Equilibrium Uniqueness in Entry Games with Private Information*

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Abstract

We study conditions under which a static entry game with privately-informed firms has a unique equilibrium. Our framework embeds most models commonly used in applied work. It allows rich forms of firm heterogeneity and selective entry. We introduce the notion of strength, which summarizes a firm's ability to endure competition. In environments of applied interest, an equilibrium in which entry strategies are ordered according to strength, called herculean equilibrium, always exists. We derive simple sufficient conditions guaranteeing equilibrium uniqueness and, consequently, robust counterfactual analyses.

JEL: D21, D43, L11, L13

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1 Introduction

Understanding firms' market entry decisions is a key element of economic policy and regulation. Predicting whether there will be timely entry after a merger or regulatory change requires a framework that determines the number and types of competitors. More broadly, a model with endogenous entry, prices, product characteristics, and welfare outcomes can be used to evaluate policies prospectively. When performing such analysis, researchers use the counterfactual equilibrium of an estimated model to assess the impact of the policy under consideration. A common challenge in this setting is the existence of multiple equilibria. Under multiplicity, the model may not yield a unique prediction to the applied question, difficulting policy analysis (Berry and Tamer, 2006; Borkovsky *et al.*, 2015). Computing these multiple equilibria may also prove challenging when using numerical methods, which may limit the researcher's ability to gain a complete understanding of the impacts of a policy of interest (Iskhakov *et al.*, 2016).

We study equilibrium uniqueness in static entry games with private information. Our framework allows for rich forms of firm *heterogeneity* and *selective* entry. Our main contribution is to provide a sufficient condition that guarantees equilibrium uniqueness. The condition is solely based on the model's fundamentals, and verifying it does not require equilibrium computation. In many common applications, we can check the sufficient condition by performing a simple calculation. For example, Roberts and Sweeting (2013) and Grieco (2014) use numerical methods to show that their fitted models have a unique equilibrium. Using their estimates and our sufficient condition, we can confirm equilibrium uniqueness in their fitted models, highlighting the usefulness of our results. Thus, our findings provide new tools for applied researchers studying entry.

We characterize firms' equilibrium behavior using a simple index, called *strength*, summarizing a firm's ability to endure competition. The strength of a firm is the unique *symmetric*-threshold strategy that makes the firm indifferent to enter the market. A stronger firm is more willing to enter the market than a weaker competitor, despite facing more competition. For the class of models studied, we show that there always exists an equilibrium in which entry strategies are ordered according to strength. We call this a *herculean* equilibrium. Identifying the herculean equilibrium is the starting point for our equilibrium uniqueness result. We show that, when our sufficient condition for uniqueness holds, only one herculean equilibrium exists, and no non-herculean equilibrium is possible. Our existence result also suggests that strength and herculean equilibrium might be used as an equilibrium selection criterion when multiplicity exists. To support this idea, we argue that the herculean equilibrium is an incomplete information analog to the risk-dominant equilibrium in complete information games (Harsanyi and Selten, 1988). In particular, we show via a Harsanyi purification argument that, as the sensitivity of payoffs to private information vanishes, the herculean equilibrium converges to the risk-dominant equilibrium.

Our proposed framework embeds static entry models commonly used in applied work. It accommodates a large variety of post-entry models, including auctions and competitions in price or quantity. The framework also allows for rich forms of firm heterogeneity. Firms are allowed to differ in their payoff functions or their distribution of types, capturing that firms might be heterogeneous in their public characteristics (e.g., firms might vary in their product characteristics, geographic locations, or levels of vertical integration). Payoffs might depend on the entry decisions and realized types of competitors, allowing a level of strategic interaction often ignored by the entry literature (auctions being an exception). For example, if firms are privately informed about their marginal costs of production, facing a potential competitor with a lower marginal cost decreases a firm's expected profit. The magnitude of this decrease depends on the firms' realized marginal costs, their degree of product substitutability, and the number of entrants. We enrich the set of models available to applied researchers by including these environments.

In the theoretical literature on market entry, Mankiw and Whinston (1986) study welfare in a symmetric model under complete information. Brock and Durlauf (2001) examine a symmetric coordination game with privately-informed agents. Our modeling shares the idea that both the action and type of an agent affect the payoffs of other agents but differs in that entry decisions are strategic *substitutes* and in that we allow for asymmetric agents. Our article generalizes the existing literature on costly entry into second-price auctions. Samuelson (1985) studies ex-ante symmetric bidders. Tan and Yilankaya (2006) study two groups of asymmetric bidders ordered by first-order stochastic dominance, whereas in Cao and Tian (2013) the two groups are ordered by entry costs. In Ye (2007), bidders are partially informed at the moment of entry and fully learn their valuations after entry occurs. Our framework allows for more general forms of bidder heterogeneity and, at the same time, embeds both informational environments. A firm's private information might correspond to its type or a signal about its type.

In the empirical literature, Bresnahan and Reiss (1990, 1991) and Berry (1992) develop the first empirical models of market entry that explicitly account for the strategic interaction between post-entry market competition and firms' entry decisions. Under complete information, the entry game often contains multiple equilibria. Tamer (2003) shows that, without further assumptions, multiple equilibria can lead to set, rather than point, identification.[1] Using numerical methods, Seim (2006) shows that firms having private information may solve the problem of equilibrium multiplicity. Berry and Tamer (2006), however, construct examples of multiple equilibria under private information, raising the question of when uniqueness can be achieved. Marcoux (2020) provides a statistical test for whether firms play the same equilibrium across a sample of entry decisions. We contribute to this discussion by identifying a testable condition guaranteeing equilibrium uniqueness and a selection criterion for when multiplicity exists.

The importance of allowing for private information in entry models lies beyond the possibility of solving the multiple equilibria problem. Using complementary methodologies, Grieco (2014) and Magnolfi and Roncoroni (2021) test and reject the hypothesis that firms possess complete information at the moment of entry. Furthermore, compared to models that allow for private information, they show that assuming complete information delivers model estimates that can lead to qualitatively different predictions. Roberts and Sweeting (2013, 2016) provide evidence of selection at the moment of entry, which cannot be accounted for by complete information models.

The article is organized as follows. For illustrative purposes, Section 2 presents our results in the context of a second-price auction. There, we introduce and discuss the notions of strength and herculean equilibrium, developing key intuitions. Section 3 introduces the general model, discusses its scope and limitations, and provides common examples used in the applied literature. Main results are presented in Section 4, which shows that the existence of a herculean equilibrium is guaranteed and provides a sufficient condition for when the herculean equilibrium is the unique equilibrium of the game. Section 5 relates the notions of strength and herculean equilibrium with risk dominance and extends the model to allow for gradual revelation of information. Finally, Section 6 concludes. All the proofs are relegated to Appendix A.

[1] Sweeting (2009) shows that multiplicity can help with the model's identification in the context of coordination games. De Paula and Tang (2012) show that multiplicity can be used to infer the signs of strategic interactions.

2 An Illustrative Example

We begin by studying an asymmetric extension of Samuelson (1985) costly entry into a second-price auction with independent private values, to provide better intuitions for our results. The definitions and results presented here, are extended to a general class of entry games in Sections 3 and 4.

2.1 Second-Price Auction with Entry Costs

Set up Consider an auction consisting of one seller, two potential bidders, and one indivisible good. Before making any entry decision, each bidder $i \in \{1, 2\}$ observes her valuation for the object v_i which is drawn from an atomless distribution function F_i with full support on $[0, \infty)$.[2] We assume that each F_i is continuously differentiable and has a finite expectation.

Upon privately observing their own valuation, each bidder, independently and simultaneously, decides whether to enter the auction. If bidder i decides to enter, she incurs in a entry cost $c_i > 0$. The tuple $(F_i, c_i)_{i=1}^2$ is commonly known by all the bidders. Observe that bidders may be asymmetric in both their distribution of valuations and entry costs. After bidders make their entry decisions, a participating bidder bids its valuation (i.e., chooses its weakly dominant strategy).

Strategies and Equilibrium An entry strategy for bidder i is called *cutoff* if there is a threshold x_i such that bidder i enters the auction whenever its valuation is higher than x_i ($v_i \geq x_i$) and stays out otherwise. Online Appendix C shows that focusing on cutoff strategies is without loss of generality and that a Bayesian equilibrium, described below, always exists.

Given the opponent's entry cutoff x_j, bidder i's expected profit of entering the auction with a valuation v_i is equal to

$$\Pi_i(v_i, x_j) = v_i F_j(\max\{v_i, x_j\}) - \int_{x_j}^{\max\{v_i, x_j\}} y dF_j(y) - c_i. \tag{1}$$

The first term shows that bidder i wins the object with probability $F_j(\max\{v_i, x_j\})$, the second term denotes the expected price paid by bidder i, and the last term denotes its entry cost. Because $\Pi_i(v_i, x_j)$ is strictly increasing in both dimensions, bidder i's best response to x_j (a cutoff) is the unique value $b_i(x_j)$ that solves

[2] Our results still apply if the support of F_i were an interval $[0, b]$ with $b > 0$. The current formulation is chosen to avoid the existence of corner solutions in which a bidder never enters.

$\Pi_i(b_i(x_j), x_j) = 0$. Using implicit differentiation, we can readily verify $b_i'(x_j) < 0$; as bidder j enters less often (higher x_j), bidder i is more willing to enter the auction. A Bayesian *equilibrium* is a pair of cutoff strategies (x_1, x_2) such that every bidder i is indifferent to enter the auction when draws a valuation equal to its equilibrium cutoff. That is, (x_1, x_2) is an equilibrium vector if an only if $\Pi_1(x_1, x_2) = \Pi_2(x_2, x_1) = 0$.

2.2 Strength and Herculean Equilibrium

We now introduce the main two definitions of the article: bidder *strength* and *herculean* equilibrium. Strength uses the game fundamentals, $(F_i, c_i)_{i=1}^2$, to rank bidders according to their ability to endure competition. We use strength to identify the equilibrium that remains when the game has a unique equilibrium: the herculean equilibrium. Identifying the herculean equilibrium is the starting point to develop our sufficient condition for equilibrium uniqueness.

Definition (Strength)**.** The strength of bidder i, is the unique number s_i that solves $\Pi_i(s_i, s_i) = 0$; that is, the unique s_i satisfying:

$$s_i F_j(s_i) = c_i. \tag{2}$$

We say that bidder i is stronger than j if $s_i < s_j$.

Strength is well defined, as it assigns a unique scalar s_i to each bidder i, delivering a complete ranking of the bidders.[3] A lower s_i means a stronger bidder. The strength of bidder i is the unique cutoff s_i that is a best response to the other bidder playing the same cutoff strategy s_i. That is, strength ranks bidders by using the unique symmetric strategy that makes each bidder indifferent to enter the auction. Because bidders can be asymmetric, the symmetric strategy that makes a bidder indifferent might differ across bidders. The importance and usefulness of strength relies on summarizing the multidimensional characteristics of bidders, $(F_i, c_i)_{i=1}^2$, into a single scalar.

In intuitive terms, strength ranks firms according to their ability to endure competition. The strength of bidder i encompasses information about a bidder's willingness to enter the auction, relative to that of its competitor. Start by observing that a lower cutoff strategy for bidder i means that bidder i is more willing to enter the auction, as it enters for lower valuations. A lower entry cutoff by

[3] Strength is well defined as $sF(s)$ is increasing, unbounded, and equal to 0 when $s = 0$.

competitors, on the other hand, implies that bidder i is more likely to face competition, as competitors are entering more often. Thus, bidder i being stronger than j ($s_i < s_j$) indicates that i, despite facing more competition than j, is more willing to enter the auction. The next lemma shows that strength generalizes common notions of relative competitiveness used in the auctions with entry cost literature.

Lemma 1 (Strength and heuristic notions of relative competitiveness)**.**

1. *If bidders have the same entry costs and a bidder's distribution of valuations first order stochastically dominates the other bidder, the dominating bidder is stronger.*
2. *If bidders have identical distributions of valuations but different entry costs, the bidder with the lowest entry cost is stronger.*

The order provided by strength coincides with those provided by common heuristics used to determine the relative competitiveness of bidders, such as first-order stochastic domiance (FOSD) or entry-cost order.[4] Strength, however, extends the order to scenarios in which relative competitiveness is not self-evident. Take, for example, a bidder whose distribution of valuations first-order stochastically dominates the other bidder but has a higher entry cost. This scenario may arise when 'smaller' firms are subsidized to enter into the auction (c.f. Marion, 2007). In this case, the former bidder might be stronger, as it is likely to draw a higher valuation, but it might also be weaker than the latter bidder, who also has a lower entry cost. Strength not only ranks bidders in this (or any other) scenario but also, as is shown below, provides meaningful information about equilibrium behavior.

Definition (Herculean Equilibrium)**.** An equilibrium is called herculean if the equilibrium cutoffs are ordered by strength, with the stronger bidder playing the lower cutoff. That is, $x_i < x_j$ if and only if $s_i < s_j$.

Because stronger bidders are more able to endure competition, they should be more inclined than weaker bidders to enter the auction. In terms of equilibrium behavior, the previous intuition means that stronger bidders should play lower entry cutoffs.

Before presenting our main result in the context of auctions, notice that in a symmetric auction ($F_i = F$ and $c_i = c$ for all i) every bidder is equally strong; thus, in a herculean equilibrium, bidders must play symmetric strategies. Furthermore,

[4] Tan and Yilankaya (2006) calls the order induced by FOSD *intuitive*, whereas Cao and Tian (2013) calls the cost-order *monotone*.

the strength of each bidder coincides with their symmetric equilibrium cutoff; i.e., $x_i = s_i$. Therefore, in symmetric games, the notions of strength, symmetric equilibrium, and herculean equilibrium coincide. Because strength is uniquely defined, this observation trivially implies that symmetric auctions have a unique symmetric equilibrium (of course, symmetric games may still have asymmetric equilibria.)

The appeal of the notions of strength and herculean equilibrium has several layers. First, the herculean equilibrium exists. Second, as shown in Section 5, strength and herculean equilibrium are incomplete information analogs to the notions of risk factor and risk-dominant equilibrium in complete information games. In particular, in the context of a two-players entry game with symmetric CDFs, we show (via a Harsanyi purification argument) that the ranking provided by the notion of strength coincides with the order given by the players' risk factor when private information banishes. Consequently, the herculean equilibrium converges to the risk-dominant equilibrium.

Strength also has advantages over other potential candidates to rank firms, such as expected payoff or probability of entry. In the Online Appendix, we construct examples with *asymmetric* bidders in which the entry game has a unique equilibrium and: (i) bidders are equally strong, so the unique equilibrium consists of a symmetric strategy, but one bidder obtains higher expected payoff than the other, and; examples in which (ii) the stronger bidder, the one playing the lower cutoff in the unique equilibrium, can either be less or more likely to enter the auction than the weaker bidder. These examples highlight that looking and ranking bidders in the strategy space gives more meaningful information and is, thus, preferable to rank bidders by looking to other observable characteristics.

2.3 Herculean Equilibrium: Existence and Uniqueness

We now illustrate our main results in the auction example: A herculean equilibrium exists and, under a weak CDF-concavity condition, it is the unique equilibrium of the game. From now on, unless otherwise noted, we order bidders' identities by their strength, with bidder 1 being the strongest bidder in the game.

Recall that $b_i(x_j)$ is bidder i's unique best response to the cutoff x_j, which is decreasing in x_j. Then, $\overline{v}_i = b_i(c_j)$ is an upper bound to bidder i's set of feasible best responses, as, in equilibrium, no bidder will choose to enter with a valuation below its entry cost. Thus, $\overline{v}_i$ is the highest entry cutoff that bidder i may play in any equilibrium. The following proposition is our main result in the present example.

Proposition 1. *There always exists a herculean equilibrium. Moreover, the entry game has a unique equilibrium if the following condition holds for each bidder*

$$F_i(v_i) \geq v_i f_i(v_i) \quad \textit{for all } v_i \in [c_i, \overline{v}_i]. \tag{3}$$

Proposition 1 has two results. First, it establishes the existence of a herculean equilibrium, confirming the intuition that there exists an equilibrium in which the strong bidder plays a lower entry cutoff. The intuition of why a herculean equilibrium exists is quite simple. Consider a bidder's best response to the opponent's strength relative to its own strength (that is, $b_i(s_j)$ relative to $b_i(s_i)$). Because bidder 1 is stronger, she faces an opponent that enters less often than its own strength. Consequently, she best responds by playing a lower entry cutoff, entering more often. Similarly, bidder 2 faces an opponent that enters more often than its own strength. She best responds by playing a higher entry cutoff, entering less often. These incentives reinforce each other, pulling mutual best responses to be further apart than the firms' strengths, inducing a herculean equilibrium.

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Perhaps more importantly, the proposition provides a sufficient condition on the shape of the distributions of valuations for the game to have a unique equilibrium. This result is significant for applied work as it provides a testable condition that guarantees robust counterfactual analysis. In intuitive terms, (3) is an equilibrium stability condition. It guarantees that bidders do not over-react to a small change in the opponent's cutoff. The lack of over-reaction, in turn, implies that a bidder's expected payoff remains monotonically increasing in its entry cutoff, even after taking into account the opponent's best response. This monotonicity implies that

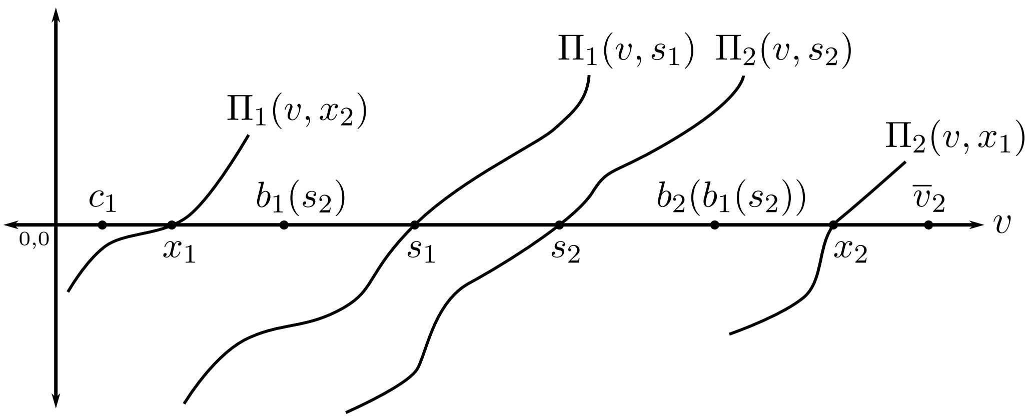


Figure 1: Construction of a herculean equilibrium from iterated best responses. Starting from firm 2's strength, s_2, firm 1's best response is lower than its own strength, s_1. Similarly, firm 2's best response to firm 1's best response is higher than s_2. These iterated best responses are monotonic and bounded, converging to a herculean equilibrium.

only one cutoff makes a bidder indifferent to enter the auction, leading to a unique equilibrium.

To see this argument formally, suppose $x_1 < x_2$. Then, using (1), the best response of bidder 1 is equal to $x_1 = b_1(x_2) = c_1/F_2(x_2)$, which in turn implies

$$-b_1'(x_2) = x_1 f_2(x_2)/F_2(x_2) < x_2 f_2(x_2)/F_2(x_2) \leq 1$$

where the first inequality follows from $x_1 < x_2$ and the last inequality from sufficient condition (3). That is, when bidder 2 increases its cutoff, bidder 1 best responds by decreasing its cutoff by a lower magnitude than bidder 2's change. Similarly, using implicit differentiation and analogous arguments, we can also show $-b_2'(x_1) = x_1 f_1(x_1)/F_1(x_2) < 1$.[5]

From bidder 2's perspective, this means that an increase in its own cutoff raises its expected payoff even after taking into account bidder 1's best response. This monotonicity in payoffs is regardless of whether bidder 2 is playing the highest or lowest entry cutoff. Because payoffs are continuous, negative when $x_2 = c_2$, and unbounded above, the monotonicity means that there is a unique x_2 (to which bidder 1 is best responding) that satisfies the equilibrium condition $\Pi_2(x_2, b_1(x_2)) = 0$.

For completeness, we show the monotonicity of $\Pi_2(x_2, b_1(x_2))$. Differentiating

[5] Sufficient condition (3), then, guarantees that every equilibrium satisfies the stability condition $b_1'(x_2)b_2'(x_1) < 1$ (see Fudenberg and Tirole, 1991, p. 24). This condition implies equilibrium uniqueness, as it is the only way to achieve stability of every equilibrium. Instead, we prove uniqueness directly by following the payoff monotonicity method, as we can scale the method to a larger set of players.

with respect to x_2 we obtain:

$$\frac{d\Pi_2(x_2, b_1(x_2))}{dx_2} = F_1(x_2) + b_1'(x_2)x_1 f_1(x_1) = F_1(x_2)\left(1 - b_1'(x_2)b_2'(x_1)\right) > 0$$

where the inequality follows from noting that condition (3) implies $-b_i'(x_j) \in (0,1)$. The following lemma helps us to further characterize the sufficient condition for equilibrium uniqueness (3).

Lemma 2. *1) If (F_1, F_2) are concave, then (3) is satisfied and the equilibrium is unique.*[6] *2) If the distributions (F_1, F_2) become concave for high valuations, there exists a pair of entry costs (c_1^*, c_2^*) such that, whenever $c_i \geq c_i^*$ for both bidders, the game has a unique equilibrium.*[7]

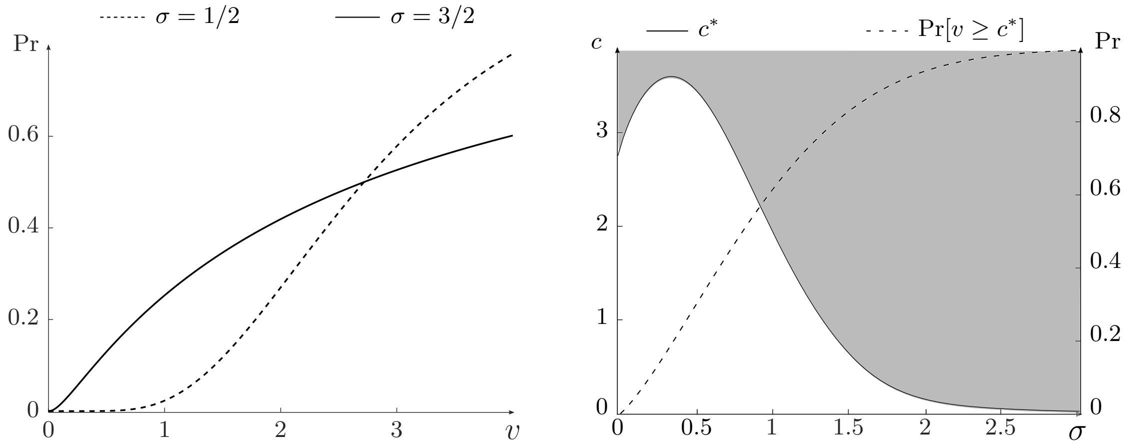
Lemma 2.1 shows that condition (3) is a weak form of CDF-concavity. In particular, auctions with concave distributions of valuations (e.g., exponential or generalized Pareto) always have a unique equilibrium. Many other distributions, such as beta, gamma, or Weibull, are concave for certain parameter specifications. Many distributions used in applications (such as the log-normal distribution) are concave for sufficiently high valuations. Lemma 2.2 shows that for these eventually-concave distributions, there exist sufficiently high entry costs guaranteeing equilibrium uniqueness. This result stands in contrast with traditional complete information intuitions, where entry by both firms leads to negative profits when entry costs are high, inducing equilibrium multiplicity. In contrast, with private types, high entry costs shift the domain of feasible strategies $[c_i, v_i]$, potentially inducing equilibrium uniqueness.

Example 1 (Log-normal valuations)**.** To illustrate the intuition behind strength, the herculean equilibrium, and sufficient condition for uniqueness (3), consider a scenario consisting of two potential bidders, with an identical entry cost c, and valuations that distribute log-normal with parameters (μ_i, σ). As illustrated by Figure 2a, this distribution family is not concave. Depending on its parameters, the entry game might have multiple or a unique equilibrium.

(a) **Uniqueness under sufficiently high entry costs:** Suppose first that bidders are symmetric, with $\mu_i = 1$. Because the log-normal distribution becomes concave for high values, by Lemma 2.2, for each value of σ we can find a threshold

[6] Tan and Yilankaya (2006) show that concave CDFs jointly with FOSD leads to a unique equilibrium when bidders have symmetric entry costs. Cao and Tian (2013) show that concavity leads to uniqueness if bidders only differ in their entry costs. Lemma 2.1 generalizes both results.

[7] The proof of the lemma shows how to find c_i^*.

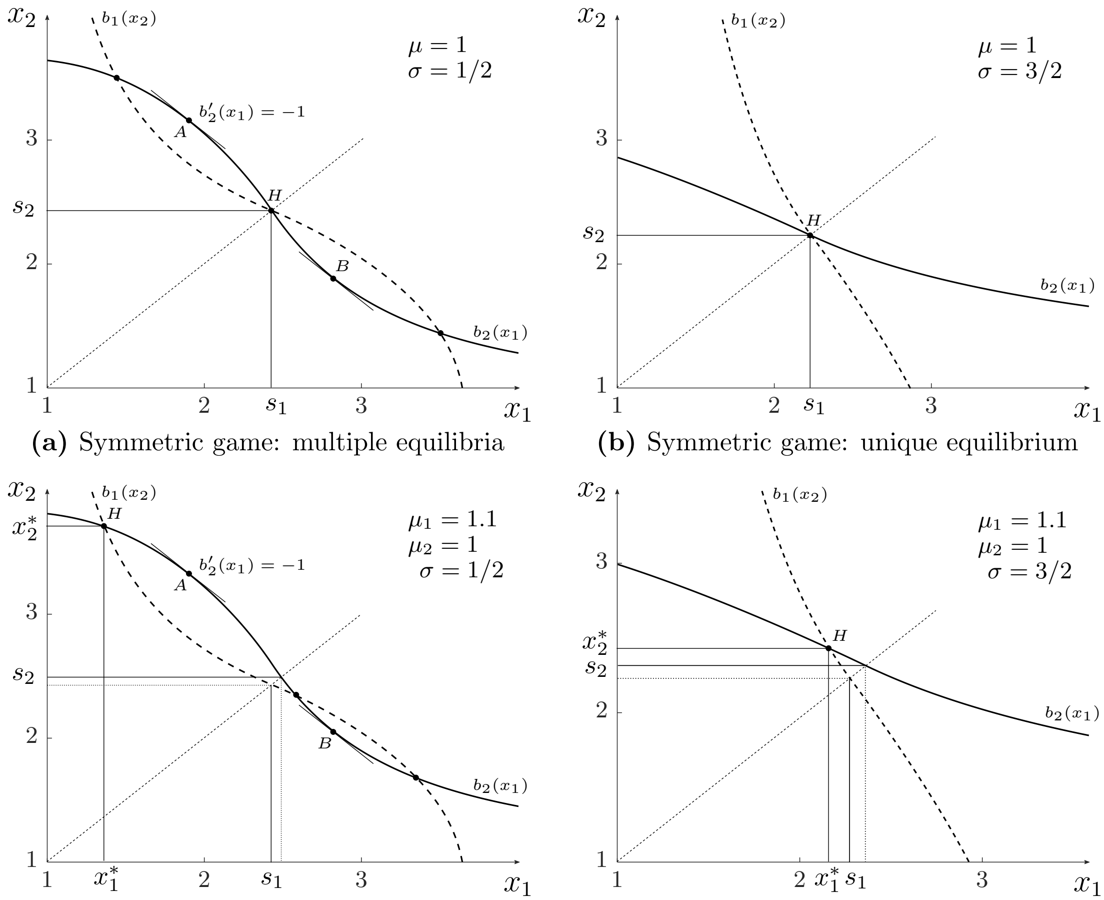


(a) Log-normal CDFs ($\mu = 1$) **(b)** c^* as a function of σ

Figure 2: Sufficiency with log-normal valuations. Panel (a) shows that log-normal CDFs are not concave. Panel (b) depicts the minimal entry cost c^* under which uniqueness condition (3) is guaranteed to hold, as a function of σ. The shaded area represents the set of entry costs under which the entry game has a unique equilibrium. $\Pr[v \geq c^*]$ represents the proportion of valuations above c^*.

c^* such that for every $c \geq c^*$ the sufficient condition for equilibrium uniqueness (3) holds. Figure 2b depicts the threshold c^* and the mass of valuations above c^*, as a function of σ. The shaded area represents the set of entry costs c under which there is a unique equilibrium. The relation between c^* and σ is non-monotonic, but c^* converges to zero when σ is high enough. The proportion of valuations above the entry cost, $\Pr[v \geq c^*]$, monotonically increases in σ. That is, the larger the dispersion of the distribution, the less demanding the condition for uniqueness becomes. When $\sigma \to 0$, the mass of valuations above c^* converges to zero. That is, as the game converges to a complete information game—where equilibrium multiplicity is known to exist —the sufficient condition for uniqueness is never met.

(b) **Multiplicity and Uniqueness:** We now illustrate the differences between multiple equilibria versus a unique equilibrium. Assume symmetric bidders, with $c = 1$ and $\mu_i = 1$. In Figures 3a and 3b we study bidders' best response functions and equilibria under two values of σ. Figure 3a shows that when $\sigma = 1/2$, the auction has three equilibria; i.e., the two best responses cross at three different points. The segment between the points A and B highlights bidder 2's violation of the sufficient condition for uniqueness (3): $-b_2'(x_1) > 1$. Because bidders are symmetric, the herculean equilibrium, denoted by H, is symmetric and equal to the bidders' strength. Incidentally, as in a risk-dominant equilibrium, the



(a) Symmetric game: multiple equilibria **(b)** Symmetric game: unique equilibrium

(c) Asymmetric game: multiple equilibria **(d)** Asymmetric game: unique equilibrium

Figure 3: Strength and herculean equilibrium under log-normal valuations. The figure depicts bidders' best response function $b_i(x_j)$, their strength s_i, and the herculean equilibrium H, when valuations distribute log-normal in four different scenarios. Panels (a) and (b) depict symmetric auctions, whereas (c) and (d) asymmetric. Scenario (a) and (c) have multiple equilibria. Sufficient condition for equilibrium uniqueness (3) does not hold between points A and B. In scenario (b) and (d), the condition (3) does hold and the game has a unique (the herculean) equilibrium.

herculean equilibrium is not stable when the game is symmetric. The slope of bidder 2's best response is lower than that of bidder 1.

In contrast, when $\sigma = 3/2$ (Figure 3b), sufficient condition (3) does hold. Best responses are flatter and the game has a unique (the herculean) equilibrium. In this case, the unique equilibrium is stable, as the slope of bidder 2 best response is greater than that of bidder 1.

(c) **Asymmetric auctions:** We now illustrate the workings of strength and the herculean equilibrium in an asymmetric context. We repeat the previous analysis but now allow bidders to differ in μ. In particular, bidder 1 is stronger, as it has higher expected valuations ($\mu_1 = 1.1 > 1 = \mu_2$). Figures 3c and 3d depict the bidders' best response functions and the strength of each bidder. Strength is computed where a bidder's best response crosses the 45° line; i.e., when $b_i(s_i) = s_i$. Because bidder 1 is stronger, herculean equilibria lie above the 45° line. Figure 3c shows that when $\sigma = 1/2$, only one equilibrium is herculean, and the other two are non-herculean. The middle equilibrium is non-herculean and unstable. The other two equilibria, one of which is herculean, are stable. Similar to a risk-dominant equilibrium in a complete information game, the herculean equilibrium has a larger basin of attraction than the stable non-herculean equilibrium. Figure 3d shows that as σ increases, best responses flatten out, the sufficient condition for uniqueness holds, and only the herculean equilibrium survives.

It is interesting to observe what happens when μ_1 increases. Comparing Figure 3a with Figure 3c, we can see that increasing the mean of bidder 1's distribution shifts bidder 2's best response upwards (same shift can be observed comparing Figures 3b and 3d). This shift implies that the non-herculean equilibrium get closer to each other. When μ_1 is sufficiently high, the upward shift of bidder 2's best response leads best responses to no longer cross at the right side of the 45° line, inducing a unique equilibrium. Sufficient condition (3) fails to capture this mechanism for equilibrium uniqueness. Condition (3) is about the *shape* of best responses, whereas means affect their *scale*.[8]

2.4 More than Two Potential Bidders

We now discuss how to extend the previous results in the context of an auction with n potential bidders. The game is characterized by the tuple $(F_i, c_i)_{i=1}^n$, where all the bidders commonly know distributions, entry costs, and the number of potential entrants. We first discuss the notion of strength and how it changes with the number of competitors. Then, we relate the robustness of strength to the existence of herculean equilibrium in environments with $n > 2$ bidders.

[8] Because i's best response is monotonically increasing in μ_j, and because an equilibrium always exists, it can be shown that there exists $\tilde{\mu}_j$ sufficiently high such that $\mu_j > \tilde{\mu}_j$ implies a unique equilibrium.

Strength In the context of n potential bidders, strength of bidder i is the unique number s_i that solves

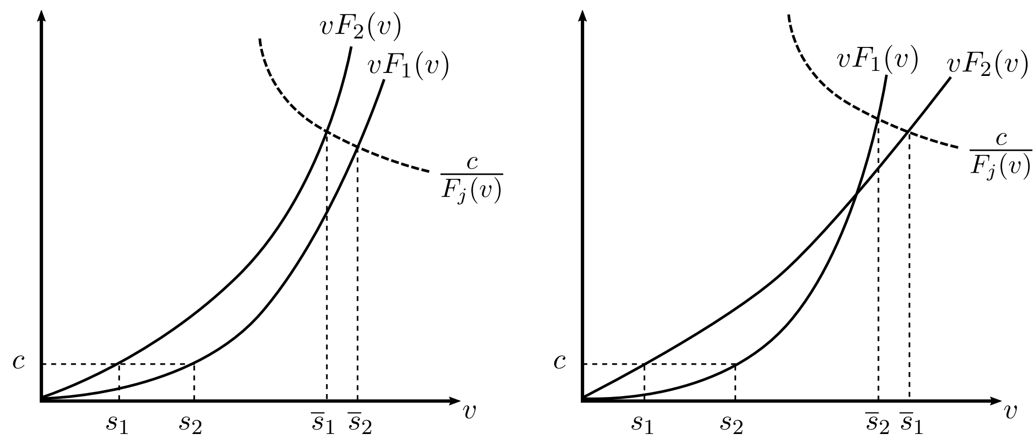
$$s_i \prod_{j \neq i} F_j(s_i) = c_i. \tag{4}$$

Definition (4) is the natural extension of (2). Strength is the unique symmetric strategy that makes a bidder indifferent to enter the auction. Observe that the relative strength between two bidders depends on the characteristics of *every* potential entrant. This dependence implies that adding a new bidder to the game might change the relative strength of existing bidders: bidder 1 being stronger than bidder 2 in a two-player game does not mean that bidder 1 will be stronger than 2 in a three-player game.

To illustrate the previous point, consider a scenario with two bidders which differ in their distribution of valuations but have identical entry costs c, as shown in Figure 4. Bidder i's strength is given by $s_i F_j(s_i) = c$. In both panels of the figure, s_i is solved by intersecting c with the curve $vF_j(v)$, with bidder 1 being stronger than bidder 2, in a two-player game. Suppose a new potential bidder joins the game. With three bidders, the strength of bidder $i \in \{1, 2\}$ is determined by $\bar{s}_i F_j(\bar{s}_i) F_3(\bar{s}_i) = c$ or, equivalently, $\bar{s}_i F_j(\bar{s}_i) = c/F_3(\bar{s}_i)$. In Figure 4, the new strength $\bar{s}_i$ is determined by the intersection of the curves $vF_j(v)$ and $c/F_3(v)$. Panel (a) depicts a situation in which bidders are ordered by FOSD. There, the relative strength of bidders 1 and 2 is invariant to adding a competitor. In contrast, Panel (b) shows a scenario where the CDFs of bidders 1 and 2 cross. In this context, adding the third bidder may change the relative strength of bidders 1 and 2.

Herculean equilibrium and uniqueness With n potential bidders, we can easily extend our methods to prove the existence of a herculean equilibrium to symmetric environments or to environments in which the n bidders can be divided into two asymmetric groups of players. Our method of showing that iterated best responses are further apart than the bidders' strength does not extend to environments with an arbitrary number of asymmetric bidders.

We can, however, show that a herculean equilibrium exists in scenarios in which bidders' strength order is robust to the opponents' behavior. We call these environments *ordered*. In the context of auctions, ordered settings satisfy one of the following two conditions: (i) bidders have identical entry costs, but their distributions of valuations are ordered by FOSD; or, (ii) bidders have identical distributions of valuations and are ordered by their entry cost. In Section 4.3, we show that



(a) First-order stochastic dominance. **(b)** General bidder heterogeneity.

Figure 4: Strength and competition. Bidder's i strength under two bidders with symmetric entry costs is given by the solution to $s_iF_j(s_i) = c$. In a three bidder game, strength is given by the solution to $s_iF_j(s_i) = c/F_3(s_i)$ (for $i \in \{1, 2\}$). Panel (a) shows that when the bidders' CDFs are ordered by FOSD, the strength order between bidders 1 and 2 is robust to adding a third player. Panel (b) shows that when the CDFs cross, the strength order can change when adding a third player.

strength is robust within this class of games and that a herculean equilibrium exists. We provide a uniqueness result in every environment for which we prove the existence of a herculean equilibrium.

3 A Model of Market Entry

We now generalize the previous framework to include several entry models studied in the applied literature, such as the linear entry and logit-demand models. We start by laying out core assumptions and describing a Bayesian equilibrium. We then discuss the generality and limitations of the proposed framework and provide examples of common models satisfying the assumptions.

3.1 The Baseline Model

Set up. Consider n firms simultaneously deciding on whether to enter a market. Firms are privately informed about their *type* v_i (a scalar), summarizing the firm's information about its profitability upon entering the market.[9] Firm i's *post-entry*

[9] In Section 5.2, we extend the model to accommodate scenarios in which v_i represents an informative *signal* about firm i's true type.

profit depends on: (i) the entry decision of every firm; (ii) firm i's type, and; (iii) the types of other entrants. The value v_i distributes according to F_i; a continuously differentiable atomless distribution, with full support on $[\alpha, \beta]$ where $\alpha, \beta \in \overline{\mathbb{R}}$ (the extended reals) and $\alpha < \beta$. The distributions of types, F_i, are independent across firms but not (necessarily) identically distributed.

Let $E = \{1, 2, \ldots, n\}$ be the set of all potential entrants and $\mathcal{E}$ its power set. The set $\mathcal{E}$ contains every potential market structure that we can observe after entry decisions are made. We denote a (realized) market structure by $e \in \mathcal{E}$. The set e lists all the firms participating in a given market structure, whereas the set $e^c = E \backslash e$ lists all the firms that are not. Let $\mathcal{E}_i = \{e \in \mathcal{E} : i \in e\}$ be the set of market structures in which firm i enters. Denote by $v_e = (v_j)_{j \in e}$ the vector of realized types for every firm participating in market structure e. For example, $v_E = (v_1, v_2, \ldots, v_n)$ denotes the vector with the realized types of every firm. As a shortcut, we denote by v_{-i} the realized types of every firm except firm i and we write v_i instead of $v_{\{i\}}$ when i is the sole entrant.

With a slight abuse of notation, let $\pi_i(v_e)$ be a real valued function representing firm i's *post-entry* profit when the realized market structure is e and the realized types are v_e. By adopting this notation, we implicitly assume that the types of non-entrants are payoff irrelevant. To illustrate the workings of the notation observe that $\pi_i(v_i)$ represents firm i's post-entry profit when i is the sole entrant and draws v_i. Similarly, $\pi_i(v_E) = \pi_i(v_i, v_{-i})$ represents i's profit when every firm enters the market and the vector of realized types is given by v_E. In our second-price auction example $\pi_i(v_e) = v_i - \max\{v_j\}_{j \in e \backslash i} - c_i$ if i is the highest valuation entrant and $\pi_i(v_e) = -c_i$ otherwise. We normalize the payoff of a non-entrant to zero. Finally, we assume that $\pi_i(v_e)$ is continuous, integrable (with finite expectation) in each dimension of v_e, and differentiable almost everywhere with respect to v_i. We denote such derivative by $\pi_i'(v_e)$.

The timing of the game is as follows. Before making any entry decision, each firm privately observes v_i. After observing v_i and without observing v_{-i}, each firm independently and simultaneously decides whether to enter the market. After entry decisions are made, market structure e is realized and each firm entering the market gets a payoff $\pi_i(v_e)$. The tuple $(F_i, \pi_i)_{i=1}^n$—which includes the number of potential entrants n—is commonly known by every potential entrant.

Main assumptions. For a given market structure e in which firm i enters the market ($e \in \mathcal{E}_i$), firm i's profit function satisfies the following three properties.

A1 (Monotonicity): The profit function $\pi_i(v_e)$ is weakly increasing in v_i and strictly increasing if firm i is the sole entrant.

Assumption A1 gives economic meaning to the firms' type. Upon entering the market, and regardless of the realized market structure e, firm i's profit increases in v_i. In terms of traditional competition models, a higher v_i can represent a lower marginal cost of production, a lower entry cost, a higher product quality, a better managerial ability, or a higher valuation for a good in an auction. In the second-price auction example, payoffs are monotone; they increase in v_i when i is the entrant with the highest valuation and are constant in v_i otherwise.

The next assumption requires the following definition. For any market structure $e \in \mathcal{E}_i$, draw of types v_e, and competitor $j \in e \backslash i$, define firm i's *profit gain* inflicted by j's exit to be

$$\delta_{i,j}(v_e) = \pi_i(v_{e\backslash j}) - \pi_i(v_e). \tag{5}$$

$\delta_{i,j}(v_e)$ represents the increase in profit that firm i attains, if firm j exits market structure e. In two-player games, $\delta_{i,j}(v_e)$ represents the difference between monopoly and duopoly profits. In our second-price auction example, with two potential bidders, $\delta_{i,j}(v_i, v_j) = \min\{v_i, v_j\}$.

A2 (Substitutes): For each competitor $j \in e \backslash i$:

(i) $\pi_i(v_e)$ is weakly decreasing in v_j.

(ii) $\delta_{i,j}(v_e) \geq 0$.

(iii) There exists a market structure and realized types v_e such that $\delta_{i,j}(v_e) > 0$.

Assumption A2 concerns the impact of competition on profits. It states that firms' entry actions are strategic substitutes, as competition decreases profits. In particular, the assumption states that $\pi_i(v_e)$ decreases when i is faced with: (i) a more productive (higher type v_j) competitor or (ii) entry ($\delta_{i,j}(v_e) \geq 0$). The second-price auction example satisfies (i) and (ii); payoffs can only decrease with entry or an increase in the valuation of an existing bidder. Part (iii) is a strengthening of (ii). It indicates that, for every potential entrant, there exist realized types v_e and a market structure e such that, when j enters the market, firm i's payoffs are strictly lower. This is a minimal assumption about the degree of competition among firms. It does not require that every pair of firms to be competitors in *every* market structure they are in. Firm j can affect firm i's profit indirectly, affecting the equilibrium behavior of other firms participating in market structure e.[10] Most competition models, including our second-price auction example, satisfy a stronger version of (iii). Entry by firm j can decrease firm i's profit in any market structure e if v_j is high enough.[11]

To state the final assumption, define $\phi(v_e) = \prod_{j \in e} f_j(v_j)$ to be the joint density of types of every firm participating in market structure e.

A3 (Costly and interior entry): There exist values $\underline{v}_i < \overline{v}_i$ in the interior of the support of $F_i(v_i)$—i.e., $\underline{v}_i, \overline{v}_i \in (\alpha, \beta)$—such that:

(i) $\pi_i(\underline{v}_i) = 0$ and,

(ii)

$$\int_{\times[\underline{v}_j,\beta]_{j\in E\setminus i}} \pi_i(\overline{v}_i, v_{-i})\phi(v_{-i})d^{n-1}v_{-i} = 0,$$

where the multiple integral is over each of the $n-1$ dimensions of v_{-i}.

Assumption A3 concerns the nature of the entry problem. Condition (i) simply states that entry is costly. Firms need a sufficiently good type, $\underline{v}_i > \alpha$, to be willing to enter the market even as the sole entrant. In the context of a second-price auction, the value $\underline{v}_i$ is equal to the bidder's entry cost c_i. Jointly with assumption A2, A3 implies that, when $v_i < \underline{v}_i$, firm i would never choose to enter the market under any market structure. That is, the value $\underline{v}_i$ represents the minimal type required to enter the market.

Condition (ii), on the other hand, states that any firm will enter the market if its type is sufficiently high. In particular, there exists a value $\overline{v}_i < \beta$ such that drawing $v_i > \overline{v}_i$ ensures entry, even if every potential competitor always enters the market. The assumption that $[\underline{v}_i, \overline{v}_i] \subset (\alpha, \beta)$ guarantees that every equilibrium is interior; i.e., no firm chooses to either never enter or always enter the market.

Strategies and equilibrium. A *cutoff* strategy for firm i is a threshold x_i such that firm i enters the market whenever $v_i \geq x_i$ and stays out otherwise. Firm i's expected profit of entering the market with type v_i when facing opponents playing cutoffs x_{-i} is

$$\Pi_i(v_i, x_{-i}) = \mathbb{E}_{\mathcal{E}_i}\left[\mathbb{E}_{v_{-i}}\left[\pi_i\left(v_e\right)|v_{-i} \geq x_{-i}\right]|x_{-i}\right]$$

[10]Consider a Hotelling model in which firms 1 and 2 are located at each end of the street. If transport costs are high, entry by 1 does not affect 2's profit if they are the only entrants. Entry by 1 can harm 2, however, if there is a third firm located in between 1 and 2.

[11]We can dispense of A2(iii) for our results, but we adopt it for brevity in the proofs.

$$= \sum_{e \in \mathcal{E}_i} \left\{ \Pr[e|x_{-i}] \int_{x_{-i}}^{\beta} \pi_i \left(v_i, v_{e\setminus i} \right) \phi(v_{-i}|v_{-i} \geq x_{-i}) d^{n-1} v_{-i} \right\}$$

where $\Pr[e|x_{-i}] = \prod_{j \in e^c} F_j(x_j) \prod_{j \in e \setminus i} (1 - F_j(x_j))$ is the probability that firm i faces market structure e given the competitors' cutoffs x_{-i}, and n_e is the number of entrants in market structure e.

Firm i's expected profit consists of an iterated expectation. First, given the opponents' strategy x_{-i}, the outer expectation is over each market structure in which firm i participates, $e \in \mathcal{E}_i$. Then, conditional on e, the expectation is over the realization of types for every competitor v_{-i}, conditional on their type being above their entry cutoff. Using that competitors out of the market, $j \in e^c$, are payoff irrelevant and simplifying the denominator of the conditional probabilities of the payoff relevant firms with the probability of observing a market structure, we obtain

$$\Pi_i(v_i, x_{-i}) = \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F_j(x_j) \right) \int_{x_{e\setminus i}}^{\beta} \pi_i \left(v_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\}. \tag{6}$$

Expression (6) is the general model analog of (1). In what follows, we will work with equation (6) as it is the most reduced expression for the expected profit. Appendix B shows that (6) is strictly increasing in firm i's type v_i and in an opponent's cutoff, x_j; a higher entry cutoff x_j lowers the competitor's probability of entry, inducing firm i to face less competition.

For notational ease, we denote by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, instead of x_E, the vector with the cutoff strategies for every potential entrant. A Bayesian *equilibrium* is a vector of cutoff strategies $\mathbf{x}$ such that, for every firm i, $\Pi_i(\mathbf{x}) = 0$. In equilibrium, when opponents play their equilibrium strategy x_{-i}, firm i is indifferent to enter the market when its type is equal to its equilibrium cutoff; i.e., when $v_i = x_i$. Online Appendix C shows that an equilibrium always exists and that *every* equilibrium is in cutoff strategies. Hence, our focus on cutoff strategies is without loss of generality. We denote the partial derivative of $\Pi_i(\mathbf{x})$ with respect to x_i by $\Pi_i'(\mathbf{x})$.

Strength and Herculean Equilibrium We now extend the notion of *strength* to the general framework. Strength uses the game fundamentals, $(F_i, \pi_i)_{j=1}^n$, to rank firms according to their ability to endure competition. As before, we use strength to identify the equilibrium that remains when the game has a unique equilibrium: the herculean equilibrium. Identifying the herculean equilibrium is the starting point to develop our sufficient condition for equilibrium uniqueness.

Definition (Strength)**.** The *strength* of firm i is the unique number $s_i \in \mathbb{R}$ that solves $\Pi_i(s_i, \dots, s_i) = 0$, where $\Pi_i(\mathbf{x})$ is given by (6). We say that firm i is *stronger* than firm j if $s_i < s_j$.

Lemma 3. $\Pi_i(s, \dots, s)$ *is strictly increasing in* s *and crosses zero once.*

The strength of firm i is the unique cutoff s_i that best responds to every competitor playing the same cutoff strategy s_i. Strength ranks firms according to their ability to endure competition. A lower value of strength for firm i ($s_i < s_j$) indicates that a firm i, despite facing more competition than j (i faces competitors with lower entry cutoffs), is more likely than j to enter the market (i plays a lower entry cutoff). Lemma 3 shows that strength is well defined, as it assigns a unique scalar s_i to each firm i, delivering a complete ranking of the firms.

As before, we call an equilibrium herculean if equilibrium cutoffs are ordered by strength, with stronger firms playing *lower* cutoffs. The intuition developed in the second-price auction example remains. Because stronger firms are more able to endure competition, they should be more inclined to enter the market than less strong firms. A herculean equilibrium should naturally emerge in entry games.

3.2 Model Discussion: Scope and Limitations

Scope An important feature of the model is that it allows for general forms of publicly-observed *ex-ante* firm heterogeneity. Firms can differ in their distribution of types F_i. The model also allows for firm heterogeneity in the profit function $\pi_i(v_e)$; i.e., even if firms face the same draws of types, profits might be different. Heterogeneity in profits may come from firms having different entry costs, production costs, production capacities, product characteristics, contracts with suppliers, or geographic locations. The heterogeneity in profitability may also be due to the way firms compete *after* entry has occurred. Firm heterogeneity can accommodate the existence of dominant firms or a predetermined order of play in the *post-entry* market, such as competition à la Stackelberg. The proposed framework can also accommodate firms receiving aggregate or idiosyncratic random shocks after entry. In such cases, $\pi_i(v_e)$ would correspond to the *expected* post-entry profit. Finally, we highlight that the model can also accommodate entry into occupied markets. That is, even though $\pi_i(v_i)$ denotes the profit of a single entrant, the market may already have firms competing in it.

Limitations The results presented in this article apply to static (binary-action) entry games with single-dimensional private types. Nonetheless, we think that the logic behind strength and herculean equilibrium could be extended to a broader class of games. We see the results presented here as a starting point to study a broader class of entry games including dynamic entry games, multi-product or multi-market decisions, multi-dimensional private characteristics, or coordination games in which entry decisions are strategic complements.

The proposed formulation of $\pi_i(v_e)$ also imposes some restrictions on the nature of *post-entry* competition. First, $\pi_i(v_e)$ is a function rather than a correspondence, imposing that either the post-entry game has a unique equilibrium or, under multiplicity of post-entry equilibria, there is market consensus about which equilibrium is played. Second, $\pi_i(v_e)$ does not depend on the profile of cutoff strategies $\mathbf{x}$, restricting the informational flow that exists between the entry and the post-entry game. In particular, this assumption rules out signaling between the entry and the post-entry stages of the game. To illustrate the restriction, consider entry into a sealed-bid first-price auction. Suppose that, after entry, but before playing the post-entry game, bidders become informed about the identities of the participating bidders e but not their valuations. In this scenario, a bidder will base its bidding strategy on its (updated) belief about the distribution of its competitors' values. Through Bayesian updating, these beliefs would depend on the entry strategy played by the opponents. For instance, an increase in the entry cutoff x_j, would lead bidder i to believe, conditional on j's entry, that j has a higher expected valuation.[12] This belief may affect i's bidding strategy. In other words, the post-entry payoff $\pi_i(v_e)$ also depends on the entry strategies $\mathbf{x}$. Although important, the analysis of such models lies outside of the scope of this article.

Given the previous discussion, a natural interpretation for the informational flow of the model is that entering firms' private information becomes public *after* entry but *before* firms compete in the post-entry game. Firms, thus, play a traditional complete information game after entry. Other environments consistent with the informational flow of the model are scenarios where: i) the strategies in the post-entry game are independent of beliefs about the opponents' types, e.g., second price auction; ii) the type is irrelevant for post-entry strategies (e.g., firms are privately informed about their entry costs); or, iii) no information is revealed between entry and the post-entry game, so that there is no belief updating.

[12] The use of a cutoff strategy is only for illustrative purposes. In this class of games, players may not play cutoff strategies.

3.3 Examples

To illustrate the breadth of the model, we present several examples of frameworks used in applied work that satisfy our assumptions. In Section 4, we use some of these examples to illustrate our results.

Example 2 (Linear model)**.** We say that the profit function is linear when

$$\pi_i(v_e) = \eta_i - h_i(e) + v_i$$

where where η_i is a scalar summarizing both market and firm characteristics.[13] The function $h_i(e)$ captures the impact of competition on firm i's profit. Observe that the profit above does not depend on the realization of v_e. In this scenario, only firm j's entry decision, but not its type, affect firm i's payoff. The most common interpretation of the linear model is that $-v_i$ represents firm i's entry cost. Variations of this model used in the empirical literature include:[14]

(i) $h_i(e) = \sum_{j \in e \setminus i} \delta_{ij}$ where the term $\delta_{ij} > 0$ represents firm i's profit gain inflicted by firm j exit (see equation (5)). Seim (2006) uses a version of this model to study entry into the video retail industry. The model captures that firms may have different degrees of substitution, as entry by different competitors may have a different impact on firm i's profitability.

(ii) $h_i(e) = \delta_i \ln(n_e)$ where n_e is the number of entrants in e. Berry (1992) uses a complete information version of this model to study entry into airlines routes. Although firm i's profit gain is independent of the competitor's identity, the model captures that the marginal entrant has a decreasing impact on profitability.

(iii) More generally, $h_i(e)$ can represent the estimated effect of competition in a complex game. For instance, Krasnokutskaya and Seim (2011) use this linear model to study entry into a first-price auction. Bidders are privately informed about their entry costs and learn their valuations and the identity of participating competitors after entry has occurred.

Example 3 (Oligopolistic competition)**.** The framework accommodates tradi-

[13] Observe that, although the term η_i is commonly known by the firms', an econometrician may not observe some elements in η_i. Typically, $\eta_i = X_i\beta_i + \zeta_i$ where X_i is a vector of observed firm and market characteristics and ζ_i is heterogeneity unobserved by the econometrician.

[14] In addition to the articles mentioned in the examples, other works involving linear entry models with private information include: Aguirregabiria and Mira (2007); Bajari *et al.* (2007); Pakes *et al.* (2007); Pesendorfer and Schmidt-Dengler (2008); Sweeting (2009); Aradillas-Lopez (2010); Bajari *et al.* (2010); De Paula and Tang (2012); Vitorino (2012); Mazzeo *et al.* (2016).

tional oligopolistic models. For instance, entry into a market in which firms compete in prices under differentiated products with a logit demand, such as

$$\pi_i(v_e) = (p_i - c_i)S_i(v_e)\mathcal{M} - K_i, \quad \text{where} \quad S_i(v_e) = \frac{D_i}{D}\frac{D^\lambda}{(1+D^\lambda)}$$

is firm i's market share, which is determined by $D_i = \exp((\eta_i + v_i - \alpha p_i)/\lambda)$ and $D = \sum_{j\in e} D_j$. The model is described by the market size $\mathcal{M}$ as well as firm i's entry costs K_i, marginal cost c_i and product/market characteristics, η_i. The parameter α captures consumers' tastes and $\lambda \in [0,1]$ captures the strength of the consumers' outside option. Every potential entrant commonly knows all these parameters. The vector of equilibrium prices, $p_e = (p_j)_{j\in e}$, and market shares are a function of the realized market structure e and the draws of types of the entrants, v_e. In this scenario, v_i represents a product characteristic (such as quality) that is privately known before entry decisions are made, but becomes publicly known after entry occurs. Complete information versions of this model (i.e., not incorporating the v_i term) have been studied by Ciliberto *et al.* (2020) in the context of entry and by Bresnahan (1987), Berry (1994), and Berry *et al.* (1995) when the number of competitors is exogenous.

4 Existence and Uniqueness

We now prove the existence of a herculean equilibrium in three common settings used in applied work. In addition, we provide a sufficient condition guaranteeing equilibrium uniqueness for each of those settings and illustrate how to use the proposed condition in practice. The following definition is instrumental for our results.

Definition (Expected profit gain)**.** For any vector of cutoff strategies $\mathbf{x}$ define firm i's *expected profit gain* inflicted by firm j's exit, $\Delta_{i,j}(\mathbf{x})$, to be

$$F_j(x_j)\sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j}\left\{\left(\prod_{k\in e^c}F_k(x_k)\right)\int_{(x_k)_{k\in e\setminus\{i,j\}}}^{\infty}\delta_{i,j}\left(x_{\{i,j\}}, v_{e\setminus\{i,j\}}\right)\phi(v_{e\setminus\{i,j\}})d^{n_e-2}v_{e\setminus\{i,j\}}\right\} \tag{7}$$

where $\delta_{i,j}(v_e) \geq 0$ is firm i's *profit gain* inflicted by firm j's exit in market structure e and types v_e, as defined in (5).

Given a vector of cutoff strategies $\mathbf{x}$, the expected profit gain (7) is the probability that firm j stays out of the market at its cutoff valuation, $F_j(x_j)$, times firm i's expected gain inflicted by firm j not entering the market. That is, the weighted sum of profit gains due to j's exit, summed over every market structure in which i and j participate, when firm i and j draw valuations equal to $x_{\{i,j\}}$, and the opponents play cutoff strategies $x_{E\setminus\{i,j\}}$. The expected profit gain captures the increase in profit that firm i experiences when firm j marginally increases its entry cutoff x_j and firm i draws type x_i. A small change in x_j only affects firm i's expected profit at firm j's pivotal draw, $v_j = x_j$. At that draw, firm j's exit occurs, inducing firm i to gain $\Delta_{i,j}(\mathbf{x})$. Although assumption A2(ii) only implies that $\delta_{i,j}(v_e) \geq 0$, together with assumption A2(iii) we have that $\Delta_{i,j}(\mathbf{x}) > 0$.

In an environment with only two potential entrants, the expected profit gain equals the probability that the opponent stays out times its induced profit gain. In our auction example, $\Delta_{i,j}(\mathbf{x}) = F_j(x_j)\delta_{i,j}(x_i, x_j) = F_j(x_j)\min\{x_i, x_j\}$. The expected profit gain will help us characterize how firm j's best response to x_i affects firm i's profitability. As we shall see below, if firm j's best response has a bounded effect in firm i's profitability (and reciprocally), the entry game has a unique equilibrium.

4.1 Ex-ante Symmetric Games

We now study the existence and uniqueness of herculean equilibrium in ex-ante symmetric environments, i.e., firms with the same ex-ante characteristics but different ex-post outcomes due to particular realizations of the firms' type. Symmetric entry games have been studied, for example, by Bresnahan and Reiss (1990, 1991), in the context of complete information, and by Brock and Durlauf (2001), Sweeting (2009), and Grieco (2014) in the context of private information.

We say that firm i's profit function is *anonymous* if, for every market structure $e \in \mathcal{E}_i$, firm i's profit function does not depend on the identities of the entrants; i.e., $\pi_i(v_e) = \pi_i(v_i, \mathbf{v}_{n_e-1})$ where $\mathbf{v}_r$ is an r-dimensional vector of realized types and n_e is the number of entrants in e. An entry game is called *symmetric* when every firm has the same distribution of types, $F_i(v_i) = F(v_i)$, and profit functions are anonymous and symmetric, $\pi_i(v_e) = \pi(v_i, \mathbf{v}_{n_e-1})$.

Proposition 2. *In symmetric entry games, there exists a unique herculean equilibrium, where a firm's cutoff is given by its strength. That is, $x_i = s$ for every firm i, where s is the unique number that solves $\Pi_i(s, \ldots, s) = 0$.*

Moreover, the entry game has a unique equilibrium if the condition

$$\frac{f(x_i)}{F(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1, \tag{8}$$

holds for any pair of firms i and j, and for every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}, \overline{v}]$.

Proposition 2 is our first generalization of Proposition 1 to the context of n symmetric firms. It shows the existence of a herculean equilibrium and provides a sufficient condition for equilibrium uniqueness. It can be readily verified that sufficient condition (8) converges to condition (3) for the case of a second-price auction with two potential entrants.[15] As in the auction scenario, condition (8) is a stability condition. It guarantees that firm i's best response does not overly react to changes in its competitors' cutoffs. In turn, this lack of overreaction guarantees that expected profits monotonically increase in firm i's cutoff, even after considering competitors' best responses.

The proof of Proposition 2 proves a stronger result. It shows that two symmetric firms that best respond to each other (and to their potentially non-symmetric competitors) must play identical strategies under sufficient condition (8). This result implies Proposition 2. If an asymmetric equilibrium exists, we would have two symmetric firms best responding to each other (and competitors) and not playing a symmetric cutoff, contradicting the previous result. Further, the result implies that in asymmetric games in which firms can be divided into symmetric groups, condition (8) will guarantee that firms play group-symmetric strategies. We will use this result in subsequent sections.

Computing equilibrium is not necessary to check whether condition (8) holds, as it only makes use of the information given in the fundamentals of the game. As shown below, depending on the application, condition (8) might require only a simple calculation. Because of symmetry, the condition only needs to hold for any pair of potential firms. To better illustrate the result, the next set of examples exploit the properties of the linear model, the most common in applications, to shows how sufficient condition (8) operates. Under linearity, the condition becomes simple, allowing us to show how uniqueness changes with the level of competition. To further illustrate the usefulness of the condition, we use it to show that the

[15] As $\Pi_i'(\mathbf{x}) = F(\max\{x_i, x_j\})$, condition (8) matches exactly when $x_j > x_i$. When $x_j \leq x_i$, the condition becomes $x_j F(x_j) f(x_i)/F(x_i)^2 < 1$. Because this condition has to hold for every $x_j \leq x_i$ and the left-hand side increases in x_j, taking $x_j = x_i$ delivers the result.

empirical model in Grieco (2014) has a unique equilibrium.

Example 4 (Symmetric linear model)**.** Consider a symmetric linear model (see Example 2), in which firm i's post-entry profit is given by

$$\pi_i(v_e) = \eta - \delta \sum_{k=1}^{n_e - 1} r^{k-1} + v_i.$$

This model captures that the marginal impact of entry is decreasing in the number of competitors. A new entrant decreases profits by a fraction $r_i \in [0, 1]$ of the previous entrant. Sufficient condition (8) holds if, for $x_i \in [\underline{v}, \overline{v}]$, the following inequality is satisfied (see Online Appendix F for a step-by-step derivation)

$$\frac{f(x_i)}{F(x_i)} < \frac{1}{\delta F(\overline{v})\,(r + F(\overline{v})\,(1-r))^{n-2}}, \tag{9}$$

where $\underline{v} = -\eta$ and $\overline{v} = \delta(1 - r^{n-1})/(1-r) - \eta$. That is, the reversed hazard rate of F needs to be bounded above by the inverse of $\delta F(\overline{v})\,(r + F(\overline{v})(1-r))^{n-2}$. We use condition (9) to illustrate relevant properties of the linear model:

(a) **Log-concave distributions.** When F is log-concave, its reversed hazard rate $f(x_i)/F(x_i)$ is decreasing in x_i.[16] Consequently, sufficient condition (9) reduces to checking the following inequality $f(\underline{v})/F(\underline{v}) < \left(\delta F(\overline{v})\,(r + F(\overline{v})(1-r))^{n-2}\right)^{-1}$. For instance, if types distribute type-I extreme value (as in Seim, 2006)[17] and $r = 1$, it can be readily checked that the sufficient condition for uniqueness becomes

$$\eta + \ln(\delta) < \exp(\eta - (n-1)\delta),$$

a restriction to the parameters of the model. Figure 5a illustrates this restriction for different number of potential entrants, n. The area outside the curves represents the combination of parameters delivering a unique equilibrium. Inside the curves, the game might have a unique or multiple equilibria.

(b) **Uniqueness and competition.** Continuing with the previous example, we exploit the properties of the linear model to illustrate the effect of competition on sufficient condition (9). In the linear model, competition manifests through two

[16] If $G(x) = \ln(F(x))$ is concave, then $G''(x) = \partial(f(x)/F(x))/\partial x < 0$. Examples of log-concave distributions include normal, exponential, extreme value, logistic, and gamma.

[17] A type-I extreme value distribution with location parameter 0 and scale parameter λ is given by $F(v) = \exp(-\exp(-v/\lambda))$. Then, its inverted hazard rate is given by $f(v)/F(v) = \exp(-v/\lambda)/\lambda$, which is decreasing in v. This distribution is called standard when $\lambda = 1$.

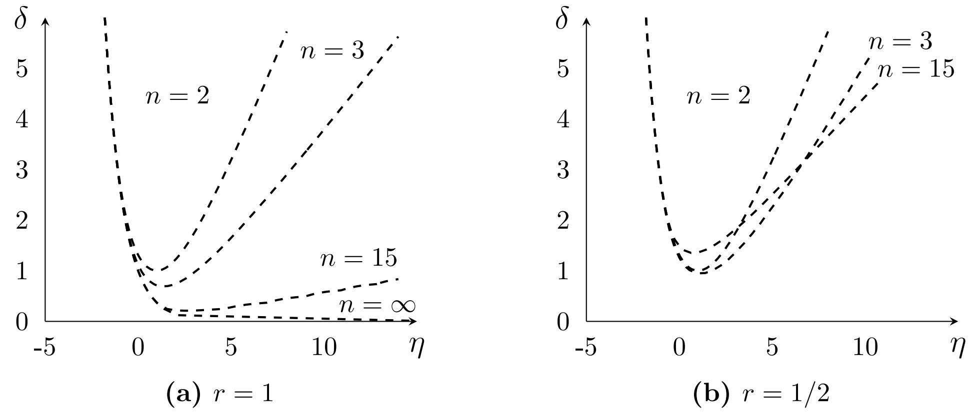


Figure 5: Equilibrium uniqueness in a symmetric linear model with a standard type-I extreme value distribution. The area outside the curves represents the set of parameters δ and η that deliver a unique equilibrium. In Panel (a), the set of parameters satisfying uniqueness shrinks with the number of potential entrants, n. In Panel (b), the set reacts non-monotonically. Restriction fades away when n becomes unboundedly large.

channels: the number of potential competitors, n, and the decreasing marginal impact of entry, r. For the latter, observe that an increase in r decreases the right-hand side of (9)—making the condition harder to satisfy—directly through r (entry has more impact) and indirectly by increasing $\overline{v}$ (enlarging the set of possible deviations). Consistent with the traditional logic of entry games, the prospects of facing multiple equilibria increase with the gains from coordinating entry, i.e., when the profit gains from avoiding competition become large. As a consequence, when comparing both panels of Figure 5, we can see that an increase in the marginal impact of entry, r, shrinks the set of parameters that deliver a unique equilibrium.

Increasing the number of competitors, n, also increases the set of possible deviations $\overline{v}$. In contrast, a larger n has the countervailing effect of increasing the expected number of entrants, captured by $(r + F(\overline{v})(1 - r))^{n-2}$. When the marginal effect of entry is decreasing (i.e., when $r < 1$), a larger number of entrants decreases the expected profit gain (7) that firm j's exit inflicts on firm i. This effect makes firm i less susceptible to entry, increasing the set of parameters for which uniqueness occurs. The interaction between these effects makes the set of parameters satisfying (9) to change non-monotonically with n (see Figure 5b). This non-monotonicity stands in contrast to the scenario with a constant marginal effect of entry ($r = 1$). There, only the effect of increasing $\overline{v}$ remains, making the set of parameters satisfying (9) shrink with the number of potential entrants n (see Figure 5a). In the limit, as n becomes unboundedly large, the restriction for $r = 1$ becomes the tightest and equal to $\eta + \ln(\delta) < 0$. In contrasts, when $r < 1$, the model always has a unique equilibrium, as the right hand side of (9) goes to infinity.

(c) **Equlibrium multiplicity and uniqueness under Normality.** Suppose types distribute $N(0, \sigma)$, which is log-concave. Berry and Tamer (2006) observe that, in a game with two entrants ($n = 2$) and under the assumption $\delta > \eta$, the entry game has multiple equilibria when it converges to a complete information game ($\sigma \to 0$) and has a unique equilibrium when the private information dominates ($\sigma \to \infty$). We can use sufficient condition (9), $\delta F(\overline{v}) f(\underline{v})/F(\underline{v}) < 1$, to provide a tighter characterization. Suppose, for instance, that $\eta = 0$ and $\delta = 1$ (so that, $\underline{v} = 0$ and $\overline{v} = 1$). Then, because both $f(0)/F(0) = 2/\sqrt{2\pi\sigma^2}$ and $F(\overline{v})$ are decreasing in σ, there is a threshold $\hat{\sigma} = 0.7298$ such that $\sigma > \hat{\sigma}$ guarantees equilibrium uniqueness.

For example, if $\sigma = 1/4$ the game has three equilibria. The herculean equilibrium, which is symmetric and given by the cutoff strategy $x_i = 0.2055$, and two asymmetric equilibria, given by $x_i = 0.041$ and $x_{3-i} = 0.435$, for $i \in \{1, 2\}$. Similarly, if $\sigma = 1$, the game has a unique equilibrium given by $x_i = 0.3596$.

(d) **A concrete application.** When studying entry of supercenters into rural grocery markets, Grieco (2014) estimates a symmetric incomplete information model with two potential entrants ($n = 2$) and $v_i \sim N(0, 1)$. In the smallest market, where coordination among entrants is more relevant, the model estimates are given by $\eta = -3.838$ and $\delta = 0.851$.[18] Using the log-concavity property of the normal distribution, in conjunction with the model estimates, sufficient condition (9) becomes $\delta F(\overline{v}) f(\underline{v})/F(\underline{v}) = 10^{-4} < 1$. The equilibrium is, thus, unique.

4.2 Two Groups of Firms

We now extend our results to games where entrants can be divided into two groups according to their public characteristics. Within each group, firms are ex-ante symmetric. Across groups, however, firms can differ in their distribution of types and profit functions. In applied work, models of two groups of entrants have been

[18] See Table 7, page 329: $\eta = \mu_0 - \mu_4 = -1.222 - 2.158 = -3.838$. Condition (8) also holds for every other specification in the paper.

used, for example, to study the timberwood industry (mills and loggers) by Athey *et al.* (2011) and Roberts and Sweeting (2013, 2016) as well as to study in highway procurement auctions (favored and non-favored bidders) by Krasnokutskaya and Seim (2011). The two-group structure may arise naturally in applications where firms can be divided into incumbents and entrants, high and low-quality firms, local and international producers, discount and traditional retailers, or legacy and low-cost airlines, among other examples.

Formally, let G_g be the set of firms belonging to group $g \in \{1, 2\}$. Group g consists of $n_g \in \mathbb{N}$ potential entrants (so that, $n_1 + n_2 = n$) described by the pair (π_g, F_g). Let $g(i)$ be the group of firm i. We assume that profits are *symmetric* and *anonymous* within a group. That is, firm i's profit under market structure e is now equal to $\pi_i(v_e) = \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k)$ where r and k are the number of entrants in $e\backslash i$ from group $g(i)$ and $-g(i)$, respectively. The vectors $\mathbf{v}_r$ and $\mathbf{v}_k$ represent the types of such entrants.

Because firms are within-group symmetric, firms in the same group have equal strength. A herculean equilibrium, thus, consists of group-symmetric strategies in which the strongest group plays the lowest cutoff. To formally characterize a group-symmetric equilibrium, define $\varphi_g(\mathbf{v}_r) = \prod_{j=1}^{r} f_g(v_j)$ to be the probability density that r firms belonging to group g draw the vector $\mathbf{v}_r$. For a pair of cutoffs $\hat{x} = (x^1, x^2)$ describing group-symmetric strategies by the opponents, firm i's expected profit of entering the market with a draw of v_i, when there are r and k entrants, other than firm i, from group $g(i)$ and $-g(i)$, is given by

$$\mathbb{E}[\pi_i(v_i, r, k)|\hat{x}] = \int_{x^1}^{b} \left(\int_{x^2}^{b} \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k)\varphi_{-g(i)}(\mathbf{v}_k) d^k\mathbf{v}_k \right) \varphi_{g(i)}(\mathbf{v}_r) d^r\mathbf{v}_r$$

where the integrals are over the r and k dimensions of $\mathbf{v}_r$ and $\mathbf{v}_k$. Then, when faced with group-symmetric strategies $\hat{x}$ (i.e., $x_j = x^{g(j)}$ for every firm $j \neq i$), firm i's expected profit of entering the market under valuation v_i is

$$\Pi_i(v_i, x_{-i}) = \sum_{k=0}^{n_j} \left\{ \binom{n_j}{k} F_j(x_j)^{n_j - k} \left[\sum_{r=0}^{n_i - 1} \binom{n_i - 1}{r} F_i(x_i)^{n_i - 1 - r} \mathbb{E}[\pi_i(v_i, r, k)|\hat{x}] \right] \right\},$$

where for ease in notation, we use i and j instead of $g(i)$ and $-g(i)$ as it leads to no confusion. The previous expression corresponds to equation (6) in the context of two groups of firms playing group-symmetric strategies. To understand the previous expression, fix a market structure in which r and k firms of group i and

j participate in the market. Because there are n_j firms in group j, there are 'n_j choose k' possibilities to obtain a market structure with k competitors from group j. Each of these possibilities occur with probability $F_j(x_j)^{n_j-k}$; i.e., the probability that $n_j - k$ firms obtain a low draw and stay out of the market. Similarly, there are '$n_i - 1$ choose r' possibilities to observe r competitors from i's group, each occurring with probability $F_i(x_i)^{n_i-1-k}$. The expression above is, thus, obtained by summing across every possible market structure.

A pair of strategies $\hat{x}$ constitutes a group-symmetric equilibrium if and only if, for each firm i, $x_i = x^{g(i)}$ and the vector of strategies $\mathbf{x}$ satisfies $\Pi_i(\mathbf{x}) = 0$. Without loss of generality, let group 1 be the strongest group. The following theorem is the main result of this subsection.

Theorem 1. *A herculean equilibrium always exists. The herculean equilibrium satisfies $x_1 < s_1 < s_2 < x_2$, where s_g and x_g are the strength and the equilibrium cutoff of group g. Moreover, the game has a unique equilibrium if these conditions*

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } j \in G_{g(i)}, \tag{10}$$

$$n_{g(j)} \frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } j \in G_{-g(i)}, \tag{11}$$

hold for any pair of firms i and j, and for every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}_{g(k)}, \overline{v}_{g(k)}]$.

Theorem 1 has two main results. First, a herculean equilibrium always exists. In this way, the theorem shows that strength is the right notion to characterize the firms' relative competitiveness. In many empirical applications, where the multiplicity of equilibria is a concern, the model estimation is based on assuming that firms play an equilibrium that is ex-ante 'intuitive' given the fundamentals of the model (c.f. Roberts and Sweeting, 2013). As we show in Example 5 below, although intuitive orders are consistent with the proposed notion of strength, they are restrictive in that many applications might not have an ex-ante 'intuitive' order. Strength, in turn, provides an order in any entry game. Theorem 1 also provides bounds on the herculean equilibrium cutoffs, $x_1 \in (\underline{v}_1, s_1)$ and $x_2 \in (s_2, \overline{v}_2)$, which speeds up numerical computation of herculean equilibria.

Second, Theorem 1 provides four conditions that need to be satisfied for equilibrium uniqueness—two conditions per group. The within-group condition (10) is analogous to the sufficient condition for uniqueness in symmetric entry games

(8).[19] This condition bounds the change in best responses due to deviations from firms within the same group, inducing group-symmetric strategies. Empirical applications usually restrict their analysis to group-symmetric strategies. Condition (10), thus, guarantees that this restriction is without loss. If an applied researcher determines that within-group *asymmetric* strategies are not relevant for the application at hand, this condition can be dispensed.

The cross-group condition (11), on the other hand, guarantees that the herculean equilibrium is the only group-symmetric equilibrium of the game. This condition bounds the best responses due to a group-symmetric deviation from the opposing group. Observe that the left-hand side of condition (11) is multiplied by the number of firms in group j. In group-symmetric strategies, there are $n_{g(j)}$ firms deviating simultaneously; thus, the condition needs to bound $n_{g(j)}$ deviations at the same time. Comparing conditions (10) and (11), we can see that the former condition does not directly depend on $n_{g(j)}$ (because $j \in G_{g(i)}$, $n_{g(j)}$ is the number of competitors in the same group as firm i). This is so, because we can exploit the within-group symmetry among firms to obtain a 'tighter' bound. Below we show that condition (11) might not necessarily be more restrictive than condition (10).

Example 5 (Linear model)**.** Consider the linear model of Example 4. In particular, assume that the marginal impact of competition is constant, $r = 1$, and that firm i's type distributes type-I extreme value with scale parameter λ_i. Firm i's post-entry profit is, then, given by

$$\pi_i(v_e) = \eta_i - (n_e - 1)\delta_i + v_i$$

where $\delta_i > 0$. In this context, the group g's strength is obtained by picking any firm $i \in G_g$ and solving $\Pi_i(s_i, \ldots, s_i) = 0$, or

$$s_i = \delta_i \sum_{k \neq i} (1 - F_k(s_i)) - \eta_i.$$

Group g's strength negatively depends (i.e., s_i increases) on the expected number of entrants when every firm plays the cutoff strategy s_i, $\sum_{k \neq i}(1 - F_k(s_i))$, weighted by the impact that each entrant has on profits, δ_i. Strength positively depends on the public characteristics of the firm η_i. Because we studied the relation between

[19] Proposition 2 is a particular case of Theorem 1 when one of the groups has no members. We chose to split the results for clarity in the exposition and because the symmetric model has value on its own.

η_i and δ_i in Example 4, we simplify the analysis by assuming $\eta_i = 0$ for both groups. Consequently, the relevant range for i's cutoffs is given by $\underline{v}_i = 0$ and $\overline{v}_i = (n-1)\delta_i$.

Let $\hat{\delta}_i = \delta_i/\lambda_i$. Using the log-concave property of extreme value distributions, conditions for uniqueness (10) and (11) become (see Online Appendix F for details)

$$C_{i,j} = \begin{cases} \exp\left(-(n-1)\hat{\delta}_i\right) > \ln\left(\hat{\delta}_i\right) & \text{if } j \in G_{g(i)} \\ \exp\left(-(n-1)\hat{\delta}_j\right) > \ln\left(n_j\hat{\delta}_i\right) & \text{if } j \in G_{-g(i)} \end{cases}. \tag{12}$$

As discussed in Example 4(b), an increase in the total number of potential entrants (from either group) make both constrains more strict (when $r = 1$). Finally, observe that the $C_{i,i}$ constraint simplifies to a threshold value for $\hat{\delta}_i$.

We use this example to illustrate how the notion of strength helps to discern the (herculean) cutoff order before computing equilibrium. Then, we show how the sufficient conditions for uniqueness restrict the set of parameters when the number of potential entrants varies.

(a) **Strength.** Suppose that $\delta_1 < \delta_2$ and $\lambda_2 > \lambda_1$. In this scenario, a firm in group 1 is less affected by competition than a firm in group 2. In addition, the types of a firm in group 1 stochastically dominate those of a firm of group 2 (in the relevant range for entry, $v_i \geq \underline{v} = 0$). In this scenario, it is 'intuitive' to think that group 1 is more competitive than group 2; thus, we expect group 1 to enter more often, i.e., to play a lower entry cutoff.

If, in turn, we have that $\delta_1 < \delta_2$ and $\lambda_2 < \lambda_1$, we cannot use an intuitive criterion because each parameter drives competitiveness in a different direction. Although group 1 is less affected by competition compared to group 2 ($\delta_1 < \delta_2$), group 2 is more likely to draw higher types ($\lambda_1 > \lambda_2$). We can use our notion of strength to discern ex-ante the cutoff order in a herculean equilibrium. Because a herculean equilibrium is guaranteed to exist, regardless of whether the game has a unique or multiple equilibria, knowing the cutoff order is helpful when searching for the equilibrium.

To illustrate the previous point, consider the case with two asymmetric firms (i.e., $n_1 = n_2 = 1$) characterized by $\delta_1 = 1$ and $\delta_2 = \lambda_2 = 5/4$. The intuitive criterion allows to rank firms, and discern a suitable equilibrium, only when $\lambda_1 \geq \lambda_2$. Figure 6 depicts the firms' strength as a function of λ_1. The strength of firm 1 is a constant ($s_1 = 0.4909$), as it does not depend on λ_1. Firm 2 becomes weaker

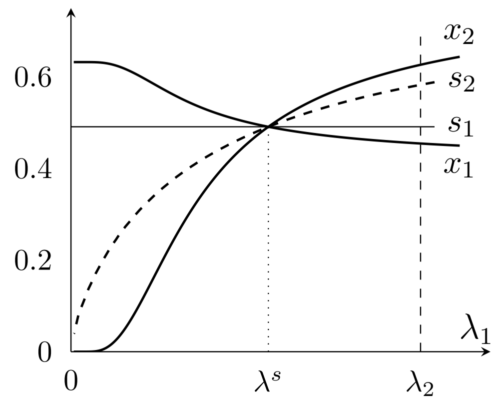
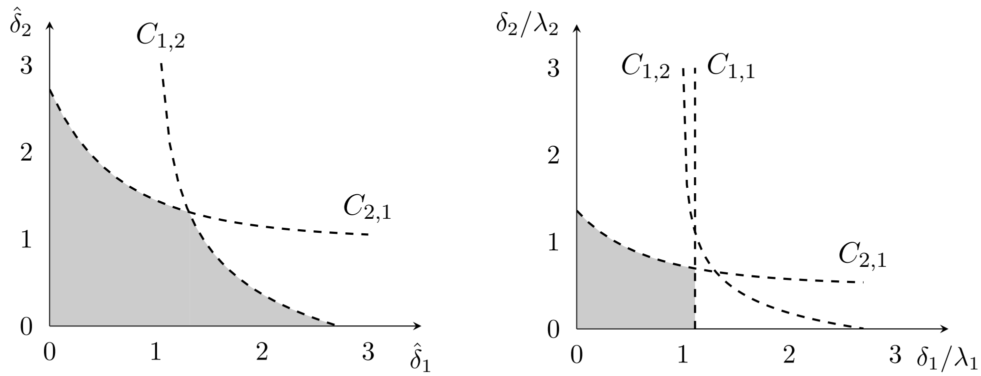


Figure 6: Strength and Herculean equilibrium in a linear model with two asymmetric firms and type-I extreme value distributions; λ_1 varies and $\delta_1 = 1$ and $\delta_2 = \lambda_2 = 5/4$.

(s_2 increases) when firm 1 becomes more competitive by drawing higher types. Consistent with the 'intuitive' criterion, firm 1 is stronger when $\lambda_1 > \lambda_2$; firm 1 is simultaneously less sensitive to entry and draws higher types. As λ_1 decreases, firm 1 remains stronger until it reaches $\lambda^s \equiv 0.7058$, the value of λ_1 which makes both firms equally strong. When $\lambda_1 < \lambda^s$, firm 2 becomes the stronger firm in the game. Figure 6 also shows the herculean equilibrium for each value of λ_1. Consistent with the previous analysis, firm 1 plays the lowest cutoff whenever it is the stronger firm in the game. In summary, for values of $\lambda_1 > \lambda_s$, firm 1 plays a lower cutoff, despite $\lambda_1 < \lambda_2$

(b) **Uniqueness with two asymmetric firms.** Continuing with the previous example of an entry game with two asymmetric firms ($n_1 = n_2 = 1$), Figure 7a depicts the set of parameters that satisfy restrictions $C_{1,2}$ and $C_{2,1}$. Aligned with the intuition that equilibrium multiplicity tends to occur when coordination among firms is important—that is, when the market is likely to support only one firm in equilibrium—multiplicity arises when the profit gain from entry by a competitor, δ_i, is high; or when a firm is unlikely to obtain a high type (low λ_i).

(c) **Uniqueness with three asymmetric firms.** To see how the sufficient condition for uniqueness changes when we increase the number of potential entrants, suppose instead that we have three firms: two belonging to group 1 and one belonging to group 2. In this scenario, conditions for uniqueness (10) and (11) become three restrictions on the model parameters. These restrictions are shown in Figure 7b. The restriction $C_{1,1}$, that entry by a firm in group 1 imposes in the other group 1 firm, becomes $\hat{\delta}_1 < 1.1138$. As can be observed in this example,



(a) $n_1 = n_2 = 1$

(b) $n_1 = 2$ and $n_2 = 1$

Figure 7: Equilibrium uniqueness (shaded area) – linear model with asymmetric firms and type-I extreme value distributions.

the restriction $C_{1,2}$ is redundant. To have a unique equilibrium, a firm in group 1 is more constrained by the behavior of firms in its own group than the behavior of the firm in the other group. The restriction that group 1 imposes in group 2, $C_{2,1}$, also tightens, as the curve shifts downwards. As mentioned above, because this example assumes a constant marginal impact of competition ($r = 1$), the set of parameters delivering uniqueness shrinks with the number of competitors. If $r < 1$, on the other hand, the set may either shrink or expand.

4.3 Ordered Games

In an entry game, there are two elements that determine payoffs: the distribution of types $F_i(v_i)$ and the profit function $\pi_i(v_e)$. A game is called *ordered* when firms differ only in one of these two elements and are ordered in this element. In this section, we extend our results to ordered environments. Formally, an entry game is called *ordered by distributions* when firms have symmetric and anonymous profit functions, and their distributions of types, $F_i(v_i)$, are ordered in terms of first-order stochastic dominance (FOSD). Similarly, a game is *ordered by profit* when firms have symmetric distributions of types and anonymous profit functions that for any realization v_e, satisfy $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$, where $\mathbf{v}_r$ is an r-dimensional vector of realized types and n_e is the number of entrants in e.[20] Without loss of generality, we order firms' identities so they satisfy $F_i(v) \leq F_{i+1}(v)$ for all v when

[20]Our results below also extend to environments in which firms are ranked consistently in both dimensions; i.e., $F_i(v) \leq F_{i+1}(v)$ for all v and $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$ for all v_e.

ordered by distributions, or $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$ when ordered by profit.

Empirical applications usually focus on ordered games. In the context of complete information entry games, the order of the model has been used as an equilibrium selection criteria when multiple equilibria exist. For example, Berry (1992) uses an ordered model, in which firms with lower entry costs are assumed to enter first (see also Jia, 2008, which uses profitability as a selection criterion). In a context in which firms have private information, Roberts and Sweeting (2013, 2016) use a model in which firms are ordered by distributions; and Vitorino (2012) uses a linear model in which firms are ordered by profits.

Lemma 4. *Suppose an entry game in which firms are ordered (either by profit or distributions). Then, the firms are also ordered by strength, with $s_i < s_{i+1}$; i.e., firm 1 is the strongest and firm n the weakest.*

The previous lemma shows that the firms' ranking provided by strength coincides with the order of the game. In ordered games, the firm ranking provided by strength is robust to adding competitors. That is, if we add a new firm to the game, the existing strength order between the firms remains unchanged, as illustrated in Figure 4a. The following theorem is the main result of this subsection.

Theorem 2. *In ordered games, there always exists a* herculean *equilibrium. Moreover, the entry game has a unique equilibrium if the following condition holds*

$$(n-1)\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \tag{13}$$

for every pair of firms i, j and every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}_k, \overline{v}_k]$, and the game is: i) ordered by profit or, ii) ordered by distributions and the profit gain does not depend on the type of competitors; i.e., $\delta_{i,j}(x_{\{i,j\}}, v_{e\setminus i}) = \delta_i(x_i, n_e)$.

As in the previous two environments, a herculean equilibrium always exists. Observe that Theorem 2 is not a particular case nor a generalization of Theorem 1. While the former can handle more than two groups of asymmetric firms, the latter allows for a larger degree of firm heterogeneity between the two groups. There are also differences in the sufficient condition for uniqueness. The induction method used in the proof of Theorem 2 needs to handle simultaneous deviations by each of the $n-1$ competitors in the game, independently of whether a subset of firms are symmetric or not. Theorem 1, on the other hand, exploits the within-group symmetry to provide a weaker sufficient condition.

Although Theorem 2 says that condition (13) needs to hold for every pair of potential entrants, the ordered structure of the game usually means that it is sufficient to check the condition for a specific pair of firms. Below, we illustrate this property in the context of the linear model. If the condition for uniqueness holds for a particular pair of firms, with the particular firms depending on the application, it holds for every other pair.

Example 6 (Linear model)**.** Continuing with the linear model of Example 4, $\pi_i(v_e) = \eta_i - (n_e - 1)\delta_i + v_i$, let assume v_i distributes $N(\mu_i, 1)$. We explore sufficient condition (13) under different orders. Start by observing that the profit gain is independent of the type and number of competitors, as $\delta_{i,j}(x_{\{i,j\}}, v_{e\setminus i}) = \delta_i$. This independence holds regardless of the game being ordered by profit or by distributions. Using the log-concavity property of the normal distribution, condition (13) holds if, for every pair of firms i and j, the following inequality is satisfied

$$(n-1)\delta_i F_j\left(\overline{v}_j\right)\frac{f_i(\underline{v}_i)}{F_i(\underline{v}_i)} < 1, \tag{14}$$

where $\underline{v}_i = -\eta_i$ and $\overline{v}_i = (n-1)\delta_i - \eta_i$. We show that, in linear-ordered environments, if condition (13) holds for one (specific) pair of firms, it holds for every pair of firms.

(a) **Ordered by distributions.** Suppose $\delta_i = \delta$ and $\eta_i = \eta$ for every firm i. That is, firms are ordered distributions by their distribution's mean, μ_i, with the strongest firm having the highest μ_i. In this scenario, sufficient condition (14) simplifies to $(n-1)\delta F_j(\overline{v}) f_i(\underline{v})/F_i(\underline{v}) < 1$. Noting: i) for a given $\underline{v}$, the inverted hazard rate increases in μ_i, and; ii) stochastic dominance ($F_n(\overline{v}) \geq F_i(\overline{v})$ for all i), the condition holds for *every* pair of firms, if it holds for $i = 1$ and $j = n$.

(b) **Ordered by profit I.** Suppose instead that $\delta_i = \delta$ and $\mu_i = \mu$ for every firm i. Firms are ordered by profit, where the strongest firms has the highest value of η_i. In this scenario, sufficient condition (14) becomes $(n-1)\delta F(\overline{v}_j) f(\underline{v}_i)/F(\underline{v}_i) < 1$. Because the inverted hazard rate is decreasing in v_i, the condition holds for every pair of firms, if it holds for $i = 1$ and $j = n$ (as $\underline{v}_1 \leq \underline{v}_i$ and $\overline{v}_n \geq \overline{v}_i$ for all i).

(c) **Ordered by profit II.** Finally, suppose that $\eta_i = \eta$ and $\mu_i = \mu$ for every firm i. Firms, then, are ordered by profit, where the strongest firm is the less sensitive to entry (has a lower δ_i). In this scenario $\underline{v}_i = -\eta$ for every i, and condition (14) becomes $(n-1)\delta_i F(\overline{v}_j) f(-\eta)/F(-\eta) < 1$. Because the two weakest firms are the ones with the highest δ_i and $\overline{v}_j$, pick $\kappa = \max\{\delta_n F(\overline{v}_{n-1}), \delta_{n-1} F(\overline{v}_n)\}$ and the condition holds for every pair of firms, if $(n-1)\kappa f(-\eta)/F(-\eta) < 1$.

5 Discussion and Extensions

In this section, we discuss how strength and herculean equilibrium relate to the notion of risk factor and risk dominant equilibrium, respectively. We, then, extend the model to include gradual revelation of information, i.e., games in which v_i represents a signal about the firm's true type.

5.1 Herculean Equilibrium and Risk Dominance

Thus far, the article's primary focus has been to identify conditions under which the entry game has a unique equilibrium. Given our existence of herculean equilibrium result, however, a natural question arises. Can we use strength and herculean equilibrium as a selection criterion when multiple equilibria exist? We provide a foundation to this approach by showing that the notion of herculean equilibrium is an incomplete information analog to the notion of risk dominance in complete information games (Harsanyi and Selten, 1988).

	E_2	N_2
E_1	π_{12}, π_{21}	$\pi_1, 0$
N_1	$0, \pi_2$	$0, 0$

(a) Generic entry game

	E_2	N_2
E_1	$-1, -3$	$1, 0$
N_1	$0, 1$	$0, 0$

(b) Example

Table 1: Game in normal form.

Consider a generic complete information entry game, as illustrated in Table 1a. If a firm does not enter, it receives zero profit. If it enters, it receives profits according to the number of entrants. We assume that $\pi_i > 0$ and $\pi_{ij} < 0$, so that the market profitably accommodates only one firm. Consequently, the game has one equilibrium in mixed-strategies and two equilibrium in pure strategies, $a_1 = (E_1, N_2)$ and $a_2 = (N_1, E_2)$. In this context, a player's *risk factor* of entering the market, r_i, is the lowest probability for which the opponent stays out and player i still wants to enter. In the generic example, $r_i = -\pi_{ij}/(\pi_i - \pi_{ij})$. The *basin of attraction* of a pure strategy equilibrium, a, is the size of the set (area) of mixed strategies for which a is a best response. The basin of attraction of a strategy profile is proportional to the Nash-product of the strategy profile. In the context of the generic entry game, the Nash-products of a_1 and a_2 are $\pi_1(-\pi_{21}) > 0$ and $\pi_2(-\pi_{12}) > 0$, respectively. Figure 8a illustrates the previous concepts. The axes represent the probability that each player enters the market. Best responses (dashed lines) are depicted using the numerical example in Table 1b. An equilibrium is called *risk dominant* if one of the following two (equivalent) conditions hold: (i) it has the lowest risk factor for a given player, or; (ii) it has the largest basin of attraction.[21]

Harsanyi and Selten (1988) argue that risk dominance is an appealing selection criterion as it has an axiomatic foundation of desirable properties.[22] We argue that strength and herculean equilibrium are incomplete information analogs to the risk factor and risk-dominant equilibrium, respectively. In particular, we show that, using a Harsanyi (1973) purification argument, as the sensitivity of payoffs to the type vanishes, the CDF evaluated at the strength of a firm converges to its risk factor in the matching complete information game.[23]

Formally, we consider an incomplete information variation of the game in Table 1a. In particular, we assume that $\pi_i(v_e) = \pi_i(e) + \varepsilon v_i$, where $\pi_i(e)$ is given by the payoffs in Table 1a, v_i is the type (private information) of firm i, and $\varepsilon \in [0, 1]$ measures the sensitivity of payoffs to the type in the game. Observe that, as $\varepsilon \to 0$, the sensitivity of payoffs to the type vanishes and the game converges to the complete information game of Table 1a.[24] We assume that the distributions of types are symmetric, that is $F_i = F$. In this context, using the definition of strength ($\Pi_i(s_i, s_i) = 0$), the strength of firm i is the unique number s_i that solves

$$F(s_i)\pi_i + [1 - F(s_i)]\pi_{ij} + \varepsilon s_i = 0 \tag{15}$$

or, equivalently, $F(s_i) = -(\pi_{ij} + \varepsilon s_1)/(\pi_i - \pi_{ij})$.

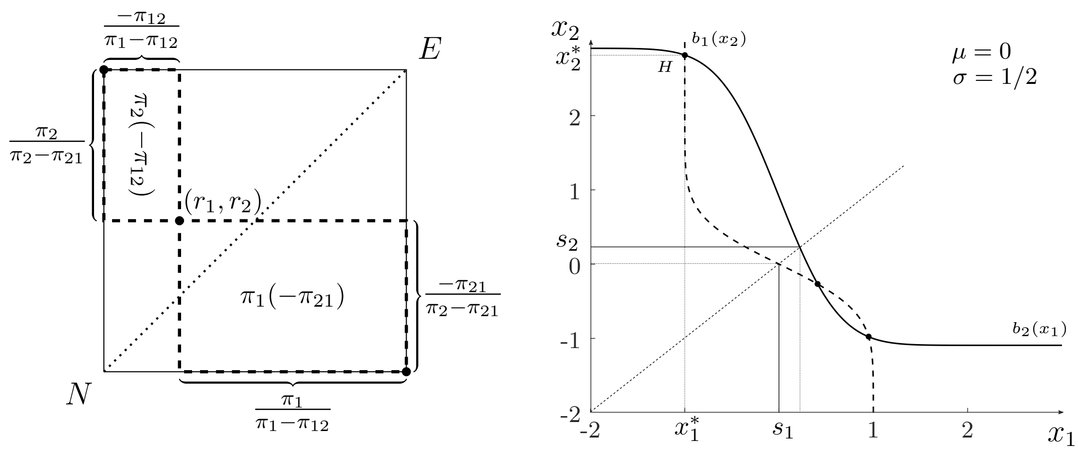
Proposition 3. *In the limit, as a firm payoff sensitivity to its type vanishes ($\varepsilon \to 0$), firm 1 is stronger than 2 if and only if firm 1 has the lowest risk factor of entry in the limiting complete information game. Consequently, as $\varepsilon \to 0$, the herculean equilibrium converges to the risk dominant equilibrium.*

[21]The equivalence between the conditions follow from observing: $r_1 = -\pi_{12}/(\pi_1 - \pi_{12}) < -\pi_{21}/(\pi_2 - \pi_{21}) = r_2$ if and only if $\pi_2(-\pi_{12}) < -\pi_1(-\pi_{21})$.

[22]Harsanyi and Selten (1988, p. 356) argue that Risk dominance is an extension of Bayesian rationality from one-person decisions to n-person games. They show that it is the only refinement invariant to relabeling strategies, affine payoff transformations, and increasing the payoffs of the selected equilibrium.

[23]We evaluate strength in the CDF, as risk factor is a probability, whereas strength is a type.

[24]Notice that this limiting exercise is different than the one we did in Example 2(a). There, we took the limit as distribution of types collapses into one point. Here, we take the limit as the sensitivity of the payoffs to the type vanishes.



(a) Risk Dominance **(b)** Herculean equilibrium

Figure 8: Risk Dominance and herculean equilibrium. Both panels illustrate best response functions and equilibria when baseline payoff are those in Table 1b. Panel (a) shows best responses in the complete information game. Panel (b) show best responses in a linear incomplete information game, when types distribute $N(0, 1/2)$ and $\varepsilon = 1$.

Both scalars, risk factor and strength, are found by computing an 'indifferent-entry' condition. The risk factor of a player is the opponent's highest entry probability for which the player enters. Similarly, the strength of a firm is the opponent's highest entry probability (lowest entry cutoff) for which the player enters if it obtains a type equal to said cutoff. Whereas risk factor summarizes information on how the opponent's behavior affects a firm's payoffs, strength summarizes how the firm's type *and* the opponent's behavior affect the firm's payoffs. As a player payoff sensitivity to its type vanishes, the importance of the own type fades away. The notion of strength converges to the risk factor and the herculean equilibrium to the risk-dominant equilibrium.

We can also show that the herculean equilibrium is the equilibrium with the largest basin of attraction. To show this, suppose that the incomplete information game described above has three equilibria: two stable and one unstable, as shown by Figure 8b. By construction, the unstable equilibrium is the one in the middle. Further, we assume that the stable equilibria satisfy the following sufficient condition for equilibrium stability $b_i'(x_j)b_j'(x_i) < 1$ and the unstable equilibria satisfies $b_i'(x_j)b_j'(x_i) > 1$. In the context of an incomplete information game, we say that the basin of attraction of a stable equilibrium is its Euclidean distance with the unstable equilibrium. In Figure 8b, which is constructed using Table 1b, types that distribute $N(0, 1/2)$, and $\varepsilon = 1$, we can see that the herculean equilibrium H has the largest basin of attraction. The (unstable) middle equilibrium is to the right of the 45-degree line.

Proposition 4. *In an asymmetric game, the herculean equilibrium has the largest basing of attraction.*

The proof of the proposition shows that increasing the strength of a player decreases its equilibrium cutoff and increases that of the opponent when the equilibrium is stable. Conversely, the stronger player increases its equilibrium cutoff when the equilibrium is unstable, and the weaker decreases it. In the case of a symmetric game, this means that the middle equilibrium, located at the 45-degree line in Figure 8b, moves below the line as we make the game asymmetric, i.e., make player 1 stronger than player 2. The middle equilibrium moves towards the other non-herculean equilibrium (and vice versa), whereas the herculean equilibrium moves further away. We illustrate this situation in Figure 8b, the two non-herculean equilibria are closer to one another, whereas the herculean equilibrium is further apart.

5.2 Gradual Revelation of Information

Selective entry occurs when firms are partially or fully informed about their type before making their entry decisions. Recent empirical work on market entry has shown the need to account for selection beyond the privately fully-informed-firms model presented in Section 3 (c.f. Roberts and Sweeting, 2013). This is so, as the gradual revelation of information allows to account for outcomes such as *ex-post* regret.[25] We can extend our framework to account for a gradual revelation of information by adding a weak affiliation assumption between firms' private information, observed before entry, and their true type, observed after entry occurs.

Let $F_i(v_i, \theta_i)$ be firm i's joint cumulative distribution of *signals* v_i and *types* θ_i with support on $[\alpha, \beta] \times [c, d]$ with $c < d$. The distributions F_i are independent across firms and not necessarily identically distributed. Before making their costly entry decisions, a firm privately observes its signal v_i, which allows it to make inferences about its true type, θ_i. Firms learn their type after entering the market. Let $F_i(v_i) = \int_c^d F_i(v_i, s)ds$ and let $F_i(\theta_i|v_i) = F_i(v_i, \theta_i)/F_i(v_i)$ be the CDF of θ_i conditional on v_i.

A4 (Affiliated Signals): For $v_i' > v_i$, $F_i(\theta_i|v_i') < F_i(\theta_i|v_i)$ for all θ_i.

[25] For instance, the gradual revelation of information can account for the realization of a market with a sole entrant (the most profitable outcome under our assumptions) having negative post-entry profit. This outcome is precluded under privately fully-informed firms.

A4 states that higher signals lead to a higher expected type in terms of first order stochastic dominance (FOSD) (c.f. Marmer *et al.*, 2013; Gentry and Li, 2014). Let $\hat{\pi}_i(\theta_e)$ be firm i's profit under market structure e and the vector of types for every participating firm is $\theta_e = (\theta_j)_{j\in e}$. Then, we re-interpret $\pi_i(v_e)$ as

$$\pi_i(v_e) = \int_c^d \hat{\pi}_i(\theta_e) \prod_{k\in e} f_k(\theta_k|v_k) d^{n_e}\theta_e$$

where the integral is across the n_e dimensions of θ_e. Given the properties of FOSD, it is straightforward to see that if the profit function $\hat{\pi}_i(\theta_e)$ satisfies analogous conditions to A1-A3, then $\pi_i(v_e)$ would also satisfy A1-A3, and the results presented above go through.

The gradual information model includes, as a limiting case, the model introduced in Section 3. This model corresponds to a scenario in which the second signal degenerates at some value of θ_i, not providing new information. It also includes, as a limiting case, models à la Levin and Smith (1994), in which no information is revealed until entry has occurred. No revelation happens when the first signal is not informative (i.e., degenerate at some value of v_i). Although our results apply when the support of the first signal is small, they do not apply in the limiting case of an uninformative degenerate signal. Our assumptions require an interval as a support with a positive measure.

To exemplify the gradual information model and to illustrate how to use our results in this context, below we present an example using the framework of Roberts and Sweeting (2013, 2016). In particular, using their estimates for the mean auction, we show that their model has a unique equilibrium.

Example 7 (Auctions with gradual information)**.** Consider a second-price auction where bidders are partially (and privately) informed about their own valuation before making entry decisions.[26] The valuation of bidder i is given by $\theta_i = v_i\varepsilon_i$, where the *signal* v_i is observed before the participation decision is made and the *noise* $\varepsilon_i \sim G_i$, which is independent from v_i, is observed after paying the participation cost $K_i > 0$ but before submitting a bid.[27]

For a given realization of signals and market structure v_e, define $\Phi(s, v_e) =$

[26]Variations of this model have been studied by Roberts and Sweeting (2013, 2016), Gentry and Li (2014), and Sweeting and Bhattacharya (2015).

[27]If we further assume that $\mathbb{E}(\varepsilon_i) = 1$, the signal v_i becomes an unbiased predictor of the valuation θ_i. That is, for a given realization of v_i, the expected valuation is given by $\mathbb{E}(\theta_i|v_i) = v_i \int_{-\infty}^{\infty} \varepsilon_i dG_i(\varepsilon_i) = v_i$.

$\prod_{j\in e} G_j(s/v_j)$ to be the probability that every firm participating in market structure e obtains a valuation less than s. Then, if $r \geq 0$ is the reserve price of the auction, the (expected) payoff of a firm that participates under v_e is:

$$\pi_i(v_e) = \int_{r/v_i}^{\beta} \left(\int_{-\infty}^{v_i\varepsilon_i} (v_i\varepsilon_i - \max\{r, s\}) d\Phi\left(s, v_{e\setminus i}\right) \right) dG_i(\varepsilon_i) - K_i.$$

A participating firm i pays the entry cost K_i and, given the signal v_i, bidder i values the good by $\theta_i = v_i\varepsilon_i$, which distributes according to $G_i(\varepsilon)$. Participating firms submit a bid equal to their valuation only if they value the good more than the reserve price r. Bidder i obtains the good when it is the highest valuation firm. The distribution of the highest valuation among i's opponents is $\Phi(s, v_e)$. It can be readily checked that this model satisfies assumptions A1-A4.

Roberts and Sweeting (2013, 2016) use the previous model to study the USFS timber auctions. The auction consists of two groups of potential entrants, millers and loggers (groups 1 and 2, respectively). Before entry, each firm observes a signal $v_i = \theta_i\varepsilon_i$, where θ_i is firm i's valuation for a tract and ε_i represents the signal's noise. For the representative (mean) auction they estimate $\ln\theta_i \sim N(\mu_{g(i)}, 0.3321)$ (with $\mu_1 = 3.9607$ and $\mu_2 = 3.5824$) and $\ln\varepsilon_i \sim N(0, 0.8579)$. The estimated entry costs is \$2.0543/mfb (dollars per thousand board foot) and the auction's reserve price is \$27.77/mfb.[28] Searching numerically, they found a single equilibrium. We prove, for the representative auction, that the game indeed has a unique equilibrium. In the scenario, with two asymmetric entrants $n_1 = n_2 = 1$, we find that the left-hand side of condition (11) for millers and loggers are 0.2104 and 0.0017 (both less than one); as a consequence, the game has a unique equilibrium. Online Appendix G offers details on the computations and a discussion of strength and herculean equilibrium for this auction.

6 Concluding Remarks

In the context of entry games with private information, this article introduces the notions of strength and herculean equilibrium. In various environments of applied interest, we showed that a herculean equilibrium always exists and developed a sufficient condition guaranteeing equilibrium uniqueness. The proposed framework embeds most of the existing models studied in the (static) entry literature,

[28]See Tables 3 or 4 in Roberts and Sweeting (2013, 2016), respectively.

accommodating firm heterogeneity and selection. With the aid of strength, we can identify the herculean equilibrium; the type of equilibrium that remains when the game has a unique equilibrium. Strength can reduce the computational burden of calculating equilibria with heterogeneous firms, as it provides bounds for the herculean equilibrium. We showed that strength and herculean equilibrium are incomplete information analogs to the risk factor and risk-dominant equilibrium in complete information games. Consequently, when the game has multiple equilibria, we can use these notions as a selection criterion.

When further exploring the set of sufficient conditions for equilibrium uniqueness, we put special emphasis to models in which private information enters the payoffs linearly. The linear model is the most common model used in the applied literature. There, the proposed conditions reduce to a set of simple calculations. The conditions provide clear intuitions on how competition among firms affects the possibility of having a unique equilibrium. We used our sufficient conditions jointly with the estimates in empirical articles to illustrate that their empirical models have a unique equilibrium, demonstrating the usefulness of the results.

The focus of this article is on static entry games with private information. We emphasized developing a framework that embeds most of the applied work on endogenous market formation. Beyond the presented results, we see these new developments as the starting point for studying equilibrium uniqueness in dynamics games with entry. We hope the tools developed here enable further research in dynamic environments.

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Appendix

A Omitted Proofs

Proof of Proposition 1. It follows from using Theorem 1 in the context of second-price auctions. A direct proof is presented in the online Appendix. ■

Proof of Lemma 2. The proof of both statements make use that a concave differentiable function is bounded above by its first-order Taylor approximation; i.e., for every x and y such that $x > y$

$$F(x) - F(y) \geq (x - y)f(x). \tag{A.1}$$

The first claim follows from taking $y = 0$ and using $F(0) = 0$. For the second statement, let y in equation (A.1) be the inflection point under which $F_i(v)$ becomes concave. Because of concavity, $f_i(x)$ is non increasing for every $x \geq y$. Because F_i is bounded above (by 1), $f_i(x)$ converges to zero as x goes to infinity. Let $c_j^* > y$ be the valuation that satisfies $F_i(y) = f_i(c_j^*)y$. Then, for every $x \geq c_j^*$ we have: $F_i(x) \geq xf_i(x) + F_i(y) - yf_i(x) \geq xf_i(x) + F_i(y) - yf_i(c_j^*) \geq xf_i(x)$. ■

Proof of Lemma 3. We show that s_i exists and that

$$\sigma_i(s) \equiv \Pi_i(s, \ldots, s) \tag{A.2}$$

single crosses zero.
Existence: Observe that assumptions A3 and A2 jointly imply $\sigma_i(\underline{v}_i) < 0$. Similarly, assumption A3 and Lemma B.1 (see Appendix B) imply, $\sigma_i(\overline{v}_i) \geq \Pi_i(\overline{v}_i, \alpha_{-i})$ > 0 (where α is the lower bound of the support of F_i). Then, by the intermediate value theorem, there exist $\hat{s}$ such that $\sigma_i(\hat{s}) = 0$.
Uniqueness: By Lemma B.1 and the chain rule, we have that $\sigma_i'(s) > 0$. Thus, $\sigma_i(s)$ single crosses zero; i.e., there is a unique value s_i satisfying $\sigma_i(s_i) = 0$. ■

Proof of Proposition 2. This proof makes use of Lemma A.1, presented below.
Lemma A.1. *Under condition (8), two symmetric firms that best respond to each other must play the same cutoff strategy.*
Proof. Consider two symmetric firms, p and q, and fix *any* profile of cutoffs strategies $x_{E\setminus\{p,q\}}$ for the rest of the firms. The equilibrium condition for firm p holds whenever there exists x_p and x_q such that $\Pi_p(x_p, x_q, x_{E\setminus\{p,q\}}) = 0$. Define $b(x_p)$ to be firm q's best response to x_p (and to $x_{E\setminus\{p,q\}}$, which is fixed throughout the proof). By Lemma B.2 in Appendix B, $b(x_p)$ exists and is uniquely defined for each x_p. To prove the Lemma we need to prove three claims.
Claim 1. There exists a unique pair of symmetric cutoffs, $x_p = x_q = z$, such that $\Pi_p(z, z, x_{E\setminus\{p,q\}}) = 0$.
Proof. Suppose firms p and q play a symmetric cutoff, $x_p = x_q = z$. Define $\hat{\sigma}(z) = \Pi_p(z, z, x_{E\setminus\{p,q\}}) = \Pi_q(z, z, x_{E\setminus\{p,q\}})$, where the last equality follows from

symmetry among firms. Thus, if the equilibrium condition is satisfied by firm p, it is also satisfied by firm q. We want to show there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Following analogous steps to those in Lemma 3, it is easy to show $\hat{\sigma}(\underline{v}_p) < 0$ and $\hat{\sigma}(\overline{v}_p) > 0$; so that, there exists $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Similarly, using Lemma B.1 and the chain rule, we can show that $\hat{\sigma}'(z) > 0$. Hence, the value $\hat{z}$ is unique. □

Claim 2. Under condition (8):[29] $0 > b'(x_p) > -\frac{f(x_p)}{F(x_p)}\frac{F(b(x_p))}{f(b(x_p))}$.

Proof. Let $\mathbf{x} = (x_p, b(x_p), x_{E\setminus\{p,q\}})$. Using implicit differentiation and equations (B.1) and (B.2) from Lemma B.1, we obtain

$$b'(x) = -\left.\frac{\partial \Pi_q(b(x_p), x_p, x_{E\setminus\{p,q\}})}{\partial x_p} \middle/ \frac{\partial \Pi_q(b(x_p), x_p, x_{E\setminus\{p,q\}})}{\partial x_q}\right. = -\frac{f(x_p)}{F(x_p)}\frac{\Delta_{q,p}(\mathbf{x})}{\Pi_q'(\mathbf{x})}$$

which is negative as the denominator and numerator are positive. To obtain the lower bound for $b'(x_p)$ simply use condition (8). □

Claim 3. An increase in x_p, when firm q best responds by playing $b(x_p)$, leads firm p to strictly increase its profit; i.e., $\Pi_p(x_p, b(x_p), x_{E\setminus\{p,q\}})$ is increasing in x_p.

Proof. Differentiating $\Pi_p(x_p, b(x_p), x_{E\setminus\{p,q\}})$ with respect to x_p, using the chain rule, and equations (B.1) and (B.2) we obtain

$$\begin{aligned}\frac{d\Pi_p}{dx_p} &= \frac{\partial \Pi_p}{\partial x_p} + \frac{\partial b(x_p)}{\partial x_p}\frac{\partial \Pi_p}{\partial x_q} \\ &= \Pi_p'(\mathbf{x}) + \frac{\partial b(x_p)}{\partial x_p}\frac{f(b(x_p))}{F(b(x_p))}\Delta_{p,q}(\mathbf{x}) > \Pi_p'(\mathbf{x}) - \frac{f(x_p)}{F(x_p)}\Delta_{p,q}(\mathbf{x}) > 0,\end{aligned}$$

where $\mathbf{x} = (x, b(x), x_{E\setminus\{p,q\}})$. The first inequality follows from Claim 2, whereas the second from condition (8); which proves the claim. □

We prove Lemma A.1 by contradiction. Recall that $x_{E\setminus\{p,q\}}$ is fixed throughout the proof. Suppose, without loss of generality, that there exists $x_q < x_p$ constituting an equilibrium. By Claim 1 there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$.

Suppose first $x_q < \hat{z} < x_p$. Because

$$\hat{\sigma}(\hat{z}) = \Pi_p(\hat{z}, \hat{z}, x_{E\setminus\{p,q\}}) = \Pi_p(\hat{z}, b(\hat{z}), x_{E\setminus\{p,q\}}) = 0,$$

and $x_p > \hat{z}$, Claim 3 implies that we must have $\Pi_p(x_p, b(x_p) = x_q, x_{E\setminus\{p,q\}}) > 0$; which contradicts (x_p, x_q) being an equilibrium.

Suppose now $x_q < x_p < \hat{z}$. Lemma B.1 and Claim 1 imply

$$0 = \hat{\sigma}(\hat{z}) > \hat{\sigma}(x_p) = \Pi_p(x_p, x_p, x_{E\setminus\{p,q\}}) > \Pi_p(x_p, b(x_p) = x_q, x_{E\setminus\{p,q\}})$$

which contradicts (x_p, x_q) being an equilibrium. Analogous argument can be constructed for the case $\hat{z} < x_q < x_p$, proving the Lemma. ■

[29]For ease in notation, we use symmetry, and drop the sub-indexes from F when referring to firms p and q.

To prove Proposition 2 observe: (i) By Lemma 3, there exists a unique value of strength and, therefore, a unique symmetric equilibrium, which also corresponds to the unique herculean equilibrium. (ii) If there is another equilibrium, it must be under asymmetric cutoffs. In the hypothetical asymmetric equilibrium, let p and q be two symmetric firms playing asymmetric cutoffs. Let $x_{E\setminus\{p,q\}}$ be the equilibrium cutoffs of the opponents. The vector $(x_p, x_q, x_{E\setminus\{p,q\}})$ contradicts Lemma A.1. ■

Proof of Theorem 1. *Proof preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume, without loss of generality, that $s_1 < s_2$. Let $\hat{\mathbf{x}} = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm in group $i \in \{1, 2\}$ and let $\hat{\Pi}_i(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ represent the expected profit of a firm in group i of entering under valuation x_i when firms play group-symmetric strategies x_1 and x_2. Observe that the function $\hat{\Pi}_i(x_1, x_2)$ has a two-dimensional domain, taking as input the group-symmetric strategy of each group. Define $b_1(x)$ to be the function that solves $\hat{\Pi}_1(b_1(x), x) = 0$. Thus, $b_1(x)$ corresponds to group one's symmetric best response to group two playing the group-symmetric cutoff x. By Lemma B.2, the value $b_1(x)$ exists and is unique; i.e., $b_1(x)$ is well defined.

Claim 4. $b_1(s_1) = s_1$ and, under condition (11), $0 > b_1'(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(b_1(x))}{f_1(b_1(x))}$.

Proof. By definition of strength we know $\hat{\Pi}_1(s_1, s_1) = 0$, therefore $b_1(s_1) = s_1$. Using implicit differentiation, the chain rule, that groups members are symmetric, and Lemma B.1

$$b_1'(x) = -\frac{\frac{\partial \hat{\Pi}_1(b_1(x),x)}{\partial x_2}}{\frac{\partial \hat{\Pi}_1(b_1(x),x)}{\partial x_1}} = \frac{-n_2 \frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_{j\in G_2}}}{\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_1} + (n_1 - 1)\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_{j\in G_1}}} = \frac{-n_2 \frac{f_2(x_2)}{F_2(x_2)}\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}}) + (n_1 - 1)\frac{f_1(x_1)}{F_1(x_1)}\Delta_{1,1}(\hat{\mathbf{x}})}$$

where $\Delta_{i,j}(\mathbf{x})$, defined in equation (7), represents the profit gain of firm i when a firm in group j exits the market with type x_j. Because numerator and denominator are positive, the equation above proves $b_1'(x) < 0$ for all x. For the lower bound of $b_1(x)$ observe that $\Delta_{1,1} > 0$. Take a lower bound for $b_1'(x)$ by making $\Delta_{1,1}$ zero. The lower bound $b_1'(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(b_1(x))}{f_1(b_1(x))}$ follows by using sufficient condition (11). □

Existence of a herculean equilibrium: Recall that $[\alpha, \beta]$ is the support of F_i. Define the function $h_2 : [\alpha, \beta] \to \mathbb{R}$ by $h_2(x) = \hat{\Pi}_2(b_1(x), x)$. This function is continuous and corresponds to the expected profit of a firm in group 2 when it enters the market under valuation x, group two plays the group-symmetric cutoff x, and group one plays their group-symmetric best response $b_1(x)$. Define x_2 to be the value satisfying $h_2(x_2) = 0$ and let $x_1 = b_1(x_2)$. The next two claims prove that an herculean equilibrium ($x_1 < x_2$) exists, $x_1 < s_1$, and $x_2 > s_2$.

Claim 5. $x_2 \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$.

Proof. Because $b_1(x)$ is decreasing in x and $b_1(s_1) = s_1$, we have that $x_1 = b_1(x_2) < s_1 < x_2$ if and only if $x_2 > s_1$. □

Claim 6. $h_2(s_2) < 0$ and there exists $\tilde{x} > s_2$ such that $h_2(\tilde{x}) > 0$. Thus, by the

intermediate value theorem, the herculean equilibrium cutoff $x_2 \in (s_2, \tilde{x})$ exists.
Proof. Because group two is weak, and $b_1(x)$ is decreasing in x, we know that $b_1(s_2) < b_1(s_1) = s_1 < s_2$ (where Claim 4 was used in the equality). Lemma B.1 and the definition of strength implies $h_2(s_2) = \hat{\Pi}_2(b_1(s_2), s_2) < \hat{\Pi}_2(s_2, s_2) = 0$, proving $h_2(s_2) < 0$. For the second part of the claim, observe that, by Lemma B.1, $\hat{\Pi}_2(x_1, x_2)$ is increasing in x_1; then, $\hat{\Pi}_2(b_1(x), x) \geq \hat{\Pi}_2(\alpha, x)$ for all x. Take $\tilde{x} = \overline{v}_2$ and observe that, by assumption A3, $\hat{\Pi}_2(\alpha, \tilde{x}) > 0$, proving the result. □

Uniqueness of equilibrium: Start by observing that, under condition (10), Lemma A.1 applies. Therefore, it is without loss to restrict the analysis to group-symmetric strategies. To prove uniqueness, then, we need to show that no other herculean equilibrium exists and that we can not have an equilibrium where $x_2 < x_1$.

Claim 7. There exists a unique herculean equilibrium.
Proof. To prove uniqueness within the herculean class, we shown $h_2'(x) > 0$ so that $h_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (b_1(x), \ldots, b_1(x), x, \ldots, x)$. Differentiating $h_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain

$$h_2'(x) = \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{\partial \Pi_2'(\hat{\mathbf{x}})}{\partial x_2} + b'(x) n_1 \frac{\partial \Pi_2'(\hat{\mathbf{x}})}{\partial x_1}$$
$$> \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) - n_1 \frac{f_2(x)}{F_2(x)}\Delta_{2,1}(\hat{\mathbf{x}}) > (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) > 0,$$

where the first inequality follows from using Lemma B.1 and the bound in Claim 4. The second inequality follows from sufficient condition (11). Proving that the derivative is positive and uniqueness within the herculean class. □

Claim 8. There is no group-symmetric equilibrium in which the strong group plays a higher cutoff than the weak group.
Proof. We show that no non-herculean equilibrium—i.e., $x_1 > x_2$ but $s_1 < s_2$—can exist. Define $b_2(x)$ to be the function that satisfies $\hat{\Pi}_2(x, b_2(x)) = 0$; $b_2(x)$ corresponds to group two's best response to the cutoff of group one when $x_1 = x$. As before, Lemma B.2 implies that $b_2(x)$ is well defined. Similarly, following the steps of Claim 4, it can be shown: $b_2(s_2) = s_2$, $b_2'(x) < 0$, and, under condition (11), $b_2'(x)$ is bounded below by $-\frac{f_1(x)F_2(b_2(x))}{F_1(x)f_2(b_2(x))}$.

Define the continuous function $h_1(x) = \hat{\Pi}_1(x, b_2(x))$ which corresponds to the expected profit of a firm in group one when entering the market under valuation x and its opponents play the pair of group-symmetric strategies $(x, b_2(x))$. We show that there is no x satisfying $x_1 = x > b_2(x) = x_2$ and $h_1(x) = 0$; i.e., no non-herculean equilibrium exists. Start by observing that $x > b_2(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 3 we showed the function $\sigma_1(s) = \hat{\Pi}_1(s, s)$ is strictly increasing in s. Then, by the definition of strength and by firm two being weak $(s_1 < s_2)$,

$$\sigma_1(s_1) = \hat{\Pi}_1(s_1, s_1) = 0 < \sigma_1(s_2) = \hat{\Pi}_1(s_2, s_2) = \hat{\Pi}_1(s_2, b_2(s_2)) = h_1(s_2),$$

showing that $h_1(s_2) > 0$. Following analogous steps to those in Claim 7, which requires using lower bound for $b_2'(x)$ and sufficient condition (11), we can show that $h_1'(x) > 0$. Then, because $h_1(s_2) > 0$ and $h_1'(x) > 0$ for all x, $h_1(x)$ never crosses zero when $x > s_2$ and the result follows. □ ■

Proof of Lemma 4. We start by showing the order in the context of ordered by profit. Let s_i be the strength of firm i, using equation (A.2) we obtain

$$\begin{aligned} 0 = \sigma_i(s_i) &= \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi_i \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} \\ &> \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi_{i+1} \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} = \sigma_{i+1}(s_i), \end{aligned}$$

where in the inequality we used $\pi_i(v, \mathbf{v}_{n_e-1}) > \pi_{i+1}(v, \mathbf{v}_{n_e-1})$. In the last equality, after changing the firm's identity, we used $\mathcal{E}_i = \mathcal{E}_{i+1}$. Then, by Lemma 3, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$.

For games ordered by distributions, rewriting equation (A.2) we obtain

$$\begin{aligned} 0 = \sigma_i(s_i) = &\sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left(\prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus \{i,i+1\}}) f_{i+1}(v_{i+1}) d^{n_e - 1} v_{e \setminus i} \right\} + \\ &\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left(F_{i+1}(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} \\ > &\sum_{e \in \mathcal{E}_i \cap \mathcal{E}_{i+1}} \left\{ \left(\prod_{j \in e^c} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus \{i,i+1\}}) f_i(v_{i+1}) d^{n_e - 1} v_{e \setminus i} \right\} + \\ &\sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}} \left\{ \left(F_i(s_i) \prod_{j \in e^c \setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in e \setminus i}}^{\beta} \pi \left(s_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} = \sigma_{i+1}(s_i), \end{aligned}$$

where the inequality uses two properties of FOSD. In the second term we used $F_i(v) < F_{i+1}(v)$. In the first term, we used $\int_{s_i}^{b} h(v) f_i(v) dv \leq \int_{s_i}^{b} h(v) f_j(v) dv$ for any non-increasing function $h(x)$. Then, by Lemma 3, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$. ■

Proof of Theorem 2. We present the proof when firms are ordered by distributions. The proof when firms are ordered by profit is, basically, identical but we can drop the subindices from the distribution functions. Using Lemma 4 we order firms using stochastic dominance, from stronger (firm 1) to weakest (firm n).

Existence of an herculean equilibrium. We prove existence by construction. For any vector of cutoff strategies $\mathbf{x}$ and $k \in \{2, \dots, n\}$ let $\mathbf{x}^k = (x_k, x_{k+1}, \dots, x_n)$. Construct the equilibrium vector sequentially, as follows:

- **Firm** 1: Define $b_1^1(\mathbf{x}^2)$ to be firm's 1 best response to $\mathbf{x}^2$; i.e., $b_1^1(\mathbf{x}^2)$ satisfies

$$\Pi_1(b_1^1(\mathbf{x}^2), \mathbf{x}^2) = 0.$$

where $\Pi_1(\mathbf{x})$ is defined in (6). By Lemma B.2 in the Auxiliary Result section, $b_1^1(\mathbf{x}^2)$ exists and (the best response) is unique and continuous.

- **Firm** 2: Let $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(b_1^1(\mathbf{x}^2), \mathbf{x}^2)$; that is, $\hat{\Pi}_2(\mathbf{x}^2)$ represents firm's 2 profit after incorporating that firm 1 is best responding to $\mathbf{x}^2$. Define $b_2^2(\mathbf{x}^3)$ to be a solution to $\hat{\Pi}_2(b_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. By Lemma B.2, $b_2^2(\mathbf{x}^3)$ exists and is continuous in each dimension of $\mathbf{x}^3$. This function represents firm's 2 best response when firms 1 and 2 are mutually best responding to each other and to $\mathbf{x}^3$. For ease in notation, denote firm's 1 best response after incorporating firm's 2 best response as $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$.[30] This function is also continuous in each dimension of $\mathbf{x}^3$.

  Claim 9. For any $\mathbf{x}^3$, $b_2^2(\mathbf{x}^3) > b_1^2(\mathbf{x}^3)$.

  Proof. Fix $\mathbf{x}^3$ and find the value $\hat{x}$ that satisfies $\hat{x} = b_1^1(\hat{x}, \mathbf{x}^3)$. The value $\hat{x}$ exists by continuity of $b_1^1(\mathbf{x}^2)$ and by $b_1^1(\mathbf{x}^2)$ being bounded below and above by $\underline{v}_1$ and $\overline{v}_1$ respectively (by assumption A3). Then by Lemma B.3 in the auxiliary results section we have $\Pi_2(\hat{x}, \hat{x}, \mathbf{x}^3) < \Pi_1(\hat{x}, \hat{x}, \mathbf{x}^3) = 0$. Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma B.2) value that solves $\Pi_2(z_m, y_{m+1}, \mathbf{x}^3) = 0$ (i.e., y_{m+1} is firm's 2 best response to the cutoffs $(z_m, \mathbf{x}^3)$) and; (iii) $z_{m+1} = b_1^1(y_{m+1}, \mathbf{x}^3)$. By definition, z_{m+1} solves $\Pi_1(z_{m+1}, y_{m+1}, \mathbf{x}^3) = 0$ and, by Lemma B.2, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_2(\hat{x}, \hat{x}, \mathbf{x}^3) < 0$ and $\Pi_2(\mathbf{x})$ being strictly increasing in the 2nd dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.1) $\Pi_1(\mathbf{x})$ is also increasing in the 2nd dimension, we have $\Pi_1(z_1, y_2, \mathbf{x}^3) > 0$, which implies $z_2 = b_1^1(y_2, \mathbf{x}^3) < z_1 = b_1^1(y_1, \mathbf{x}^3)$. This, in turn, implies (by Lemma B.3)

$$\Pi_2(z_2, y_2, \mathbf{x}^3) < \Pi_1(z_2, y_2, \mathbf{x}^3) = 0;$$

  which implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_2$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_1$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = b_1^1(y_\infty, \mathbf{x}^3)$ and; (ii) $\Pi_2(z_\infty, y_\infty, \mathbf{x}^3) = \hat{\Pi}_2(y_\infty, \mathbf{x}^3) = 0$ (i.e., $y_\infty = b_2^2(\mathbf{x}^3)$). Thus, $b_1^1(y_\infty, \mathbf{x}^3) = b_1^2(\mathbf{x}^3)$ and, as $z_\infty < \hat{x} < y_\infty$, we have $b_2^2(\mathbf{x}^3) > b_1^2(\mathbf{x}^3)$. □

- **Firm** $k \le n$: Suppose we have shown the existence of $b_\ell^\ell(\mathbf{x}^{\ell+1})$ for every $\ell \in \{1, \dots, k-1\}$, have recursively defined $b_j^\ell(\mathbf{x}^{\ell+1}) = b_j^{\ell-1}(b_\ell^\ell(\mathbf{x}^{\ell+1}), \mathbf{x}^{\ell+1})$ for $j \in \{1, \dots, \ell\}$, and that both constructions are continuous. Let $\hat{\Pi}_k(\mathbf{x}^k) =$

[30] More generally, for $j < k$ the notation $b_j^k(\mathbf{x}^k)$ represents firm's j best response to $\mathbf{x}^k$ after incorporating the best response of every firm up to k.

$\Pi_k(b_1^{k-1}(\mathbf{x}^k), \dots, b_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$ represent firm's k profit after incorporating that every firm $j \in \{1, \dots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$. Define $b_k^k(\mathbf{x}^{k+1})$ (observe that b_n^n is a number, not a function, as $\mathbf{x}^{k+1}$ is empty when $k = n$) to be a solution to $\hat{\Pi}_k(b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. By Lemma B.2, $b_k^k(\mathbf{x}^k)$ exists and is continuous in each dimension of $\mathbf{x}^k$. This function represents firm's k best response to $\mathbf{x}^{k+1}$ when every firm $j \in \{1, \dots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$.

Claim 10. For any $\mathbf{x}^{k+1}$, if firm $k-1$ is stronger than k the solution $b_k^k(\mathbf{x}^{k+1})$ satisfies $b_k^k(\mathbf{x}^{k+1}) > b_{k-1}^k(\mathbf{x}^{k+1})$.

Proof. Fix any $\mathbf{x}^{k+1}$ and let $b_k^k(\mathbf{x}^{k+1})$ be one of the solutions found in the previous step. Then define the vector of cutoffs $\mathbf{x} = \big(b_1^k(\mathbf{x}^{k+1}), \dots, b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\big)$. Throughout the proof, the vector of strategies for every firm except firm k and $k-1$, $\mathbf{x}_{-k,k-1}$, remains fixed (i.e., they are numbers not functions). Define $\hat{x}$ to be a value satisfying $\hat{x} = b_{k-1}^{k-1}(\hat{x}, \mathbf{x}^{k+1})$. The value $\hat{x}$ exists by continuity of $b_{k-1}^{k-1}(\mathbf{x}^k)$ and by $b_{k-1}^{k-1}(\mathbf{x}^k)$ being bounded below and above by $\underline{v}_{k-1}$ and $\overline{v}_{k-1}$ respectively (by assumption A3). By definition of best response $\hat{x}$ satisfies $\Pi_{k-1}(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) = 0$. Then, by Lemma B.3, we have

$$\Pi_k(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) < \Pi_{k-1}(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) = 0.$$

Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma B.2) value that solves $\Pi_k(z_m, y_{m+1}, \mathbf{x}_{-k,k-1}) = 0$ (i.e., y_{m+1} is firm's k best response to the cutoffs $(z_m, \mathbf{x}_{-k,k-1})$) and; (iii) $z_{m+1} = b_{k-1}^{k-1}(y_{m+1}, \mathbf{x}^{k+1})$. By definition, z_{m+1} solves $\Pi_{k-1}(z_{m+1}, y_{m+1}, \mathbf{x}_{-k,k-1}) = 0$ and, Lemma B.2, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_k(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) < 0$ and $\Pi_k(\mathbf{x})$ being strictly increasing in the kth dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.1) $\Pi_{k-1}(\mathbf{x})$ is also increasing in the kth dimension, we have $\Pi_{k-1}(\hat{x}, y_2, \mathbf{x}_{-k,k-1}) > 0$, which implies $z_2 = b_{k-1}^{k-1}(y_2, \mathbf{x}^{k+1}) < b_{k-1}^{k-1}(y_1, \mathbf{x}^{k+1}) = z_1$. This, in turn, implies (by Lemma B.3)

$$\Pi_k(z_2, y_2, \mathbf{x}_{-k,k-1}) < \Pi_{k-1}(z_2, y_2, \mathbf{x}_{-k,k-1}) = 0,$$

which, in turns, implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A3, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_k$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_{k-1}$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = b_{k-1}^{k-1}(y_\infty, \mathbf{x}^{k+1})$ and; (ii) $\Pi_k(z_\infty, y_\infty, \mathbf{x}_{-k,k-1}) = \hat{\Pi}_k(y_\infty, \mathbf{x}_{-k,k-1}) = 0$ (i.e., $y_\infty = b_k^k(\mathbf{x}^{k+1})$). Thus, $b_{k-1}^{k-1}(y_\infty, \mathbf{x}^{k+1}) = b_{k-1}^k(\mathbf{x}^{k+1})$ and, as $z_\infty < \hat{x} < y_\infty$, we have $b_k^k(\mathbf{x}^{k+1}) > b_{k-1}^k(\mathbf{x}^{k+1})$. □

Thus, we have constructed an equilibrium vector $\mathbf{x} = (b_1^n(x_n), \dots, b_{n-1}^n(x_n), x_n)$ with the property that $x_i < x_{i+1}$; i.e., a Herculean equilibrium.

Uniqueness within the herculean-equilibrium class: We show that at each step k of the previous construction there is a unique best response $x_k = b_k^k(\mathbf{x}^{k+1})$ to $\mathbf{x}^{k+1}$.

- **Firm** 1: The uniqueness of $b_1^1(\mathbf{x}^2)$ follows from Lemma B.2. The next result is needed for subsequent steps.

Claim 11. Under condition (13), for every $j \in \{2,\dots,n\}$, $\partial b_1^1(\mathbf{x}^2)/\partial x_j$ satisfies:

$$0 > \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} = -\frac{f_j(x_j)}{F_j(x_j)}\frac{\Delta_{1,j}(\mathbf{x})}{\Pi_1'(\mathbf{x})} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_1(x_1)}{f_1(x_1)}\frac{1}{n-1}. \tag{A.3}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{2,\dots,j-1\} \tag{A.4}$$

Proof. Let $\mathbf{x} = (b_1^1(\mathbf{x}^2), \mathbf{x}^2)$; using implicit differentiation and Lemma B.1 we obtain

$$\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} = -\frac{\partial \Pi_1(\mathbf{x})/\partial x_j}{\partial \Pi_1(\mathbf{x})/\partial x_1} = -\frac{f_j(x_j)}{F_j(x_j)}\frac{\Delta_{1,j}(\mathbf{x})}{\Pi_1'(\mathbf{x})}, \tag{A.5}$$

which is negative as, $\Delta_{1,j}(\mathbf{x}) > 0$ and $\Pi_{1,j}(\mathbf{x}) > 0$ for every $\mathbf{x}$. The lower bound in equation (A.3) follows from applying condition (13) into equation (A.5). Property (A.4) follows from observing

$$\frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} = \frac{1}{\Pi_1'(\mathbf{x})}\left(\Delta_{1,j}(\mathbf{x}) - \frac{\Delta_{1,q}(\mathbf{x})}{n-1}\right) > 0,$$

where the equality follows from substituting in equation (A.5), and the inequality follows from Lemma B.4 and the fact that $q \in \{2,\dots,j-1\}$. □

- **Firm** 2: Fix $\mathbf{x}^3$ and let $\mathbf{x} = (b_1^1(\mathbf{x}^2), \mathbf{x}^2)$, we need to show that the best response $b_2^2(\mathbf{x}^3)$ is unique. We do this by showing that $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(b_1^1(\mathbf{x}^2), \mathbf{x}^2)$ is strictly increasing in x_2; so that, $\hat{\Pi}_2(x_2, \mathbf{x}^3)$ single crosses zero and there is a unique value $b_2^2(\mathbf{x}^3)$ satisfying $\hat{\Pi}_2(b_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. Using the chain rule and equation (B.2)

$$\begin{aligned}\hat{\Pi}_2'(\mathbf{x}^2) = \Pi_2'(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial \Pi_2}{\partial x_1} &= \Pi_2'(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{f_1(b_1^1(\mathbf{x}^2))}{F_1(b_1^1(\mathbf{x}^2))}\Delta_{2,1}(\mathbf{x}) \\ &> \Pi_2'(\mathbf{x}) - \frac{f_2(x_2)}{F_2(x_2)}\frac{\Delta_{2,1}(\mathbf{x})}{n-1} > 0,\end{aligned} \tag{A.6}$$

where in the first inequality follows from the lower bound in equation (A.3) and the second inequality follows from sufficient condition (13). This proves uniqueness of the best response. The next result is needed for the induction argument in the proof.

Claim 12. Let $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$. Under condition (13), for every $j \in \{3,\dots,n\}$ and $\ell \in \{1,2\}$, $\partial b_\ell^2(\mathbf{x}^3)/\partial x_j$ satisfies:

$$\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_j} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{2,j}(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})} \tag{A.7}$$

$$0 > \frac{\partial b_\ell^2(\mathbf{x}^3)}{\partial x_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_\ell(x_\ell)}{f_\ell(x_\ell)}\frac{1}{n-1} \quad \text{and,} \tag{A.8}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{3,\ldots,j-1\} \tag{A.9}$$

Proof. To show equation (A.7) use implicit differentiation, the chain rule, and equation (B.2) to obtain

$$-\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_j} = \frac{\frac{\partial \hat{\Pi}_2}{\partial x_j}}{\frac{\partial \hat{\Pi}_2}{\partial x_2}} = \frac{\frac{\partial \Pi_2}{\partial x_j} + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}\frac{\partial \Pi_2}{\partial x_1}}{\Pi_2'(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial \Pi_2}{\partial x_1}} = \frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{2,j}(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}$$

Observe, by equation (A.6), that the denominator is positive. Using lower bound (A.3) and Lemma B.4 we can see that the numerator is also positive, implying that $\partial b_2^2(\mathbf{x}^3)/\partial x_j$ is negative; which proves the upper bound of (A.8) when $\ell = 2$. For the lower bound of equation (A.8) when $\ell = 2$, using equation (A.7), observe that equation (A.8) holds if and only if the following expression is positive:

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\left(\frac{F_2(x_2)}{f_2(x_2)}\frac{\Pi_2'(\mathbf{x})}{n-1} - \Delta_{2,j}(\mathbf{x})\right) + \frac{f_1(x_1)}{F_1(x_1)}\left(\frac{F_2(x_2)}{f_2(x_2)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}\right)\Delta_{2,1}(\mathbf{x})\right].$$

The first round bracket is positive by sufficient condition (13). The second round bracket is positive by property (A.4). Thus, the expression is indeed positive and the lower bound in equation (A.8) holds.

We now prove the bounds of (A.8) when $\ell = 1$. Using $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$, observe

$$\frac{\partial b_1^2(\mathbf{x}^3)}{\partial x_j} = \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_j}. \tag{A.10}$$

Using (A.5) to substitute for $\partial b_1^1(\mathbf{x}^2)/\partial x_\ell$ with $\ell \in \{2, j\}$ and using the lower bound in equation (A.8) when $\ell = 2$, we obtain the following upper bound:

$$\frac{\partial b_1^2(\mathbf{x}^3)}{\partial x_j} < \frac{f_j(x_j)}{F_j(x_j)}\frac{1}{\Pi_1'(\mathbf{x})}\left[\frac{\Delta_{1,2}(\mathbf{x})}{n-1} - \Delta_{1,j}(\mathbf{x})\right] < 0,$$

the inequality follows from Lemma B.4; proving the upper bound. The lower bound in equation (A.8) follows from using equation (A.10) and observing

$$\frac{\partial b_1^2(\mathbf{x}^3)}{\partial x_j} > \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j} > -\frac{f_j(x_j)}{F(x_j)}\frac{F_1(x_1)}{f_1(x_1)}\frac{1}{n-1},$$

where the inequalities follow from $\partial b_2^2(\mathbf{x}^3)/\partial x_j \cdot \partial b_1^1(\mathbf{x}^2)/\partial x_2 > 0$ and equation (A.3), respectively.

Finally, to prove property (A.9) use equation (A.7) to write

$$\frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_q}\frac{1}{n-1}-\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_2^2(\mathbf{x}^3)}{\partial x_j}=\frac{1}{D_2}\Bigg[\Delta_{2,j}(\mathbf{x})-\frac{\Delta_{2,q}(\mathbf{x})}{n-1}+ \frac{f_1(x_1)}{F_1(x_1)}\left(\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}-\frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_q}\frac{1}{n-1}\right)\Delta_{2,1}(\mathbf{x})\Bigg],$$

where $D_2=\Pi_2'(\mathbf{x})+\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})>0$. We show that a lower bound of this expression is positive. Taking $-\partial b_1^1(\mathbf{x}^2)/\partial x_q>0$ to zero, we obtain

$$\frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x})-\frac{\Delta_{2,q}(\mathbf{x})}{n-1}+\frac{f_1(x_1)}{F_1(x_1)}\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_j}\Delta_{2,1}(\mathbf{x})\right] > \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x})-\frac{\Delta_{2,q}(\mathbf{x})}{n-1}-\frac{\Delta_{2,1}(\mathbf{x})}{n-1}\right] > \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x})-\frac{2\Delta_{2,q}(\mathbf{x})}{n-1}\right]>0.$$

The first inequality follows from using lower bound (A.3). The other two inequalities follow from Lemma B.4 and the fact that $q\in\{2,\ldots,j-1\}$. ☐

- **Firm** $k\in\{3,\ldots,n\}$: Suppose that, for every $p\in\{1,\ldots,k-1\}$ and $j\in\{p+1,\ldots,n\}$, we have proven that: $b_p^p(\mathbf{x}^{p+1})$ is unique;

$$0>\frac{\partial b_p^p(\mathbf{x}^k)}{\partial x_j}=-\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{p,j}(\mathbf{x})+\sum_{\ell=1}^{p-1}\frac{\partial b_\ell^{p-1}(\mathbf{x}^p)}{\partial x_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{p,\ell}(\mathbf{x})}{\Pi_p'(\mathbf{x})+\sum_{\ell=1}^{p-1}\frac{\partial b_\ell^{p-1}(\mathbf{x}^p)}{\partial x_p}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{p,\ell}(\mathbf{x})}; \tag{A.11}$$

$$0>\frac{\partial b_q^p(\mathbf{x}^k)}{\partial x_j}>-\frac{f_j(x_j)}{F_j(x_j)}\frac{F_q(x_q)}{f_q(x_q)}\frac{1}{n-1}\quad \text{for } q\in\{1,\ldots,p\} \text{ and;} \tag{A.12}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_j}<\frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_q}\frac{1}{n-1}\quad \text{for } q\in\{p+1,\ldots,j-1\}. \tag{A.13}$$

Fix $\mathbf{x}^{k+1}$ and let $\mathbf{x}=(b_1^{k-1}(\mathbf{x}^k),\ldots,b_{k-1}^{k-1}(\mathbf{x}^k),\mathbf{x}^k)$. We show that the best response $b_k^k(\mathbf{x}^{k+1})$ is unique by showing that $\hat{\Pi}_k(\mathbf{x}^k)$ is strictly increasing in x_k. Differentiating,

$$\hat{\Pi}_k'(\mathbf{x}^k)=\Pi_k'(\mathbf{x})+\sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x}) > \Pi_k'(\mathbf{x})-\frac{f_k(x_k)}{F_k(x_k)}\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}>\Pi_k'(\mathbf{x})-\frac{f_k(x_k)}{F_k(x_k)}\frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1}>0,$$

where the inequalities follow from lower bound (A.12), Lemma B.4 and, sufficient condition (13), respectively. This proves uniqueness of the best response. The next result completes the induction argument.

Claim 13. Under condition (13), for every $j\in\{k+1,\ldots,m\}$ and $p\in\{1,\ldots,k\}$,

$\partial b_p^k(\mathbf{x}^{k+1})/\partial x_j$ satisfies

$$\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_j} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})} \tag{A.14}$$

$$0 > \frac{\partial b_p^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1} \quad \text{and,} \tag{A.15}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1} \quad \text{for } q \in \{k+1,\dots,j-1\} \tag{A.16}$$

Proof. To show equation (A.14) use the implicit differentiation, the chain rule, and equation (B.2) to obtain

$$\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_j} = -\frac{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_j}{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_k} = -\frac{\frac{\partial \Pi_k(\mathbf{x})}{\partial x_j} + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{\partial \Pi_k(\mathbf{x})}{\partial x_\ell}}.$$

$$= -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}.$$

We already showed that the denominator is positive. We show that a lower bound of the numerator is positive, which immediately implies the upper bound in equation (A.15) for the case when $p = k$. Using equation (A.12) a lower bound for the numerator is

$$\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) - \frac{f_j(x_j)}{F_j(x_j)}\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1} > \frac{f_j(x_j)}{F_j(x_j)}\left[\Delta_{k,j}(\mathbf{x}) - \frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1}\right] > 0,$$

where both inequalities follows from Lemma B.4. Thus, the numerator is positive.

For the lower bound in equation (A.15) in the case $p = k$, replace (A.14) into (A.15) and observe that the inequality holds if and only if the following expression is positive

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{\Pi_k'(\mathbf{x})}{n-1} - \Delta_{k,j}(\mathbf{x})\right) + \right.$$
$$\left.\sum_{\ell=1}^{k-1}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right)\Delta_{k,\ell}(\mathbf{x})\right]. \tag{A.17}$$

The first term in round brackets is positive due to sufficient condition (13). We now work with the summation and show that it is also positive. Before doing this, observe that, by definition, for every $\ell \in \{1, \ldots, k-1\}$

$$b_\ell^k(\mathbf{x}^{k+1}) = b_\ell^\ell(b_{\ell+1}^k(\mathbf{x}^{k+1}), b_{\ell+2}^k(\mathbf{x}^{k+1}), \ldots, b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}).$$

Then, for any $j \in \{k+1, \ldots, m\}$

$$\frac{\partial b_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} = \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j} + \sum_{q=\ell+1}^{k} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q} \frac{\partial b_q^k(\mathbf{x}^{k+1})}{\partial x_j}. \tag{A.18}$$

For a given ℓ in the summation in equation (A.17), we use equation (A.18) to write the round bracket as

$$\left(\frac{F_k(x_k)}{f_k(x_k)} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) + \sum_{q=\ell+1}^{k-1} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_q} \left(\frac{F_k(x_k)}{f_k(x_k)} \frac{\partial b_q^{k-1}(\mathbf{x}^k)}{\partial x_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{\partial b_q^{k-1}(\mathbf{x}^k)}{\partial x_j}\right). \tag{A.19}$$

Substitute equation (A.19) when $\ell = 1$ into the summation in equation (A.17) to obtain

$$\sum_{\ell=2}^{k-1} \left(\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \Delta_{k,\ell}(\mathbf{x}) + \frac{\partial b_1^1(\mathbf{x}^2)}{\partial x_2} a_1\right) \left(\frac{F_k(x_k)}{f_k(x_k)} \frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j}\right) + a_1 \left(\frac{F_k(x_k)}{f_k(x_k)} \frac{\partial b_1^1(\mathbf{x}^k)}{\partial x_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{\partial b_1^1(\mathbf{x}^k)}{\partial x_j}\right), \tag{A.20}$$

where $a_1 = \Delta_{k,1}(\mathbf{x}) \frac{f_1(x_1)}{F_1(x_1)} > 0$. Then, substituting (in increasing order) into equation (A.20) the expression (A.19) for $\ell = 2$, $\ell = 3$ until $\ell = k-1$, we obtain that the summation in equation (A.17) is equal to

$$\sum_{\ell=1}^{k-1} a_\ell \left(\frac{F_k(x_k)}{f_k(x_k)} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{\partial b_\ell^\ell(\mathbf{x}^{\ell+1})}{\partial x_j}\right) > 0, \tag{A.21}$$

where

$$a_\ell = \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \Delta_{k,\ell}(\mathbf{x}) + \sum_{p=1}^{\ell-1} \frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_\ell} a_p \tag{A.22}$$

is defined recursively. The parenthesis in equation (A.21) is positive by equation (A.13). We show that each a_ℓ is positive, which proves the lower bound in equation (A.15) when $p = k$. By induction, suppose that for every $p \in \{1, \ldots, \ell-1\}$ we have shown that $(f_p(x_p)/F_p(x_p))\Delta_{k,p}(\mathbf{x}) \leq a_p > 0$ (we already showed this for a_1). We need to show that the same inequalities hold for equation (A.22). First, because $\partial b_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$ and $a_p > 0$ (by induction hypothesis) it is easy to see that $a_\ell < (f_\ell(x_\ell)/F_\ell(x_\ell))\Delta_{k,\ell}(\mathbf{x})$. Using the lower bound in equation

(A.12) and the upper bound for a_p we obtain the following lower bound for equation (A.22)

$$a_l > \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left[\Delta_{k,\ell}(\mathbf{x}) - \sum_{p=1}^{\ell-1}\frac{\Delta_{k,p}(\mathbf{x})}{n-1}\right] > \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left[1-\frac{(\ell-1)}{n-1}\right]\Delta_{k,\ell}(\mathbf{x}) > 0,$$

where the second inequality follows from Lemma B.4; which proves the result.

To prove the upper bound in equation (A.15) for $p \in \{1,\dots,k-1\}$ we proceed by induction downwards. Suppose that for every firm $\ell \in \{p+1,\dots,k\}$ we have proven

$$0 > \frac{\partial b_\ell^k(\mathbf{x}^{k+1})}{\partial x_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_\ell(x_\ell)}{f_\ell(x_\ell)}\frac{1}{n-1} \tag{A.23}$$

we prove that equation (A.15) holds for p. Observing that, in equation (A.18), $\partial b_p^p(\mathbf{x}^{p+1})/\partial x_\ell < 0$, we can construct an upper bound for $\partial b_p^k(\mathbf{x}^{k+1})/\partial x_j$ using the induction hypothesis (A.23)

$$\frac{\partial b_p^k(\mathbf{x}^{k+1})}{\partial x_j} < \frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_j} - \sum_{\ell=p+1}^{k}\frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_\ell}\frac{f_j(x_j)}{F_j(x_j)}\frac{F_\ell(x_\ell)}{f_\ell(x_\ell)}\frac{1}{n-1}$$

Using equation (A.11), the upper bound for $\partial b_p^k(\mathbf{x}^{k+1})/\partial x_j$ is equal to

$$\frac{1}{D_p}\frac{f_j(x_j)}{F_j(x_j)}\sum_{\ell=p+1}^{k}\left(\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{p,\ell}(\mathbf{x}) + \sum_{q=1}^{p-1}\frac{\partial b_q^{p-1}(\mathbf{x}^p)}{\partial x_\ell}\frac{f_q(x_q)}{F_q(x_q)}\Delta_{p,q}(\mathbf{x})\right)\frac{F_\ell(x_\ell)}{f_\ell(x_\ell)}\frac{1}{n-1}$$
$$-\frac{1}{D_p}\frac{f_j(x_j)}{F_j(x_j)}\left(\Delta_{p,j}(\mathbf{x}) + \frac{F_j(x_j)}{f_j(x_j)}\sum_{q=1}^{p-1}\frac{\partial b_q^{p-1}(\mathbf{x}^p)}{\partial x_j}\frac{F_q(x_q)}{f_q(x_q)}\Delta_{p,q}(\mathbf{x})\right)$$

where $D_p = \Pi_p'(\mathbf{x}) + \sum_{q=1}^{p-1}\frac{db_q^{p-1}(\mathbf{x}^p)}{dx_p}\frac{f_q(x_q)}{F_q(x_q)}\Delta_{p,q}(\mathbf{x}) > 0$. Taking $\partial b_q^{p-1}(\mathbf{x}^p)/\partial x_\ell < 0$ equal to zero and $\partial b_q^{p-1}(\mathbf{x}^p)/\partial x_j < 0$ to the lower bound in equation (A.12), we build the following upper bound for the previous expression (and omitting D_p, as it does not affect the sign)

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\sum_{\ell=p+1}^{k}\frac{\Delta_{p,\ell}(\mathbf{x})}{n-1} + \sum_{q=1}^{p-1}\frac{\Delta_{p,q}(\mathbf{x})}{n-1} - \Delta_{p,j}(\mathbf{x})\right] < \frac{f_j(x_j)}{F_j(x_j)}\left[\frac{k-1}{n-1}-1\right]\Delta_{p,j}(\mathbf{x})$$
$$\leq 0.$$

The inequality follows from equation Lemma B.4; proving $\partial b_p^k(\mathbf{x}^{k+1})/\partial x_j < 0$.

The lower bound for $\partial b_p^k(\mathbf{x}^{k+1})/\partial x_j$ follows from equation (A.18) and observing

$$\frac{\partial b_p^k(\mathbf{x}^{k+1})}{\partial x_j} > \frac{\partial b_p^p(\mathbf{x}^{p+1})}{\partial x_j} > -\frac{f_j(x_j)}{F(x_j)}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1}$$

where the first inequality follows from $(\partial b_p^p(\mathbf{x}^{k+1})/\partial x_\ell)\cdot(\partial b_\ell^k(\mathbf{x}^{k+1})/\partial x_j) > 0$ for every ℓ, and the second from the lower bound in equation (A.12).

Finally, we prove equation (A.16) using equation (A.14) to write

$$\frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_q}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_k^k(\mathbf{x}^{k+1})}{\partial x_j} = \frac{1}{D_k}\Bigg[\Delta_{k,j}(\mathbf{x}) - \frac{\Delta_{k,q}(\mathbf{x})}{n-1} + \sum_{\ell=1}^{k-1}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left(\frac{F_j(x_j)}{f_j(x_j)}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_j} - \frac{F_q(x_q)}{f_q(x_q)}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_q}\frac{1}{n-1}\right)\Delta_{k,\ell}(\mathbf{x})\Bigg],$$

where $D_k = \Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{\partial b_\ell^{k-1}(\mathbf{x}^k)}{\partial x_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x}) > 0$. We show that a lower bound of this expression is positive. Taking $-\partial b_\ell^{k-1}(\mathbf{x}^k)/\partial x_q > 0$ to zero and $\partial b_\ell^{k-1}(\mathbf{x}^k)/\partial x_j < 0$ to the lower bound in equation (A.12), we obtain

$$\frac{1}{D_k}\left[\Delta_{k,j}(\mathbf{x}) - \frac{\Delta_{k,q}(\mathbf{x})}{n-1} - \sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right] > \frac{1}{D}\left[\Delta_{k,j}(\mathbf{x}) - \frac{k\Delta_{k,q}(\mathbf{x})}{n-1}\right] > 0.$$

The inequalities follow from Lemma B.4 and the fact that $q \in \{k,\dots,j-1\}$. □

Because at each step, best responses are unique and at $k = n$ the firm has only one best response when every firm $k < n$ is best responding to x_n and to each other, there is a unique Herculean equilibrium within the herculean class.

No non-herculean equilibria exists: By contradiction. Suppose $\mathbf{x}$ represents a non-herculean equilibrium. Order firms from smallest cutoff x_1 to largest, x_n. Let p be the first instance (smallest cutoff) that a strength reversal occurs. That is, $x_p < x_{p+1}$ but $s_p > s_{p+1}$. Because every firm $k \in \{1,\dots,p\}$ is ordered by strength, they satisfy conditions (A.14), (A.12), and (A.13). We show that x_{p+1} cannot lie above x_p (i.e, a contradiction). Fix $\mathbf{x}$ and let $\hat{x}$ be the value that satisfies $b_p(\hat{x},\mathbf{x}_{-p,p+1}) = \hat{x}$, where $b_p(\mathbf{x}_{-p})$ is firm's p best response to $\mathbf{x}_{-p}$. This best response exists (and is unique) by Lemma B.2. The value $\hat{x}$ exists because $b_p(\mathbf{x}_{-p})$ is continuously decreasing in x_{p+1}. In addition, following analogous steps to those in Claim 11, we can show that $\partial b_p(\mathbf{x}_{-p})/\partial x_{p+1} > -\frac{f_{p+1}(x_{p+1})}{F_{p+1}(x_{p+1})}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1}$. Then, by Lemma B.3, $\Pi_p(\hat{x},\hat{x},\mathbf{x}_{-p,p+1}) = 0 < \Pi_{p+1}(\hat{x},\hat{x},\mathbf{x}_{-p,p+1})$. Also, letting $\hat{\mathbf{x}} = (b_p(\mathbf{x}_{-p}),\mathbf{x}_{-p})$ observe that

$$\begin{aligned}\frac{d\Pi_{p+1}(\hat{\mathbf{x}})}{dx_{p+1}} &= \Pi_{p+1}'(\hat{\mathbf{x}}) + \frac{\partial b_p(\mathbf{x}_{-p})}{\partial x_{p+1}}\frac{\partial \Pi_{p+1}(\hat{\mathbf{x}})}{\partial x_p}\\ &> \Pi_{p+1}'(\hat{\mathbf{x}}) - \frac{f_{p+1}(x_{p+1})}{F_{p+1}(x_{p+1})}\frac{\Delta_{p+1,p}(\hat{\mathbf{x}})}{n-1} > 0\end{aligned}$$

Thus, $\Pi_{p+1}(\hat{\mathbf{x}})$ is strictly increasing in x_{p+1} which implies that $\Pi_{p+1}(\hat{\mathbf{x}}) > 0$ for every $x_{p+1} \geq \hat{x}$, which implies that no $x_{p+1} > b_p(\mathbf{x}_{-p}) = x_p$ exists. ■

Proof of Proposition 3 . For the first claim, observe that when $\varepsilon \to 0$ we have $F(s_i) \to r_i$. Thus, $s_1 < s_2$ if and only if $r_1 < r_2$. For the second claim, suppose

$s_1 < s_2$. A herculean equilibrium is given by a solution $x_1 < x_2$ of

$$F(x_j)\pi_i + (1 - F(x_j))\pi_{ij} + \varepsilon x_i = 0 \quad \text{for } i \in \{1, 2\}, \tag{A.24}$$

which implies $x_i = b_i(x_j) = -(\pi_{ij} + F(x_j)(\pi_i - \pi_{ij}))/\varepsilon$. As $\pi_i - \pi_{ij} > 0$, if the opponent's cutoff increases, firm i best responds by decreasing its own cutoff. As $\varepsilon \to 0$ the herculean cutoffs satisfy $(x_1, x_2) \to (-\infty, \infty)$ or firm 1 entering with probability 1 and firm 2 staying out. ■

Proof of Proposition 4. Start by observing that the only source of asymmetries among firms is their payoffs (π_i or π_{ij}). We first analyze how equilibria changes with a small increase in π_1. The proof is analogous for any change in any other parameter. An increase in π_1 raises the strength of firm 1. From the proof of Proposition 3 we know $b_i(x_j) = -(\pi_{ij} + F(x_j)(\pi_i - \pi_{ij}))$. Consequently, $b_i'(x_j) = -f(x_j)(\pi_i - \pi_{ij}) < 0$. Using implicit differentiation of equilibrium condition (A.24) at a given equilibrium (x_1, x_2), we obtain

$$\begin{pmatrix} \frac{\partial x_1}{\partial \pi_1} \\ \frac{\partial x_2}{\partial \pi_1} \end{pmatrix} = \begin{pmatrix} \frac{-F(x_2)}{1 - b_1'(x_2) b_2'(x_1)} \\ \frac{-F(x_2) b_1'(x_2)}{1 - b_1'(x_2) b_2'(x_1)} \end{pmatrix}$$

An increase in firm 1's strength, decreases its equilibrium cutoff when the equilibrium is stable ($b_1'(x_2)\, b_2'(x_1) < 1$) and increases the cutoff when unstable. The converse is true for firm 2.

Then the results follow from the following three observations: (i) In the linear model, the only way for two stable equilibria to have the same basin of attraction is if the game is symmetric. (ii) Given the proof above, in a symmetric game, an increase in firm 1's strength increases the basin of attraction of the equilibrium that becomes herculean after the change. This follows from the (new) herculean equilibrium being stable, thus it moves away from the center; the symmetric (non-stable) equilibrium moving towards the stable non-herculean equilibrium, and; the asymmetric non-herculean equilibrium moving towards the symmetric. (iii) For any asymmetric game, there is a symmetric game with payoffs $\hat{\pi}_m = \min\{\pi_1, \pi_2\}$ and $\hat{\pi}_d = \min\{\pi_{12}, \pi_{21}\}$ such that, we can construct a continuous path of payoff converging to the asymmetric game, maintaining the relative strength of the firms—thus, preserving the largest basing of attraction for the herculean equilibrium—along the path.[31] ■

[31] Let i be the strongest firm in the original game. Then either $\pi_i > \pi_j$ or $\pi_{ij} > \pi_{ji}$ (or both) hold. Pick the characteristic that holds, say $\pi_i > \pi_j$, and move from the symmetric game described by $(\hat{\pi}_m, \hat{\pi}_d)$ towards the game described by $(\pi_i, \pi_j, \hat{\pi}_d)$. Along this path firm i is always stronger. Then, moves toward the original game, and i remains stronger along the path as firm i was stronger to start with and no reversal can occur.

B Auxiliary Results

Lemma B.1. $\Pi_i(\mathbf{x})$ *is strictly increasing in every dimension of* $\mathbf{x}$.

Proof of Lemma B.1. Start with the derivative of Π_i with respect to x_i, then

$$\frac{\partial \Pi_i}{\partial x_i} \equiv \Pi_i'(\mathbf{x}) = \sum_{e \in \mathcal{E}_i} \left\{ \left(\prod_{j \in e^c} F_j(x_j) \right) \int_{\{x_j\}_{j \in e \setminus i}}^{\beta} \pi_i' \left(x_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} > 0, \quad \text{(B.1)}$$

which is positive as, by assumption A3, there is a positive probability that firm i is the sole entrant.

For the derivative of Π_i with respect to x_j, pick a market structure e such that $j \in e$. Conditional on e, the derivative of Π_i with respect to x_j is equal to

$$-f_j(x_j) \left(\prod_{k \in e^c} F_k(x_k) \right) \int_{\{x_k\}_{k \in e \setminus \{i,j\}}}^{\beta} \pi_i \left(x_{\{i,j\}}, v_{e \setminus \{i,j\}} \right) \phi(v_{e \setminus \{i,j\}}) d^{n_e - 2} v_{e \setminus \{i,j\}}.$$

Now take market structure e, from above, and using Leibnitz differentiation, compute the derivative of Π_i with respect to x_j conditional on market structure $e \setminus j$; i.e., entry decisions by every firm remain the same as in e except that of firm j, which stays out

$$f_j(x_j) \left(\prod_{k \in e^c} F_k(x_k) \right) \int_{\{x_k\}_{k \in e \setminus \{i,j\}}}^{\beta} \pi_i \left(x_i, v_{e \setminus \{i,j\}} \right) \phi(v_{e \setminus \{i,j\}}) d^{n_e - 2} v_{e \setminus \{i,j\}}.$$

where in the product we used $e^c = (e \setminus j)^c \setminus j$ and in the integral $e \setminus \{i, j\} = (e \setminus j) \setminus i$. Observe that both expressions from above only differ in sign and in the profit function that is integrated over. Summing both equations delivers

$$f_j(x_j) \left(\prod_{k \in e^c} F_k(x_k) \right) \int_{\{x_k\}_{k \in e \setminus \{i,j\}}}^{\beta} \delta_{i,j} \left(x_{\{i,j\}}, v_{e \setminus \{i,j\}} \right) \phi(v_{e \setminus \{i,j\}}) d^{n_e - 2} v_{e \setminus \{i,j\}}.$$

where $\delta_{i,j}(v_e) \geq 0$ is defined in equation (5). Summing across every market structure and using equation (7) we obtain

$$\begin{aligned} \frac{\partial \Pi_i}{\partial x_j} &= \sum_{e \in \mathcal{E}_i \cap \mathcal{E}_j} \left\{ \left(\prod_{k \in e^c} F_k(x_k) \right) \int_{(x_k)_{k \in e \setminus \{i,j\}}}^{\beta} \delta_{i,j} \left(x_{\{i,j\}}, v_{e \setminus \{i,j\}} \right) \phi(v_{e \setminus \{i,j\}}) d^{n_e - 2} v_{e \setminus \{i,j\}} \right\} \\ &= \frac{f_j(x_j)}{F_j(x_j)} \Delta_{i,j}(\mathbf{x}) > 0 \qquad \text{(B.2)} \end{aligned}$$

Thus, the derivative is positive. ■

Lemma B.2. *Let* Π_i *be defined by* (6)*. Let* A *and* B *be disjoint sets of* k *and* r *firms, where* $k + r < n$*, such that* $i \in A$*. Define* $f : [\alpha, \beta]^{k+r} \to [\alpha, \beta]^{n-k-r}$ *to be a continuous function and let* x_B *be any vector of cutoff strategies for firms in* *set B. Then, there exist a value $\tilde{x}$ such that the symmetric k-dimensional vector $\tilde{x}_A = (\tilde{x})_{i\in A}$ satisfies $\Pi_i(\tilde{x}_A, f(\tilde{x}_A, x_B), x_B) = 0$. The vector $\tilde{x}_A$ is a continuous in each dimension of x_B. When the function f is constant—i.e., when $\mathbf{x}_{E\setminus A} = (f(\tilde{x}_A, x_B), x_B)$ does not change with $\tilde{x}_A$—the value of $\tilde{x}$ is unique.*

Proof. Fix x_B, because f is continuous, the function $\Pi_i(x_A, f(x_A, x_B), x_B)$ is continuous in the input value x of the symmetric vector x_A. Let $\underline{v}_A = (\underline{v}_i)_{i\in A}$ and $\overline{v}_A = (\overline{v}_i)_{i\in A}$. Observe that assumptions A3 and A2 jointly imply

$$\Pi_i(\underline{v}_A, f(\underline{v}_A, x_B), x_B) \leq \pi_i(\underline{v}_i) < 0.$$

Similarly, assumption A3 and Lemma B.1 together imply,

$$\Pi_i(\overline{v}_A, f(\overline{v}_A, x_B), x_B) \geq \Pi_i(\overline{v}_i, \alpha_{-i}) > 0.$$

Then, by the intermediate value theorem, there exist $\tilde{x} \in (\underline{v}_i, \bar{v}_i)$ such that

$$\Pi_i(\tilde{x}_A, f(\tilde{x}_A, x_B), x_B) = 0.$$

Because the functions Π_i and f are continuous, the value $\tilde{x}_A$ is continuous in each dimension of x_B. For uniqueness when f is constant, by the chain rule, $d\Pi_i/dx = \sum_{k\in A} \partial\Pi_i/\partial x_k > 0$ where the inequality follows from Lemma B.1. Therefore $\Pi_i(x_A, f(x_A, x_B), x_B)$, as a function of the value x for the symmetric vector x_A, is increasing and crosses zero once. ■

Lemma B.3. *Consider an ordered game, in which the firms' identities are ordered by strength, with firm 1 being the strongest. Then, for any firm $i < j$, valuation y, and vector of strategies for the other firms $x_{E\setminus\{i,j\}}$, we have*

$$\Pi_i(y, y, x_{E\setminus\{i,j\}}) > \Pi_j(y, y, x_{E\setminus\{i,j\}}).$$

Proof. If firms are ordered by profit, the inequality follows by definition. Recall $\phi(v_e) = \prod_{j\in e} f_j(v_j)$. For games ordered by distribution, observe

$$\begin{aligned}
\Pi_i(y, y, x_{E\setminus\{i,j\}}) = & \sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left(F_j(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{\beta} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
& \sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left(\prod_{k\in e^c} F_k(x_k) \right) \int_y^{\beta} \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{\beta} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_j(v) d^{n_e-1} v_{e\setminus i} \right\} \\
> & \sum_{e\in\mathcal{E}_i\setminus\mathcal{E}_j} \left\{ \left(F_i(y) \prod_{k\in e^c\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in e\setminus i}}^{\beta} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
& \sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j} \left\{ \left(\prod_{k\in e^c} F_k(x_k) \right) \int_y^{\beta} \int_{(x_k)_{k\in e\setminus\{i,j\}}}^{\beta} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus\{i,j\}}) f_i(v) d^{n_e-1} v_{e\setminus i} \right\}
\end{aligned}$$

$$= \Pi_j(y, y, x_{E\setminus\{i,j\}}),$$

where the inequality uses two properties of FOSD. The first term uses that $F_i(x) \leq F_j(x)$ for all x. The second term uses that $\int_y^\beta h(x) f_i(x)dx \leq \int_y^\beta h(x) f_j(x)dx$ for any non-increasing function $h(x)$. ■

Lemma B.4. *Let firm k be stronger than firm j. Suppose the firms play cutoffs $x_k < x_j$; then, for any firm i, $\Delta_{i,j}(\mathbf{x}) \geq \Delta_{i,k}(\mathbf{x})$ if: (i) firms are ordered by profits, or; (ii) firms are ordered by distribution and the profit gain only depends on the number of entrants.*

Proof. Start by observing that, in the expression for $\Delta_{i,j}(\mathbf{x})$ (see equation (7)), the sum over market structures $\mathcal{E}_i \cap \mathcal{E}_j$ can be divided into two disjoints sets: $\mathcal{E}_i \cap \mathcal{E}_j \cap \mathcal{E}_k$ and $(\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$. Using these sets subtract $\Delta_{i,j}(\mathbf{x}) - \Delta_{i,k}(\mathbf{x})$ to obtain

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_i\cap\mathcal{E}_j\cap\mathcal{E}_k} \left\{ \left(\prod_{\ell\in e^c} F_\ell(x_\ell) \right) \left[F_j(x_j) \int_{x_k}^\beta \int_{(x_\ell)_{\ell\in e\setminus\{i,j,k\}}}^\beta \delta_i(x_{\{i,j\}}, v_{e\setminus\{i,j\}}) \phi(v_{e\setminus\{i,j\}}) d^{n_e-2} v_{e\setminus i} \right.\right. \\
&\qquad \left.\left. - \; F_k(x_k) \int_{x_j}^\beta \int_{(x_\ell)_{\ell\in e\setminus\{i,j,k\}}}^\beta \delta_i(x_{\{i,k\}}, v_{e\setminus\{i,k\}}) \phi(v_{e\setminus\{i,k\}}) d^{n_e-2} v_{e\setminus\{i,k\}} \right] \right\} \\
&+ \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_j)\setminus\mathcal{E}_k} \left\{ \left(\prod_{\ell\in e^c\cup j} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in e\setminus\{i,j\}}}^\beta \delta_i(x_{\{i,j\}}, v_{e\setminus\{i,j\}}) \phi(v_{e\setminus\{i,j\}}) d^{n_e-2} v_{e\setminus\{i,j\}} \right\} \\
&- \sum_{e\in(\mathcal{E}_i\cap\mathcal{E}_k)\setminus\mathcal{E}_j} \left\{ \left(\prod_{\ell\in e^c\cup k} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in e\setminus\{i,k\}}}^\beta \delta_i(x_{\{i,k\}}, v_{e\setminus\{i,k\}}) \phi(v_{e\setminus\{i,k\}}) d^{n_e-2} v_{e\setminus\{i,k\}} \right\}
\end{aligned} \tag{B.3}$$

where, by profits being anonymous, we dropped the second sub index from the profit gain $\delta_i(x_{\{i,j\}}, v_{e\setminus i})$. Equation (B.3) has three sumations. For the last two summations observe that, for each market structure $e \in (\mathcal{E}_i \cap \mathcal{E}_j) \setminus \mathcal{E}_k$ in the first summation, the outer productory and the integral are over the same set of firms than the market structure $\hat{e} = (e \setminus j) \cup k$ (where $\hat{e} \in (\mathcal{E}_i \cap \mathcal{E}_k) \setminus \mathcal{E}_j$) in the second summation. Thus, subtracting the payoffs in those two market structures, we obtain

$$\left(\prod_{\ell\in e^c\cup j} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in e\setminus\{i,j\}}}^\beta \left(\delta_i(x_{\{i,j\}}, v_{e\setminus\{i,j\}}) - \delta_i(x_{\{i,k\}}, v_{e\setminus\{i,j\}}) \right) \phi(v_{e\setminus\{i,j\}}) d^{n_e-2} v_{e\setminus\{i,j\}}$$

but the term inside the integral is non-negative as

$$\delta_i(x_{\{i,j\}}, v_{e\setminus i}) - \delta_i(x_{\{i,k\}}, v_{e\setminus i}) = \pi_i(x_{\{i,k\}}, v_{e\setminus i}) - \pi_i(x_{\{i,j\}}, v_{e\setminus i}) \geq 0$$

where the last inequality follow from assumption A2 and $x_k < x_j$. Summing across every market structure shows that the last two summations in (B.3) are non-negative.

We now show that the square bracket in the first summation of (B.3) is non-negative. Fix a market structure $e \in \mathcal{E}_i \cap \mathcal{E}_j \cap \mathcal{E}_k$, we show that a lower bound of the first term is square bracket is equal to the subtracting term. Thus, the subtraction is non-negative.

When the game is ordered by profit, we can drop the sub-index from the distributions of type. Bounding the first term

$$F(x_j) \int_{x_k}^{\beta} \int_{(x_\ell)_{\ell \in e\setminus\{i,j,k\}}}^{\beta} \delta_i(x_{\{i,j\}}, v_{e\setminus\{i,j\}}) \phi(v_{e\setminus\{i,j,k\}}) f(v_k) d^{n_e-2} v_{e\setminus\{i,j\}}$$
$$> F(x_k) \int_{x_j}^{\beta} \int_{(x_\ell)_{\ell \in e\setminus\{i,j,k\}}}^{\beta} \delta_i(x_{\{i,k\}}, v_{e\setminus\{i,k\}}) \phi(v_{e\setminus\{i,j,k\}}) f(v_j) d^{n_e-2} v_{e\setminus\{i,k\}}$$

where in the inequality we used $x_j > x_k$ in three places: (i) in the probability of firm j being out of the market; (ii) in the domain of integration over k's types, which jointly $\delta_i(x_{\{i,j\}}, v_{e\setminus i}) \geq 0$ implies that we are integrating over a smaller domain, decreasing the value of the integral, and; (iii) $\delta_i(x_i, s, v_{e\setminus i})$ being increasing in s (by assumption A2). Finally, we use that payoffs are anonymous to re-arrange indexes for integrating variables, inverting the roles of firm k and j. Because the inequality holds for every market structure $e \in \mathcal{E}_i \cap \mathcal{E}_j \cap \mathcal{E}_k$, the inequality proves the result.

When the game is ordered by distributions and the profit gain only depends on the number of entrants, the second term becomes

$$F_j(x_j) \left((1 - F_k(x_k)) \prod_{\ell \in e\setminus\{i,k\}} (1 - F_\ell(x_\ell)) \right) \delta_i(x_i, n_e)$$
$$> F_k(x_k) \left((1 - F_j(x_j)) \prod_{\ell \in e\setminus\{i,k\}} (1 - F_\ell(x_\ell)) \right) \delta_i(x_i, n_e)$$

where the first inequality uses stochastic dominance and the fact that $x_k < x_j$ (so that $F_j(x_j) \geq F_j(x_k) \geq F_k(x_k)$). The equality follows by re-arranging indexes, noticing that the lower bound above is identical to the third term in equation (B.3), proving the result. ■

Online Appendix

Equilibrium Uniqueness in Entry Games with Private Information

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Supplemental Material –Not for Publication

C Equilibrium Exists and is in Cutoff Strategies

An entry strategy for firm i is a mapping from the firm's type v_i to a probability of entering in the market $\tau_i : [\alpha, \beta] \to [0, 1]$. We assume that the strategy of firm i is an integrable function with respect to its own type v_i. We study the Bayesian Equilibria of the entry game. Denote by $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ the vector of entry strategies. Given a strategy profile τ, the expected profit of firm i *after* drawing the type v_i but *before* entry decisions are realized is

$$\Pi_i(v_i, \tau) = \tau_i(v_i) \left[\sum_{e \in E_i} \left\{ \int_{[\alpha,\beta]^{n-1}} \pi_i(v_e) \Pr[e|\tau_{-i}, v_{-i}] \phi(v_{-i}) d^{n-1} v_{-i} \right\} \right] \tag{C.1}$$

where $\Pr[e|\tau_{-i}, v_{-i}]$ is the probability of observing market structure e, given the vector of strategies τ_{-i} and the realizations of types v_{-i}. The integral is over each of the $n-1$ dimensions of firm i's competitors types, v_{-i}. Conditional on i's entry, which occurs with probability $\tau_i(v_i)$, the expected profit of firm i consists of the expected sum of profit that firm i would get under each feasible market structure, which is induced by the vector of strategies τ and the realization of types v_{-i}, integrated over all possible realizations of the competitors' types, $\phi(v_{-i})$.

Definition (Cutoff Strategy)**.** A strategy $\tau_i(v_i)$ is called *cutoff* if there exists a threshold $x > 0$ such that

$$\tau_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases} .$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on whether its type is above or below some given threshold. In any best response, there exists a type, v_i, that makes a firm indifferent to enter the market. We break this indifference by assuming that firms enter. For a cutoff strategy, this means that a firm enters when its type is greater or equal to its cutoff. Given a vector τ_{-i}, a best response is given by the strategy $\hat{\tau}_i$ that maximizes (C.1) at every value of v_i.

A Bayesian Nash equilibrium is defined by a vector of strategies τ in which every firm best respond to each other. The next proposition establishes the existence of an equilibrium and that, without loss of generality, we can restrict our analysis to cutoff strategies.

Lemma C.1. *For any game $(\pi_i, F_i)_{i=1}^n$ satisfying assumptions A1-A3, there exists an equilibrium. For any vector τ_{-i}, firm i's best response is a cutoff strategy. Therefore, every equilibrium of the game is in cutoff strategies.*

Proof of Lemma C.1.

best responses are cutoff strategies: Fix any firm i and vector of strategies τ. By assumptions A3 and A2, we know that in equilibrium no firm will enter if they draw $v_j < \underline{v}_j$. For relevance, impose that τ satisfies the restriction $\tau_j(v_j) = 0$ in that range. Because firm i's profit is linear in τ_i, firm i's best response is to participate with probability one whenever there is a positive payoff of doing so. Suppose firm i enters the market with certainty ($\tau_i(v_i) = 1$). The restriction above implies that there is positive probability that firm i is the sole entrant to the market and, consequently, by A1, profits are strictly increasing in v_i. By A3, $\Pi_i(\underline{v}_i, \tau) < 0$, and $\Pi_i(\overline{v}_i, \tau) > 0$. Thus, $\Pi_i(v_i, \tau)$ single crosses zero and i's best response to τ_{-i} is the cutoff strategy defined by the value x_i that satisfies $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$.

Existence: We check the conditions of Brouwer's fixed-point theorem. Because F_i is atomless and has full support and $\pi_i(v_e)$ being continuous and differentiable in v_i, firm i's best response lies in the compact and convex set $[\underline{v}_i, \bar{v}_i]$. Thus the n-dimensional function of best responses is a continuous mapping from $\times_{i=1}^n[\underline{v}_i, \bar{v}_i]$ to itself and the conditions for the theorem are met. ■

Existence follows from Brouwer's fixed-point theorem. The restriction to cutoff strategies is quite intuitive: regardless of which strategy competitors are playing, assumption A1 implies that firm i's expected profit is strictly increasing in its type. Because i's expected profit is linear in its entry probability (see eq. (C.1)), i either prefers to enter with certainty, when it is profitable to do so, or to stay out. The next Lemma characterizes all cutoff equilibria.

Lemma C.2. *The vector* $\mathbf{x}$ *of cutoff strategies constitutes an equilibrium if and only if* $\Pi_i(\mathbf{x}) = 0$ *for every firm* i.

Proof of Lemma C.2. By the previous proof a cutoff strategy is defined as the value x_i satisfying $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$. Because in a cutoff equilibrium $\Pr[e|\tau, v_i]$ is either zero or one. Integrating (C.1) over payoff-irrelevant firms delivers (6). ■

Lemma C.2 characterizes every equilibrium of the entry game. Firm i's best response to $\mathbf{x}_{-i}$ is defined by a cutoff x_i equal to the value of v_i that satisfies $\Pi_i(v_i, \mathbf{x}_{-i}) = 0$. A profile of equilibrium cutoffs $\mathbf{x}$ is, thus, constructed by the collection of functions $\Pi_i(\mathbf{x})$ evaluated at a point in which every firm i is indifferent between entering the market when drawing type x_i.

D Alternative notions for Strength

In this section, we explore alternative notions for strength. In particular, we study the relationship between: (i) the cutoff strategies, x_i; (ii) the *ex-ante* probability of participating in the auction, $1 - F_i(x_i)$; and (iii) the *ex-ante* expected payoff of each bidder; which, for a given vector of cutoffs strategies $\mathbf{x} = (x_1, x_2)$, is equal to:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} sdF_j(s) - c_i \right) dF_i(v). \tag{D.1}$$

That is, for each valuation v_i under which bidder i participates (i.e., for each $v_i > x_i$), the expected payoff of participating in the auction, weighted by the probability that v_i occurs.

We explore the relation between the previous objects by means of an example. Consider two asymmetric bidders whose distribution of valuations follows a Generalized Pareto distribution (GPD) with shape parameter κ and scale parameter σ.[32] The choice of GPD yields a simple concave distribution with positive support that is flexible enough to change its mean and variance. Suppose both bidders have a symmetric participation cost c, but bidder 1 is characterized by $(\kappa_1, \sigma_1) = (0, 1)$ and bidder 2 by $(\kappa_2, \sigma_2) = (0.25, 0.75)$. Both distributions have the same mean but the second distribution has twice the variance. That is, the second distribution is a mean-preserving spread of the first. Because the CDFs cross, distributions are *not* ordered by FOSD. Consequently, the game is not *ordered* and it is not self-evident which bidder is stronger.

Intuitively, the stronger bidder would be the one whose distribution of valuations has more mass to the right of the equilibrium cutoffs strategies, as this implies the bidder is more likely to obtain higher valuations. If the equilibrium cutoff strategies are high, then bidder 2 would have more mass to the right of the cutoffs, and thus bidder 2 would be the stronger bidder. High equilibrium cutoff strategies are likely to occur when participation costs are high. Conversely, if the cutoff strategies are low, then bidder 1 would have more probability mass to the right of the cutoffs, and thus bidder 1 would be the stronger bidder. Low equilibrium cutoff strategies are likely to occur when participation costs are low.

This situation is illustrated in Figure 9. Panel (a) shows that both distributions are concave, thus Lemma 2 implies that the participation game has a unique equilibrium for any participation costs $c > 0$. Panel (a) also shows that both distributions cross at $v^\circ = 2.2007$. Panel (b) depicts the bidders' strength. It shows that bidders are equally strong when $c^\circ = 1.957$. For participation costs above c°, bidder 2 is stronger ($s_2 < s_1$) and, in the unique equilibrium, bidder 2 plays a lower cutoff strategy ($x_2 < x_1$). For instance, if $c_a = 2 > c^\circ$, then the vector of equilibrium cutoffs is $\mathbf{x} = (2.241, 2.238)$. Alternatively, when $c < c^\circ$, bidder 1 is stronger ($s_1 < s_2$) and plays a lower equilibrium cutoff strategy ($x_1 < x_2$). For

[32] For $\kappa \in \mathbb{R}$ and $\sigma \in (0, \infty)$, the Generalized Pareto CDF is defined over $\mathbb{R}_+$ and given by

$$F(x|\kappa,\sigma) = \begin{cases} 1 - \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1}{\kappa}} & \kappa \neq 0 \\ 1 - e^{-\frac{x}{\sigma}} & \kappa = 0 \end{cases}.$$

The CDF is concave whenever $\kappa > -1$, its mean is well defined for $\kappa < 1$ and given by $\sigma/(1-\kappa)$, whereas its variance is defined for $\kappa < 1/2$ and given by $\sigma^2/(1-\kappa)^2(1-2\kappa)$.

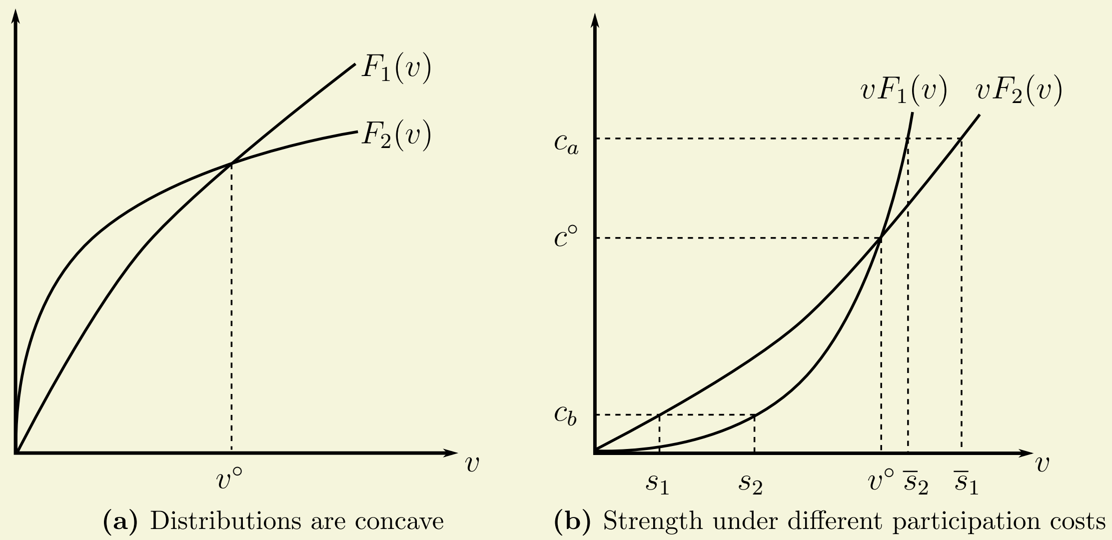


Figure 9: Strength under second-order stochastic dominance. Distributions are Generalized Pareto with parameters $(\kappa_1, \sigma_1) = (0, 1)$ and $(\kappa_2, \sigma_2) = (0.25, 0.75)$ respectively. Panel (a) shows that distributions cross at $v^\circ = 2.2007$. Panel (b) shows that, depending on the entry cost, either bidder can be stronger.

example, if $c_b = 1 < c^\circ$, then the equilibrium is $\mathbf{x} = (1.281, 1.383)$.

When the participation cost is equal to c°, bidders are equally strong ($s_i = v^\circ$). Because the CDFs are concave, the unique equilibrium is given by the symmetric cutoffs equal to the bidders' strength ($x_i = v^\circ$). The expected payoff of bidder 2, however, is greater than the expected payoff of bidder 1. Using equation (D.1), we obtain $(U_1, U_2) = (0.103, 0.185)$. This means that although bidders' cutoffs are not ranked, their expected profits are. The intuition in this scenario follows from $F_2(v) < F_1(v)$ for every $v > v^\circ$. Relative to bidder 1, bidder 2's valuations (distributed according to $F_2(v)$) are skewed to the right tail of the distribution, whereas their expected payment price (distributed according to $F_1(v)$) is skewed towards the left (see Figure 9a). In other words, for valuations greater than v°, bidder 2's conditional distribution of valuations FOSD the bidder 1's conditional distribution.

Beginning from the previous example, we construct an equilibrium in which bidder 1 receives a lower expected payoff than bidder 2, despite playing a lower participation cutoff and having a higher participation probability. By decreasing bidder 1's participation cost, bidder 1 becomes stronger than bidder 2 and will play a lower cutoff in the unique equilibrium of the game. By continuity, if the decrease in bidder 1's cost is small, we can construct an equilibrium with said characteristics. Take for example $(c_1, c_2) = (1.9, c^\circ)$, then bidder 1 is stronger and plays a lower cutoff—in this case $\mathbf{x} = (2.1327, 2.2196)$—but also receives lower expected payoffs $(U_1, U_2) = (1.11, 1.83)$. At a cutoff equal to v°, both bidders are equally likely to enter. Thus, $x_1 < v^\circ < x_2$ implies that bidder 1 is simultaneously more likely to participate and receive a lower expected payoff.

Finally, to show that cutoff order need not coincide with entry-probability order, modify the participation costs to $(c_1, c_2) = (1.1, 1)$. In this scenario, bidder 1 plays a higher entry cutoff $x_1 = 1.434 > 1.313 = x_2$ while also participating more frequently $1 - F_1(x_1) = .238 > .234 = 1 - F_2(x_2)$.

E Proof of Proposition 1

We begin by proving existence of a herculean equilibrium. If bidders are equally strong ($s_1 = s_2 = s$), their strength corresponds to their herculean equilibrium.

Suppose w.l.o.g. that $s_1 < s_2$; i.e., bidder 1 is the strong bidder of the game. We construct an equilibrium where $x_1 < x_2$. Define $g(v) = c_1/F_2(v)$ to be the equilibrium cutoff played by bidder 1 when bidder 2 plays $x_2 = v$. Observe that $g(v) > c_1$ and $g'(v) = -g(v)f_2(v)/F_2(v) < 0$. Define the function $h : [\alpha, \beta) \to \mathbb{R}$ by

$$h(v) = vF_1(v) - \int_{g(v)}^{v} x f_1(x)\, dx - c_2$$

which is a continuous function of v. The function $h(v)$ represents bidder 2's revenue of drawing valuation v when she plays a cutoff $x_2 = v$ and 1 plays the cutoff $x_1 = g(v)$. To have a herculean equilibrium we need a value x_2 satisfying $h(x_2) = 0$ and $x_2 > g(x_2)$. The next two claims prove the result.

Claim 14. $x_2 \in (s_1, \beta]$ is necessary and sufficient to have herculean cutoffs.

Proof. Observe that $g(v)$ is weakly decreasing in v and, by definition of strength, $g(s_1) = s_1$. Therefore, $x_2 > g(x_2)$ if and only if $x_2 \in (s_1, \infty)$. □

Claim 15. $h(s_1) < 0$ and $h(v)$ is unbounded above.

Proof. Bidder 2 being weak ($s_1 < s_2$) implies

$$h(s_1) = s_1 F_1(s_1) - c_2 < s_2 F_1(s_2) - c_2 = 0.$$

On the other hand, $h(v)$ is unbounded above as $vF_1(v)$ is unbounded and the finite expectation assumption of F_i. □

Claim 15 plus continuity imply that there exists $x^* > s_1$ such that $h(x^*) = 0$. Therefore, $x_1 = g(x^*)$ and $x_2 = x^*$, constitute a herculean equilibrium.

Now we prove uniqueness. We begin by showing that among the herculean class the equilibrium is unique. Then we extend the uniqueness result among all equilibria. In order to have a unique equilibrium in the herculean class it is sufficient to show that $h'(v) > 0$ for all $v > s_1$, so that $h(v)$ single crosses zero at x^* from below. Differentiating and then using $g'(v)$

$$h'(v) = F_1(v) + g'(v)g(v)f_1(g(v)) = F_1(v) - g(v)^2 \frac{f_2(v)}{F_2(v)} f_1(g(v)).$$

We show that a lower bound for $h'(v)$ is positive. Using $vf_2(v) \le F_2(v)$ for $v > s_1$, and $vf_1(v) \leq F_1(v)$ for $v > c_2$ we can write $h'(v)$ as

$$h'(v) > F_1(v) - \frac{g(v)}{v} F_1\left(g(v)\right).$$

Since we are only interested in $v \geq s_1$, Claim 14 implies $v > g(v)$. Thus $h'(v) > 0$ proving uniqueness within the herculean class.

To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium; i.e., $x_1 \geq x_2$. Define $\bar{g}(v) = c_2/F_1(v)$ to be the equilibrium cutoff played by bidder 2 when bidder 1 plays $x_1 = v$, and let

$$\bar{h}\left(v\right) = vF_2\left(v\right) - \int_{\bar{g}(v)}^{v} xf_2\left(x\right)dx - c_1$$

represent bidder 1's revenue of drawing valuation v when she plays a cutoff $x_1 = v$ and 2 plays the cutoff $x_2 = \bar{g}(x_1)$. As before, because $\bar{g}(s_2) = s_2$ and $\bar{g}(v)$ being decreasing, in order to have an non-herculean equilibrium $\bar{h}$ has to be defined on $[s_2, \infty)$. Now observe that $\bar{h}\left(s_2\right) = s_2 F_2\left(s_2\right) - c_1 > 0$. By repeating the argument above $\bar{h}'(v) > 0$ and $\bar{h}(v) > 0$ for all $v \in (s_2, \infty)$, so there is no x^* such that $\bar{h}\left(x^*\right) = 0$ and no non-herculean equilibrium exists.

F Uniqueness in the Linear Model

In this section we derive the condition for uniqueness used in Examples 4, 5 and 6. Consider the following linear model

$$\pi_i(v_e) = \eta_i - \delta_i \sum_{k=1}^{n_e-1} r_i^{k-1} + v_i.$$

In this context, for a given vector of cutoff strategies $\mathbf{x}$, equation (6) is given by

$$\Pi_i(v_i, \mathbf{x}_{-i}) = \eta_i + v_i - \delta_i \mathbb{I}_{n_e>1} \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \left(\prod_{\ell \in I_i(e)} (1 - F_\ell(x_\ell)) \right) r_i^{n_e-2} \right\}$$

and $\Pi'(\mathbf{x}) = 1$. Similarly, noticing that $\pi(v_i, v_{e\setminus i}) - \pi(v_i, v_j, v_{e\setminus i}) = r^{n_e-1}\delta$ we obtain

$$\Delta_{i,j}(\mathbf{x}) = \delta F_j(x_j) \prod_{\ell \neq i,j} \left(r + F_\ell(x_\ell)(1-r)\right).$$

Then,

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} = \delta F_j(x_j) \prod_{\ell \neq i,j} \left(r + F_\ell(x_\ell)(1-r)\right).$$

Noticing that $F_\ell(x_\ell)$ for $\ell \neq i$ increases in x_ℓ we can replace $x_\ell = \overline{v}_\ell$ in the previous expression, which leads to

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} \leq \delta F_j(\overline{v}_j) \prod_{\ell \neq i,j} \left(r + F_\ell(\overline{v}_\ell)(1-r)\right).$$

When firms are symmetric, the previous expression can be used to derive equation (9). When $r = 1$, the expression simplifies to:

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} \leq \delta F_j(\overline{v}_j).$$

which can be used to construct conditions (12) and (14).

G Uniqueness with Partially Informed Bidders

Here we describe the model and derive the sufficient condition to determine whether the gradual information model of Roberts and Sweeting (2013, 2016) has a unique equilibrium when there are one potential entrant from each group.

There are two bidders, a logger and a miller, which for simplicity we call $i \in \{1,2\}$. Each bidder observes a signal $v_i = \theta_i \varepsilon_i$ where $\varepsilon_i \sim LN\left(0, \sigma_\varepsilon^2\right)$ and $\theta_i \sim LN\left(\mu_i, \sigma_\theta^2\right)$. Consequently, $v_i \sim LN\left(\mu_i, \sigma_\theta^2 + \sigma_\varepsilon^2\right)$. We call the CDF of this distribution $F_i\left(v_i\right)$. Conditional on v_i the posterior of $\theta_i | v_i \sim LN(\alpha\mu_i + (1-\alpha)\ln\left(v_i\right), \alpha\sigma_\theta^2)$, where

$$\alpha = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\theta^2}.$$

We denote the PDF of this distribution $h_i\left(\theta_i | v_i\right)$. For completeness

$$\begin{aligned} h_i\left(\theta_i | v_i\right) &= \frac{1}{\theta_i \sqrt{2\alpha\sigma_\theta^2 \pi}} \exp\left(-\frac{\left(\ln\theta_i - \left(\alpha\mu_i + (1-\alpha)\ln\left(v_i\right)\right)\right)^2}{2\alpha\sigma_\theta^2}\right) \\ f_i\left(v_i\right) &= \frac{1}{v_i \sqrt{2\left(\sigma_\theta^2 + \sigma_\varepsilon^2\right)\pi}} \exp\left(-\frac{\left(\ln v_i - \mu_i\right)^2}{2\left(\sigma_\theta^2 + \sigma_\varepsilon^2\right)}\right) \end{aligned}$$

The sufficient condition for uniqueness hold if, for every $x_k \in \left[\underline{v}_k, \overline{v}_k\right]$,

$$\frac{f_i\left(x_i\right)}{F_i\left(x_i\right)} \frac{\Delta_{i,j}\left(x_i, x_j\right)}{\Pi'_i\left(x_i, x_j\right)} < 1.$$

In the current scenario, these expressions become

$$\begin{aligned} \Delta_{i,j}\left(x_i, x_j\right) &= F_j\left(x_j\right)\left(\pi_i\left(x_i\right) - \pi_i\left(x_i, x_j\right)\right) \\ \Pi'_i\left(x_i, x_j\right) &= F_j\left(x_j\right)\pi'_i\left(x_i\right) + \int_{x_j}^{\infty} \pi'_i\left(x_i, v_j\right) dF_j\left(v_j\right) \end{aligned}$$

Table 2: Roberts and Sweeting (2013, 2016) estimates.

μ_1	μ_2	σ_ε^2	σ_θ^2	α	K	r
3.9607	3.5824	0.8578	0.3321	0.689	2.0543	27.77

Note: From Tables 3 and 4 from the cited papers.

where

$$\begin{aligned}\pi_i(x_i) &= \int_r^\infty (\theta_i - r)\, h_i(\theta_i|x_i)\, d\theta_i - K \\ \pi_i(x_i, x_j) &= \int_r^\infty \left(\int_0^{\theta_i} (\theta_i - \max\{r, \theta_j\})\, h_j(\theta_j|x_j)\, d\theta_j\right) h_i(\theta_i|x_i)\, d\theta_i - K \\ \pi_i'(x_i) &= \int_r^\infty (\theta_i - r) \frac{\partial h_i(\theta_i|x_i)}{\partial x_i} d\theta_i \\ \pi_i'(x_i, v_j) &= \int_r^\infty \left(\int_0^{\theta_i} (\theta_i - \max\{r, \theta_j\})\, h_j(\theta_j|v_j)\, d\theta_j\right) \frac{\partial h_i(\theta_i|x_i)}{\partial x_i} d\theta_i\end{aligned}$$

and

$$\frac{\partial h_i(\theta_i|x_i)}{\partial x_i} = h_i(\theta_i|x_i) \frac{(1-\alpha)(\ln\theta_i - (\alpha\mu_i + (1-\alpha)\ln x_i))}{x_i \alpha \sigma_\theta^2}$$

Using the estimates provided in Table 2, we can now compute all the necessary elements to verify sufficient condition (11).

Cutoffs lower bound: The lower bound for a firm's feasible cutoff, $\underline{v}_i$, is given by the unique solution to:

$$\int_r^\infty (\theta_i - r)\, h_i(\theta_i|\underline{v}_i)\, d\theta_i = K.$$

Computing, we obtain $\underline{v}_1 = 1.978$ and $\underline{v}_2 = 4.573$.

Upper bound: The upper bound for a firm's feasible cutoff, $\underline{v}_i$, is given by the unique solution to:

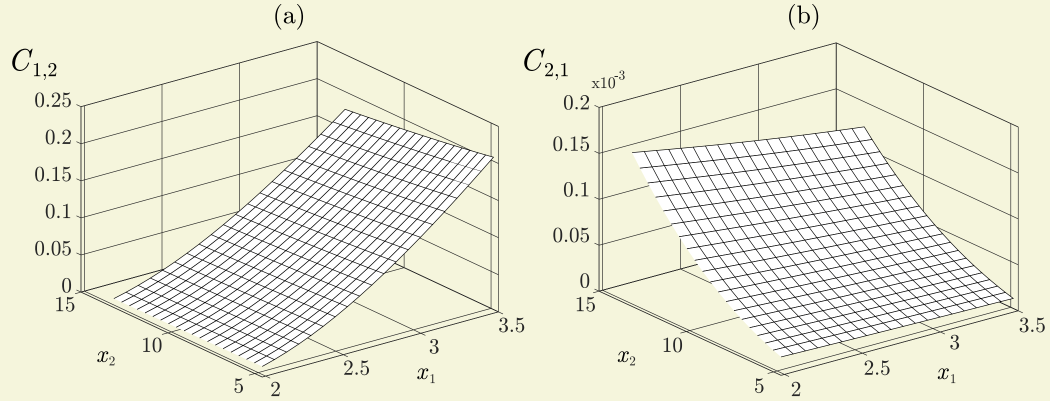
$$\int_0^\infty \pi(\overline{v}_i, v_j)\, dF_j(v_j) = 0$$

Computing, we obtain $\overline{v}_1 = 3.468$ and $\overline{v}_2 = 13.109$.

Sufficient condition: The left-hand side of sufficient condition is plotted for the relevant range of cutoff. As it can be observed in the figure below, the conditions are always less than one.

Strength and herculean equilibrium: For completeness, we also present the strength of each firm. Firm i's strength is given by the unique solution to $\sigma_i(s_i) = 0$, where

$$\sigma_i(s_i) = F_j(s_i)\, \pi_i(s_i) + \int_{s_i}^\infty \pi_i(s_i, v_j)\, dF_j(v_j) - K$$



The strength of each firm is given by $s_1 = 3.465$ and $s_2 = 12.146$. Finally, the herculean equilibrium is given by the unique solution to the system $\Pi_1(x_1, x_2) = 0$ and $\Pi_2(x_1, x_2) = 0$, where

$$\Pi_i(x_i, x_j) = F_j(x_j)\,\pi_i(x_i) + \int_{x_j}^{\infty} \pi_i(x_i, v_j)\, dF_j(v_j)$$

We find that $x_1 = 3.314 \in (\underline{v}_1, s_1)$ and $x_2 = 12.999 \in (s_2, \overline{v}_2)$.