded below by

$$-\frac{f_2(x)F_1(g(x))}{F_2(x)f_1(g(x))}. \quad (13)$$

Proof. By definition of strength $H_1(s_1, s_1) = 0$. Then, when $x = s_1$, $g(s_1) = s_1$. Using the implicit function theorem:

$$g'(x) = -\frac{f_2(x)\Delta_1(g(x), x)}{F_2(x)\pi'_1(g(x)) + \int_v^{\infty} \pi'_1(g(x), y) dF_2(y)}$$

which is non-positive as the denominator is positive and the numerator is non-negative. For the lower bound of $g'(x)$ observe that the integral term in the denominator is positive. Then, taking the integral to zero:

$$g'(x) \geq -\frac{f_2(x)\Delta_1(g(x), x)}{F_2(x)\pi'_1(g(x))} \geq -\frac{f_2(x)F_1(g(x))}{F_2(x)f_1(g(x))},$$

where condition (3), $\Delta_1(g(x), x) \leq F_1(g(x))\pi'_1(g(x))/f_1(g(x))$, was used in the second inequality. \square

¹⁸The notation $x_i = -\infty$ is used to denote that firm i always enters the market; i.e., plays the entry cutoff $-\infty$. Thus, $-\infty_{-i}$ denotes when every firm but i always enter the market.

Existence: Define the function $h : [s_1, \infty) \rightarrow \mathbb{R}$ by $h(x) = H_2(x, g(x))$. This function is continuous and corresponds to the expected profits of firm two entering in the market when firm 1 best responds to x . Define x_2 to be the value satisfying $h(x_2) = 0$, we prove that x_2 exists and that is an *herculean* equilibrium. The next two claims prove the result.

Claim 2. $x_2 \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$ (*herculean* cutoffs).

Proof. $g(x)$ is weakly decreasing in x and $g(s_1) = s_1$. Therefore, $x_1 = g(x_2) < x_2$ if and only if $x_2 \in (s_1, \infty)$. \square

Claim 3. $h(s_1) < 0$ and there exists \hat{x} such that $h(\hat{x}) > 0$.

Proof. Because firm two is weak, Lemma 3 and the definition of *strength* implies $h(s_1) = H_2(s_1, s_1) < H_2(s_2, s_2) = 0$. For the second part of the claim start by observing that $H_2(x, y)$ is increasing in y by Lemma 2. Then $H_2(x, g(x)) \geq H_2(x, -\infty)$ for all x . By A3, there exists \hat{x} such that $H_2(\hat{x}, -\infty) > 0$ and the result follows. \square

Claim 3 and the Intermediate Value Theorem imply that exists $x_2 > s_1$ such that $h(x_2) = 0$. Claim 2 implies that $(g(x_2), x_2)$ constitute an *herculean* equilibrium.

Uniqueness: The uniqueness proof is divided in two claims. Condition (3) is used in each of them.

Claim 4. There exists a unique *herculean* equilibrium.

Proof. To prove uniqueness within the *herculean* class, it is shown that $h'(x) > 0$ so that $h(x)$ single crosses zero from below. The derivative of $h(x)$ is:

$$h'(x) = \pi_2'(x)F_1(g(x)) + \int_{g(x)}^{\infty} \pi_2'(x, y) dF_2(y) + g'(x)f_1(g(x))\Delta_2(x, g(x)).$$

The first two terms of $h'(x)$ are positive. The term containing $g'(x)$ is non-positive. Replacing the lower bound (13), which only needs to hold (by construction of a best response) for values of $g(x)$ and x greater than \underline{v}_1 and \underline{v}_2 respectively, for $g(x)$ we find

$$h'(x) > \pi_2'(x)F_1(g(x)) + \int_{g(x)}^{\infty} \pi_2'(x, y) dF_1(y) - \frac{f_2(x)\Delta_2(x, g(x))F_1(g(x))}{F_2(x)}$$

Condition (3) implies $f_2(x)\Delta_2(x, g(x)) < F_2(x)\pi_2'(x)$, then:

$$h'(x) > \int_{g(x)}^{\infty} \pi_2'(x, y) dF_1(y) > 0$$

proving uniqueness within the *herculean* class. \square

Claim 5. There is no equilibrium in which the strong firm plays a higher cutoff than the weak firm.

Proof. To prove that the only equilibrium is the *herculean*, suppose we have a non-*herculean* equilibrium—i.e., $x_1 > x_2$ but $s_1 < s_2$. Define $\bar{g}(x)$ to be the function that satisfies $H_2(\bar{g}(x), x) = 0$. $\bar{g}(x)$ corresponds to firm two's best response to the cutoff of firm one when $x_1 = x$. As before, Lemma 7 implies that $\bar{g}(x)$ is well defined. Similarly,

following the steps of Claim 1 , it can be shown: $\bar{g}(s_2) = s_2$, $\bar{g}'(x) < 0$, and, under (3), $\bar{g}'(x)$ is bounded below by

$$-\frac{f_1(x)F_2(g(x))}{F_1(x)f_2(g(x))}. \quad (14)$$

Define the continuous function $\bar{h}(x) = H_1(x, \bar{g}(x))$ which corresponds to firm one's expected profits of entering in the market under valuation x when firm two best responds to x . We show that there is no x such that $x_1 = x > \bar{g}(x) = x_2$ and $\bar{h}(x) = 0$ exists; i.e., no non-*herculean* equilibrium exists. Start by observing that $x > \bar{g}(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 3 we showed the function $\sigma_1(s) = H_1(s, s)$ is strictly increasing in s . Then, by the definition of strength and by firm two being weak,

$$\sigma_1(s_1) = H_1(s_1, s_1) = 0 < \sigma_1(s_2) = H_1(s_2, s_2) = H_1(s_2, \bar{g}(s_2)) = \bar{h}(s_2).$$

Following analogous steps to those in Claim 4 (which requires to use the lower bound (14)) it is possible to show $\bar{h}'(x) > 0$. Then, because $\bar{h}(s_2) > 0$ and $\bar{h}'(x) > 0$, $\bar{h}(x)$ never crosses zero when $x > s_2$ and the result holds. \square ■

Proof of Proposition 2. We start proving statement (i). Define $\mathbf{H} : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2$ where each dimension is defined according to $H_i(x_i, x_j)$. An equilibrium is, thus, defined by $\mathbf{H}(\mathbf{x}) = \mathbf{0}$. We make use of implicit differentiation. Let \mathbf{J} be the Jaccovian of \mathbf{H} . By Lemma 2, the terms in the diagonal of \mathbf{J} are positive and the off-diagonal terms are non-negative. Then,

$$\mathbf{J}^{-1} = \det(\mathbf{J}) \begin{pmatrix} + & \leq 0 \\ \leq 0 & + \end{pmatrix} \quad \text{where} \quad \det(\mathbf{J}) = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \\ \frac{\partial H_1}{\partial x_2} & \frac{\partial H_2}{\partial x_1} \end{pmatrix},$$

\mathbf{J}^{-1} consists of positive terms along the diagonal and non-positive terms off the diagonal. The derivatives in the determinant are computed in Lemma 2. Using condition (3) we can bound $\det(\mathbf{J})$ below and show it is strictly positive.

The comparative static with respect parameter $\kappa \in \{\omega_i, \theta_i\}$ is defined by $\mathbf{x}_\kappa = -(\mathbf{J}^{-1})\mathbf{H}_\kappa$, where subscripts denote derivatives. The next lemma will be used in this proof and in the proof of other propositions.

Lemma 5. *Let $(F_i, \pi_i)_{i=1}^n$ be (respectively) parametrized by $(\omega_i, \theta_i)_{i=1}^n$ where ω_i orders F_i in terms of FOSD and where π_i is uniformly increasing in θ_i . Let $H_i(\mathbf{x})$ be the function defined in (2). Then, for every i and $j \neq i$*

$$(i) \quad \frac{\partial H_i}{\partial \theta_i} > 0; \quad (ii) \quad \frac{\partial H_i}{\partial \theta_j} = \frac{\partial H_i}{\partial \omega_i} = 0, \text{ and}; \quad (iii) \quad \frac{\partial H_i}{\partial \omega_j} < 0. \quad (15)$$

Proof. Statement (ii) directly follows from θ_j and ω_i not being in H_i . (i) Follows from π_i being increasing in θ_i . (iii) follows from observing that when $j \notin e$, $F_j(x_j|\omega_j)$ decreases in ω_j . When $j \in e$, then

$$\int_{(x_k)_{k \in I_i(e)}}^{\infty} \pi_i(x_i, v_e) \phi_i(v_{e \setminus j}) f_j(s|\omega_j) d^{m_e-1} v_{e \setminus i}$$

decreases in ω_j by integrating over a decreasing function under FOSD. \square

To complete proof of the first statement, by the lemma above, the signs of the derivatives with respect the parameters are $\mathbf{H}_{\omega_1} = [0, -]$, and $\mathbf{H}_{\theta_1} = [+ , 0]$ (similarly for derivatives with respect parameters of firm two), the result follows from multiplying and checking the sign. For statement (ii) simply note:

$$\begin{aligned}\frac{d\bar{H}_i}{d\theta_i} &= \int_{x_i}^{\infty} \left(\frac{\partial H_i(s, x_j)}{\partial \theta_i} + f_j(x_j) \frac{dx_j}{d\theta_i} \Delta_i(s, x_j) \right) dF_i(s) > 0 \\ \frac{d\bar{H}_i}{d\theta_j} &= \int_{x_i}^{\infty} f_j(x_j) \frac{dx_j}{d\theta_j} \Delta_i(s, x_j) dF_i(s) < 0 \\ \frac{d\bar{H}_i}{d\omega_i} &= \int_{x_i}^{\infty} f_j(x_j) \frac{dx_j}{d\omega_i} \Delta_i(s, x_j) dF_i(s) + \int_{x_i}^{\infty} H_i(s, x_j) \frac{df_i(s)}{d\omega_i} ds > 0 \\ \frac{d\bar{H}_i}{d\omega_j} &= \int_{x_i}^{\infty} \left(\frac{\partial H_i(s, x_j)}{\partial \omega_j} + f_j(x_j) \frac{dx_j}{d\omega_j} \Delta_i(s, x_j) \right) dF_i(s) < 0\end{aligned}$$

where the signs follow from statement (i), Lemma 5 and by integrating over an increasing function under FOSD. \blacksquare

Proof of Proposition 3. For claim (i) suppose without loss $x_1 < x_2$. Subtracting $\bar{H}_2(\mathbf{x})$ to $\bar{H}_1(\mathbf{x})$, see (5), under the assumption that firms are homogeneous we obtain:

$$\int_{x_1}^{x_2} \left(H_1(x, x_2) + \int_{x_2}^{\infty} \Delta(y, x) dF(y) \right) dF(x).$$

In equilibrium $H_1(x_1, x_2) = 0$ and, by Lemma 2, $H_1(x, x_2) > 0$ for $x > x_1$. Also, $\Delta_1(x, y) \geq 0$. Thus, $x_1 < x_2$ implies $\bar{H}_1(\mathbf{x}) > \bar{H}_2(\mathbf{x})$. For claim (ii), suppose that firms are ordered by θ_i with $\theta_1 > \theta_2$ (similar proof applies when firms are ordered by ω_i). Let $x_1 < x_2$ be a herculean equilibria. Consider an alternative entry game, where firms are homogeneous and equal to firm 1 (i.e., $\theta_i = \theta_1$) but $x_1 < x_2$ is played. Although this is not an equilibrium, the same steps as in claim (i) imply $\bar{H}_1(\mathbf{x}, \theta_1) > \bar{H}_2(\mathbf{x}, \theta_1)$. Then, because of the profit order, $\bar{H}_2(\mathbf{x}, \theta_2) < \bar{H}_2(\mathbf{x}, \theta_1)$ and the result follows. \blacksquare

Proof of Proposition 1A. This proof makes use of Lemma 6, presented below.

Lemma 6. *Under condition (7), two homogeneous firms that best respond to each other must play the same cutoff strategy.*

Proof. Consider two homogeneous firms, p and q , and fix *any* profile of cutoffs strategies $(x_j)_{j \neq p, q}$ for the rest of the firms. For ease in notation we drop sub-indexes from π and F when referring to firms p and q . Define $H_{p, q}(x, y) \equiv H_p(x_p = x, x_q = y, \mathbf{x}_{-\{p, q\}})$ where H_p is the function defined in (2). $H_{p, q}(x, y)$ represents p 's expected profit of entering the market under valuation x when q plays the entry cutoff y and all other firms play according to $(x_j)_{j \neq p, q}$. By Lemma 1, the equilibrium condition for firm p holds whenever there exists x_p and x_q such that $H_{p, q}(x_p, x_q) = 0$. Define $g(x)$ to be the value of x_p such that $H_{p, q}(g(x), x) = 0$; i.e., $g(x)$ is p 's best response to $x_q = x$. By Lemma 7, $g(x)$ is well defined; i.e., there is a unique value $g(x)$ for each x . To prove the Lemma we need to prove three claims.

Claim 6. There exists a unique equilibrium such that $x_p = x_q$.

Proof. Start by assuming that homogeneous firms play symmetric cutoffs; i.e., $x_p = x_q = y$. Define $\hat{\sigma}_p(y) = H_{p,q}(y, y)$ and observe that $\hat{\sigma}_p(y) = \hat{\sigma}_q(y)$. By the discussion above, a p, q -symmetric equilibrium exists whenever $\hat{\sigma}_p(y) = 0$. We need to show that there exists a unique value of y such that $\hat{\sigma}_p(y) = 0$. Following analogous steps to those in Lemma 3, it is easy to show $\hat{\sigma}_p(\underline{v}_p) < 0$ and that there exists \hat{y} such that $\hat{\sigma}_p(\hat{y}) > 0$. Using Lemma 2, we can show that $\hat{\sigma}'_p(y) > 0$ so that the value of y is unique. \square

Claim 7. Under condition (7): $0 \geq g'(x) > -f(x)F(g(x))/(F(x)f(g(x)))$.

Proof. To simplify notation we use x_q to denote x and x_p to denote $g(x)$. Implicitly differentiating $H_{p,q}(x_p, x_q) = 0$ and using equations (11) and (12) from Lemma 2, we obtain that $g'(x)$ is equal to:

$$-\frac{f(x_q) \sum_{e \in E_p \setminus E_q} \left\{ \left(\prod_{j \in O_p(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_p(e)}}^{\infty} \Delta_{p,q}(x_p, x_q, v_e) \phi_p(v_e) d^{n_e-1} v_{e \setminus p} \right\}}{F(x_q) \sum_{e \in E_p} \left\{ \left(\prod_{j \in O_p(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_p(e)}}^{\infty} \pi'(x_p, v_e) \phi_p(v_e) d^{n_e-1} v_{e \setminus p} \right\}},$$

which is non-positive as the denominator is positive and numerator is non-negative. To show the lower bound for $g'(x)$ first, make the denominator smaller by taking $\pi'(x_p, v_e) = 0$ for every $e \in E_p \cap E_q$. After this step, both numerator and denominator are sums of market structures in $E_p \setminus E_q$. Then, use condition (7) to substitute for $\Delta_{p,q}(x_p, x_q, v_e)$ in the numerator and obtain the mentioned lower bound. \square

Claim 8. An increase in x_q , which p best responds by playing $g(x_q)$, leads firm q to strictly increase its profits; i.e., $H_{q,p}(x, g(x))$ is increasing in x .

Proof. Differentiating $H_{q,p}(x, g(x))$ with respect to x we obtain

$$\begin{aligned} & \sum_{e \in E_q \setminus E_p} \left\{ \sum_{j \in O_q(e)} \left[\frac{f(g(x))}{F(g(x))} g'(x) \Pr[O_q(e)] \int_{(x_j)_{j \in I_p(e)}}^{\infty} \Delta_{q,p}(x, g(x), v_e) \phi_q(v_e) d^{n_e-1} v_{e \setminus q} \right] \right\} \\ & + \sum_{e \in E_q} \left(\prod_{j \in O_q(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_p(e)}}^{\infty} \pi'(x, v_e) \phi_q(v_e) d^{n_e-1} v_{e \setminus q}. \end{aligned}$$

where $\Pr[O_q(e)] = \prod_{j \in O_q(e)} F_j(x_j)$. Because $g'(x) \leq 0$, the first summation is non-positive. The second summation is positive. Take a lower bound for the first summation using Claim 7 and condition (7) to get

$$\sum_{e \in E_q \setminus E_p} \left\{ \sum_{j \in O_q(e)} \left[\frac{f(x)}{F(x)} \left(\prod_{j \in O_q(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_p(e)}}^{\infty} \pi'(x, v_e) \frac{F(x)}{f(x)} \phi_q(v_e) d^{n_e-1} v_{e \setminus q} \right] \right\}$$

Subtracting to the second summation above we obtain

$$H_{q,p}(x, g(x)) \geq \sum_{e \in E_q \cap E_p} \left(\prod_{j \in O_q(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_p(e)}}^{\infty} \pi'(x, v_e) \phi_q(v_e) d^{n_e-1} v_{e \setminus q} > 0$$

Proving the result. \square

We prove the lemma by contradiction. Recall that $(x_j)_{j \neq p, q}$ is fixed. Suppose there exists $x_p < x_q$ constituting an equilibrium. By Claim 6 there exists a unique value y such that $\hat{\sigma}_j(y) = 0$ with $j \in \{p, q\}$. Suppose first $x_p < y < x_q$. Because

$$\hat{\sigma}_q(y) = H_{q,p}(y, y) = H_{q,p}(y, g(y)) = 0,$$

Claim 8 implies that we must have $H_{q,p}(x_q, g(x_q)) > 0$ as $x_q > y$, which contradicts (x_q, x_p) being an equilibrium. Suppose now $x_p < x_q < y$. Lemma 2 and Claim 6 imply:

$$0 = \hat{\sigma}_p(y) > \hat{\sigma}_p(x_q) = H_{p,q}(x_q, x_q) > H_{p,q}(x_p, x_q)$$

which contradicts (x_q, x_p) being an equilibrium. Similar argument can be constructed for the case $y < x_p < x_q$, proving the Lemma. ■

To prove the proposition observe: (i) By Lemma 3, there exists a unique value of strength and, therefore, a unique symmetric equilibrium, which also corresponds to the unique herculean equilibrium. (ii) If firms are not playing a symmetric equilibrium, then there must exist two homogeneous firms best-responding to each other but playing different cutoffs, contradicting Lemma 6. ■

Proof of Proposition 1B. *Preliminaries:* If $s_1 = s_2$ the *herculean* equilibrium corresponds to the *strength* of the firms. Assume $s_1 < s_2$ and define $H_i(x_i, x_j)$ to be equal to:

$$\sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k] \right] \right\}. \quad (16)$$

The function $H_i(x_i, x_j)$ represents the expected profits of entering the market for a firm in group i when the firm draws x_i and the other firms in group i play the cutoff x_i and the firms in group j play the cutoff x_j . Define $g(x)$ to be the function that solves $H_1(g(x), x) = 0$. Thus, $g(x)$ corresponds to group one's best response to every firm in group two playing the cutoff strategy x . By Lemma 7, the value $g(x)$ exists and is unique; i.e., $g(x)$ is well defined.

Claim 9. $g(s_1) = s_1$, $g'(x) \leq 0$ and, under (9), $g'(x)$ is bounded below by (13).

Proof. By definition of *strength* we know $H_1(s_1, s_1) = 0$, therefore $g(s_1) = s_1$. Using the implicit function theorem

$$g'(x) = - \frac{\partial H_1(g(x), x) / \partial x_2}{\partial H_1(g(x), x) / \partial x_1},$$

which is negative by Lemma 8 in Appendix B. Define

$$\mathbb{E}_i [\mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k]] = \sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k].$$

For the lower bound of $g(x)$ observe that the $\Delta_{1,1}$ terms in the denominator of $g'(x)$ are

non-negative. Taking a lower bound by making them zero delivers

$$\begin{aligned} g'(x) &\geq -\frac{f_2(x) \sum_{k=0}^{m_2-1} \left\{ \binom{m_2}{k} (m_2 - k) F_2(x)^{m_2-k-1} \mathbb{E}_1 [\mathbb{E}[\Delta_{1,2}(g(x), x) | \mathbf{x}, r, k]] \right\}}{\sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} F_2(x)^{m_2-k} \mathbb{E}_1 [\mathbb{E}[\pi'_1(g(x)) | \mathbf{x}, r, k]] \right\}} \\ &= -\frac{f_2(x) \sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} (m_2 - k) F_2(x)^{m_2-k} \mathbb{E}_1 [\mathbb{E}[\Delta_{1,2}(g(x), x) | \mathbf{x}, r, k]] \right\}}{F_2(x) \sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} F_2(x)^{m_2-k} \mathbb{E}_1 [\mathbb{E}[\pi'_1(g(x)) | \mathbf{x}, r, k]] \right\}}, \end{aligned}$$

where in the last expression we multiplied and divided by $F_2(x)$. Using equation (18) (see Lemma 9 in Appendix B) in the denominator, we obtain the lower bound (13). \square

Existence: Define the function $h : [s_1, \infty) \rightarrow \mathbb{R}$ by $h(x) = H_2(x, g(x))$. This function is continuous and corresponds to the expected profits of entering the market for a firm in group 2 when it draws the valuation x , every other firm in group two plays the cutoff x , and every firm in group one play their best response to x ; i.e., $g(x)$. Define x_2 to be the value satisfying $h(x_2) = 0$. Because the statements and the proofs of Claims 2 and 3 apply directly (see the proof of Proposition 1), it follows that there exist $x_2 > s_1$ such that $h(x_2) = 0$. Therefore, the pair $(g(x_2), x_2)$ constitute a *herculean* equilibrium of the game.

Uniqueness. Start by observing that if condition (8) and (9) hold, then condition (7) holds when applied to firms in the same group. By the Lemma 6, when condition (7) holds, homogeneous firms will always play (in equilibrium) the same entry cutoff. Therefore, is without loss to restrict the analysis to within-group symmetric strategies. To prove uniqueness, then, we need to show that no other *herculean* equilibrium exists and that we can not have an equilibrium where $x_2 < x_1$.

Claim 10. There exists a unique *herculean* equilibrium.

Proof. To prove uniqueness within the *herculean* class, we shown $h'(x) > 0$ so that $h(x)$ single crosses zero from below. Differentiating $h(x)$ we obtain:

$$\begin{aligned} &\sum_{k=0}^{m_1} \left\{ \binom{m_1}{k} F_1(g(x))^{m_1-1} \left(\mathbb{E}_2 [\mathbb{E}[\pi'(x) | \mathbf{x}, r, k]] + \right. \right. \\ &\quad \left. \left. \sum_{r=0}^{m_2-2} \binom{m_2-1}{r} (m_2 - r - 1) F_2(x)^{m_2-2-r} \mathbb{E}[\Delta_{2,2}(x_i, x_i) | \mathbf{x}, r, k] \right) \right\} + \\ &f_1(g(x)) g'(x) \sum_{k=0}^{m_1-1} \left\{ \binom{m_1}{k} (m_1 - k) F_1(g(x))^{m_1-k-1} \mathbb{E}_2 [\mathbb{E}[\Delta_{2,1}(x, g(x)) | \mathbf{x}, r, k]] \right\} \end{aligned}$$

The first two terms of $h'(x)$ are positive, and only the third term (containing $g'(x)$) is negative. We work on bounding below the third term. Using the lower bound (13) for $g(x)$ and equation (18) in Lemma 9 we obtain the following lower bound for the third term:

$$-\sum_{k=0}^{m_1-1} \left\{ \binom{m_1}{k} F_1(g(x))^{m_1-k} \mathbb{E}_2 [\mathbb{E}[\pi'_2(x) | \mathbf{x}, r, k]] \right\}$$

Substituting back in the expression for $h'(x)$ we obtain:

$$h'(x) \geq \mathbb{E}_2 [\mathbb{E}[\pi'(x)|\mathbf{x}, r, m_1]] + \sum_{k=0}^{m_1} \left\{ \binom{m_1}{k} F_1(g(x))^{m_1-1} \times \sum_{r=0}^{m_2-2} \binom{m_2-1}{r} (m_2-r-1) F_2(x)^{m_2-2-r} \mathbb{E}[\Delta_{2,2}(x_i, x_i)|\mathbf{x}, r, k] \right\} > 0$$

which is positive, proving uniqueness within the *herculean* class. \square

Claim 11. There is no equilibrium in which the strong firm plays a higher cutoff than the weak firm.

Proof. To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium—i.e., $x_1 > x_2$ but $s_1 < s_2$. Define $\bar{g}(x)$ to be the function that satisfies $H_2(\bar{g}(x), x) = 0$ were H_2 is defined by (16). $\bar{g}(x)$ corresponds to group two's best response to the cutoff of group one when $x_1 = x$. As before, Lemma 7 implies that $\bar{g}(x)$ is well defined. Similarly, following the steps of Claim 9, it can be shown: $\bar{g}(s_2) = s_2$, $\bar{g}'(x) < 0$, and, under (9), $\bar{g}'(x)$ is bounded below by

$$-\frac{f_1(x)F_2(g(x))}{F_1(x)f_2(g(x))}. \quad (17)$$

Define the continuous function $\bar{h}(x) = H_1(x, \bar{g}(x))$ which corresponds to firm one's expected profits of entering in the market under valuation x when firm two best responds to x . We show that there is no x such that $x_1 = x > \bar{g}(x) = x_2$ and $\bar{h}(x) = 0$ exists; i.e., no non-herculean equilibrium exists. Start by observing that $x > \bar{g}(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 3 we showed the function $\sigma_1(s) = H_1(s, s)$ is strictly increasing in s . Then, by the definition of strength and by firm two being weak ($s_1 < s_2$),

$$\sigma_1(s_1) = H_1(s_1, s_1) = 0 < \sigma_1(s_2) = H_1(s_2, s_2) = H_1(s_2, \bar{g}(s_2)) = \bar{h}(s_2).$$

Following analogous steps to those in Claim 10 (which requires to use the lower bound (17)) it is possible to show $\bar{h}'(x) > 0$. Then, because $\bar{h}(s_2) > 0$ and $\bar{h}'(x) > 0$, $\bar{h}(x)$ never crosses zero when $x > s_2$ and the result holds. \square \blacksquare

Proof of Lemma 4. Suppose that firms compete in prices (analogous proof apply in Cournot-type of games). Variable profits are given by $\tilde{\pi}_i(v_i) = (p_i(v_i) - c_i(v_i))q_i$ where q_i is a function of price. Differentiating variable profits with respect to v_i

$$\tilde{\pi}'_i(v_i) = \left((p_i^* - c_i) \frac{dq_i}{dp_i} + q_i \right) \frac{dp_i}{dc_i} c'_i(v_i) - q_i c'_i(v_i)$$

By the Envelop Theorem, the parenthesis above is zero, and $\tilde{\pi}'_i(v_i) = -q_i c'_i(v_i)$. Taking the ratio $\tilde{\pi}'_i(v_i)/\tilde{\pi}_i(v_i)$ and rearranging, we obtain (10). \blacksquare

Proof of Proposition 4. Let $p_i(q_1, \dots, q_{n_e})$ be firm i 's price under market structure

e. Differentiating $\tilde{\pi}_i(v_e)$ with respect v_i and using the envelope theorem

$$\tilde{\pi}'_i(v_e) = -c'_i(v_i) q_i \left(1 - \sum_{j \in e-i} \underbrace{\frac{dp}{dq_j}}_{(-)} \underbrace{\frac{dq_j}{dq_i}}_{(-)} \underbrace{\frac{dq_i}{dc_i}}_{(-)} \right) = -c'_i(v_i) q_i (1 + K) > 0$$

where K is a positive constant summarizing the effect of a change in i 's marginal cost on the opponents decisions. Under standard assumptions $dp/dq_j < 0$ by downward sloping demand, $dq_j/dq_i < 0$ by strategic substitutes, and $dq_i/dc_i < 0$ by implicitly differentiating the firms first order condition. Then, it is not hard to check

$$\frac{\tilde{\pi}'_i(v_e)}{\tilde{\pi}_i(v_e)} = (1 + K) \frac{-c'_i(v_i)}{c_i(v_i)} \frac{1}{\mu(v_e)} > \frac{-c'_i(v_i)}{c_i(v_i)} \frac{1}{\mu(v_e)} \geq \frac{-c'_i(v_i)}{c_i(v_i)} \frac{1}{\mu(v_i)} = \frac{\tilde{\pi}'_i(v_i)}{\tilde{\pi}_i(v_i)}$$

where in the last step $\mu_i(v_i) \geq \mu_i(v_e)$, markup decreasing with competition, was used. ■

B Auxiliary Results

Lemma 7. *Let H_i be defined by (2). For any profile \mathbf{x}_{-i} of cutoff strategies by the competitors, there exist a unique value \hat{x} such that $H_i(\hat{x}, \mathbf{x}_{-i}) = 0$.*

Proof. We start by showing existence of \hat{x} using the Intermediate Value Theorem. Fix \mathbf{x}_{-i} and observe that A3 jointly with A2 implies $H_i(v_i, \mathbf{x}_{-i}) \leq \pi_i(v_i) < 0$. On the other hand, Lemma 2 implies $H_i(\hat{x}, \mathbf{x}_{-i}) \geq H_i(\hat{x}, -\infty_{-i})$. By A3, there exist \bar{v}_i such that $H_i(\bar{v}_i, -\infty_{-i}) > 0$ and, therefore, $H_i(\bar{v}_i, \mathbf{x}_{-i}) > 0$. Then, by the Intermediate Value Theorem there exist \hat{x} such that $H_i(\hat{x}, \mathbf{x}_{-i}) = 0$. For uniqueness, by Lemma 2 $\partial H_i / \partial x_i > 0$. Therefore $H_i(x_i, \mathbf{x}_{-i})$ single crosses zero. ■

Lemma 8. *Let $H_i(x_i, x_j)$ be defined by equation (16). The partial derivatives of H_i with respect to x_i and x_j are positive and equal to:*

$$\begin{aligned} \frac{\partial H_i}{\partial x_j} &= \sum_{k=0}^{m_j-1} \left\{ \binom{m_j}{k} (m_j - k) F_j(x_j)^{m_j-k-1} f_j(x_j) \mathbb{E}_i [\mathbb{E}[\Delta_{i,j}(x_i, x_j) | \mathbf{x}, r, k]] \right\} \\ \frac{\partial H_i}{\partial x_i} &= \sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\mathbb{E}_i [\mathbb{E}[\pi'_i(x_i) | \mathbf{x}, r, k]] \right. \right. \\ &\quad \left. \left. + f_i(x_i) \sum_{r=0}^{m_i-2} \binom{m_i-1}{r} (m_i-1-r) F_i(x_i)^{m_i-2-r} \mathbb{E}[\Delta_{i,i}(x_i, x_i) | \mathbf{x}, r, k] \right] \right\}. \end{aligned}$$

Proof. We prove the result for the derivative of H_i with respect to x_j . The proof for the derivative of H_i with respect to x_i follows the similar steps. Using Leibnitz differentiation:

$$\frac{\partial H_i}{\partial x_j} = \sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} (m_j - k) F_j(x_j)^{m_j-k-1} f_j(x_j) \mathbb{E}_i [\mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k]] \right\}$$

$$- \sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} (k) F_j(x_j)^{m_j-k} f_j(x_j) \mathbb{E}_i [\mathbb{E}[\pi_i(x_i, x_j) | \mathbf{x}, r, k-1]] \right\}$$

Observe that the first summation becomes zero when $k = m_j$, and that the second summation becomes zero when $k = 0$. Changing the second summation's range to go from 0 to $m_j - 1$ (instead of going from 1 to m_j) we obtain:

$$\begin{aligned} \frac{\partial H_i}{\partial x_j} &= \sum_{k=0}^{m_j-1} \left\{ \binom{m_j}{k} (m_j - k) F_j(x_j)^{m_j-k-1} f_j(x_j) \mathbb{E}_i [\mathbb{E}[\pi_i(x_i) | \mathbf{x}, r, k]] \right\} \\ &\quad - \sum_{k=0}^{m_j-1} \left\{ \binom{m_j}{k+1} (k+1) F_j(x_j)^{m_j-k-1} f_j(x_j) \mathbb{E}_i [\mathbb{E}[\pi_i(x_i, x_j) | \mathbf{x}, r, k]] \right\}. \end{aligned}$$

The result follows from using the identity $\binom{m_j}{k+1} (k+1) = \binom{m_j}{k} (m_j - k)$ and subtracting both expressions.¹⁹ ■

Lemma 9. *In a two-group entry model, suppose firms play according to $\mathbf{x} = (x_1, x_2)$ where x_i is the cutoff strategy of a firm in group i . Then, for any market structure $e \in E_i$ and $j \notin e$, condition (9) implies:*

$$(m_j - k) f_i(x_i) \mathbb{E}[\Delta_{i,j}(x_i, x_j) | \mathbf{x}, r, k] < F_i(x_i) \mathbb{E}[\pi'_i(x_i) | \mathbf{x}, r, k]. \quad (18)$$

Proof. Fix market structure e and pick $j \notin e$. Let r be the number of firm in group i firms entering in e minus one, and let k be the number of j firms entering in e . Rewrite condition (9) as:

$$f_i(x) \Delta_{i,j}(x, y, v_e) (m_j - k) < F_i(x) \pi'_i(x, v_e)$$

Because condition holds for every x, y and v_e , we could integrate on both sides with respect to any measure and the inequality would be preserved. In particular, because firms play according to \mathbf{x} , for a given market structure e we have that:

$$\int_{\{x_j\}_{j \in I_i(e)}}^{\infty} \Delta_{i,j}(x_i, x_j, v_e) \phi_i(v_e) d^{n_e-1} v_e = \mathbb{E}[\Delta_{i,j}(x_i, x_j) | \mathbf{x}, r, k],$$

where r and k are the number of firms (other than i) in group g_i and $3 - g_i$, respectively, participating in e . Repeating steps with respect to the integral of $\pi'_i(x, v_e)$ we obtain the right hand side of (18) and the result follows. ■

¹⁹For the combinatorial identity simply observe that $\binom{m}{k+1} (k+1)$ is equal to:

$$\frac{m!}{(k+1)!(m-k-1)!} (k+1) \frac{m-k}{m-k} = \frac{m!}{k!(m-k)!} (m-k) = \binom{m}{k} (m-k).$$