Second-Price Auctions with Participation Costs*

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Abstract

We study equilibria and efficiency in a second-price auction with public participation costs. We generalize previous results by allowing arbitrary heterogeneity in the bidders' distribution of valuations and in their participation costs. We apply the notion of bidder *strength*, and show that an equilibrium where *stronger* bidders have a lower participation threshold than weaker bidders, called *herculean* equilibrium, always exists. We provide a sufficient condition for equilibrium uniqueness. Even though all equilibria are *ex-post* inefficient, an *ex-ante* efficient equilibrium always exists. Therefore, when conditions for uniqueness hold, the *herculean* equilibrium is unique and *ex-ante* efficient.

JEL: D21, D44, D61, D82

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1 Introduction

The empirical auction literature's recent growth is due, in part, to increased data availability. In order to be suitable for applied work, auction models should reflect the richness of real world data. For instance, as studied by Krasnokutskaya and Seim (2011), some auction formats intentionally discriminate among groups of bidders, creating *de facto* heterogeneity among bidders even if they were ex-ante homogeneous. In these environments, accurate estimates are not feasible without a model that properly captures such heterogeneity. The theoretical literature, however, has not kept pace with the empirical needs. In this article, we introduce an auction model with endogenous participation that accommodates rich forms of bidder heterogeneity. The proposed model allows empiricists to perform more precise estimations and obtain reliable counterfactuals.

We study the existence, uniqueness and efficiency of equilibria in a second-price auction (SPA) with independent private values and public participation costs. Our main contribution is to provide a general characterization of equilibria in environments that allow general forms of *bidder heterogeneity* and provide a weak sufficient condition for *equilibrium uniqueness*. We show that an equilibrium always exists, and that every equilibrium is in cutoff strategies. That is, bidders participate in the auction with certainty if and only if their private value is above some given threshold. Using the game fundamentals, we further characterize the auction's set of equilibria. We develop a notion of *strength*, which ranks the potential bidders of the game using *publicly known* characteristics of all potential bidders. We show that, in very general environments, an equilibrium where participation cutoffs are ordered by *strength*—or *herculean* equilibrium—always exists. Furthermore, we generalize existing results in Tan and Yilankaya (2006) and show that *concavity* of the distribution functions is a sufficient condition for the game to have a unique equilibrium.

For any participation game, the notion of *strength* ranks bidders by building upon the idea of symmetric equilibrium. The *strength* of a bidder corresponds to the value of the symmetric strategy that makes the bidder indifferent to participate in the auction; i.e., *strength* is the hypothetical cutoff value that would make the bidder indifferent between participating or not, conditional on all other bidders using the same cutoff strategy. This strategy is not generally an equilibrium because a symmetric cutoff may not be the optimal strategy for the opponents. A lower participation cutoff by an opponent means more competition. Thus, the bidder with the lowest symmetric cutoff—i.e., the *stronger* player—is, intuitively, more able to endure competition than other bidders. Thus, there should be an equilibrium in which a *stronger* bidder participates more often than its competitors. We show that an equilibrium in which bidders' participation cutoffs are ordered according to their *strength*, called *herculean* equilibrium, always exists. Thus, when conditions for uniqueness hold, the *herculean* equilibrium is the unique equilibrium of the game.

The ranking provided by *strength* coincides with intuitive orders, including bidders with symmetrically-distributed valuations who have different participation costs (Cao and Tian, 2013) and bidders with valuations ordered by first-order stochastic dominance (FOSD) with identical participation costs (Tan and Yilankaya, 2006). The main advantage of *strength* is it ranks bidders in situations where an intuitive order does not exist. For instance, a bidder's distribution of valuations may first-order stochastically dominate those of a competitor while having higher participation costs. In general, bidders' distributions of valuations need not be ordered in a FOSD sense. Another important feature of *strength* is that it can be generalized to rank bidders when the source of heterogeneity is not in the bidders' fundamentals. The analysis here can be easily extended to rank bidders with participation fees or subsidies and bid handicaps as in Marion (2007) and Krasnokutskaya and Seim (2011).

This article contributes to the auctions with participation costs literature. There are two broad classes of models that describe bidders' own information about their valuations: S models (Samuelson (1985)) and LS models (Levin and Smith (1994)). Levin and Smith (1994) study auction participation in enviroments in which no private information is revealed before the entry decicion is made (see also McAfee and McMillan, 1987; Tan, 1992; Jehiel and Lamy, 2015). In this framework, participation becomes a coordination game, which generally leads to multiple equilibria.[1] In contrast, our framework builds upon Samuelson (1985) who studied a symmetric environment where information is revealed prior to the entry decision.[2] Campbell (1998) uses Samuelson's framework to study coordinated entry, whereas Tan and Yilankaya (2007) examine collussive outcomes and Menezes and Monteiro (2000) study optimal auction design. Tan and Yilankaya

[1] Environments where signals are informative but public; i.e., they are observed by all bidders, would also resemble a coordination game as in Levin and Smith (1994).

[2] More generally, bidders may receive signals of their valuations before participating in the auction (c.f. Gentry and Li, 2014; Sweeting and Bhattacharya, 2015). We defer the discussion of such models to Section 5.

(2006) show that equilibrium multiplicity is guaranteed under convex distribution of valuations.

This article puts special emphasis on equilibrium uniqueness. Our interest in uniqueness stems from the rise of structural estimation methods in auction markets. There, the existence of multiple equilibria weakens empirical identification (Tamer, 2003) and precludes robust counterfactual analysis (Berry and Reiss, 2007). A common problem in the empirical auction literature is allowing for an endogenous number of participants while maintaining a framework flexible enough for bidder (self)-*selection* (Roberts and Sweeting, 2013). Although Tan and Yilankaya (2006) and Cao and Tian (2013) identify conditions for a unique equilibrium, their frameworks do not allow enough bidder heterogeneity for applied work.

We show that in symmetric games—and in auctions with restricted heterogeneity where bidders are ordered by participation costs or FOSD— bidders' probability of participating, equilibrium cutoffs, and expected revenues are intrinsically linked to one another. Thus, bidders with lower entry cutoffs are more likely to enter and obtain higher revenues. In line with Maskin and Riley (2000), we show this link may be broken in heterogeneous games. For example, in games with heterogeneous bidders, bidders can be more likely to participate in the auction and, at the same time, have lower expected revenues. Therefore, models that allow for broader bidder heterogeneity could better capture the diversity of outcomes observed in practice.

Our welfare analysis expands the early work of Stegeman (1996) (see also Lu, 2009). Although every equilibrium is *ex-post* inefficient, Stegeman (1996) shows that SPA with participation costs have one equilibrium that is *ex-ante* efficient. We contribute to this literature by providing a direct proof of Stegeman's result. Furthermore, we show that each equilibrium corresponds to a critical point on the social welfare function. Finally, by identifying the equilibrium that survives when the uniqueness condition holds, we provide a partial characterization of the efficient equilibria.

Finally, this article is closely related to the literature on entry games with private information. Particularly to Espín-Sánchez and Parra (2018). However, this article differs from our previous work in a few important aspects. First, second-price auctions do not satisfy the (strict) payoff-monotonicity assumption made in that article. Second, and more importantly, we exploit the linear structure of the bidders' payoffs to obtain more general and sharper results here. Our uniqueness condition depends only on the concavity of the distribution functions whereas, in the general setting, uniqueness also depends on the curvature of the profit function and the number of potential entrants. We also present a uniqueness result for *herculean* equilibria in an n-bidder scenario. Lastly, we provide welfare results that are not applicable in the general setting.

The rest of the article is organized as follows. Section 2 presents the model. Section 3 characterizes all equilibria, establishes existence and discusses efficiency. Section 4 defines *strength*, *herculean* equilibria and presents the main results of the article. Section 5 discusses some generalizations of the model and Section 6 concludes. All proofs are relegated to the Appendix.

2 Setup

Consider a sealed-bid second-price auction with no reservation price in an independent private values environment.[3] The auction consists of one seller, n potential bidders, and one indivisible good. Before making any participation decision, each bidder i observes her valuation for the object v_i which is drawn from an atomless distribution function F_i with full support on $\mathbb{R}_+$. We assume that each F_i is continuously differentiable and has finite expectation.[4]

Upon privately observing their own valuation, each bidder, independently and simultaneously, decides whether to participate in the auction. If bidder i decides to participate, she incurs a cost $c_i \geq 0$. The tuple $(F_i, c_i)_{i=1}^n$, which includes the number of potential bidders n, is commonly known by all the bidders. A game is called *symmetric* if $F_i = F$ and $c_i = c$ for all i.

After participation decisions are made, bidders observe the identities of the other participating agents. Afterwards, everyone submit their bids simultaneously. We simplify the bidding stage by assuming that each player bids their valuation; i.e., they play their weakly dominant strategy.[5] Therefore, we restrict attention to participation strategies. A participation strategy for bidder i is a mapping from bidder i's valuation to a probability of participating in the auction $\tau_i : \mathbb{R}_+ \to [0, 1]$. We assume that bidder i's strategy is an integrable function with respect to her

[3]For results on a common value setting see Murto and Välimäki (2015)

[4]Our results would go through if the support of F_i were the interval $[0, b_i]$ with $b_i > 0$. This, however, would complicate the exposition as we would have to deal with corner solutions.

[5]Tan and Yilankaya (2006) model non-participation as submitting a zero bid. Technically, their model is a one-stage game in which for a bidder is no longer a dominant to bid its valuation. In contrast, we explicitly model the sequential process. Both formulations are equivalent.

own type v_i. The equilibrium concept used is Perfect Bayesian Equilibrium.

Given a strategy profile $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, define

$$T_i(v) = F_i(v) + \int_v^{\infty} (1 - \tau_i(s))\, dF_i(s)$$

to be the *ex-ante* probability that bidder i does not obtain the object when the highest bid among her opponents is v. Observe that $T_i(v) > 0$ whenever $v > 0$. The expected utility of a bidder who participates in the auction with probability $\tau_i(v)$, faces opponents playing τ_{-i}, and values the good by v is:

$$u_i(\tau, v) = \tau_i(v) \left[vG_i(v) - \int_0^v s dG_i(s) - c_i \right], \tag{1}$$

where $G_i(v) = \prod_{k \neq i} T_k(v)$ is the probability that bidder i obtains the object when her valuation is v. In words, conditional on participating, the expected utility of bidder i consists of the expected value of getting the good $vG_i(v)$, minus the participation costs c_i, minus the expected price paid, which distributes according to $dG_i(v)$ and is equal to the second highest bid in the auction.

3 Characterization of Equilibria

In this section we provide a general characterization of the equilibria in the game. We establish the existence of an equilibrium and we show that, without loss of generality, we can restrict attention to cutoff strategies. In addition, we prove that, although every equilibrium of the game is *ex-post* inefficient, the game possesses an *ex-ante* efficient equilibrium.

3.1 Equilibrium Existence

Definition. A strategy $\tau_i(v)$ is called *cutoff* if there exists $x > 0$ such that

$$\tau_i(v) = \begin{cases} 1 & \text{if } v \geq x \\ 0 & \text{if } v < x \end{cases}.$$

A cutoff strategy specifies whether a bidder participates in the auction with certainty depending on her valuation being above some given threshold. The next Lemma shows that, without loss of generality, we can restrict our attention to study cutoff strategies.

Lemma 1. *For each profile of opponent's strategies τ_{-i}, bidder i has a unique best response. Bidder i's best response is a cutoff strategy given by the unique value of v that solves $u_i(\tau_i = 1, \tau_{-i}, v) = 0$.*

Lemma 1 follows from showing that, conditional on participation and regardless of opponents' strategies, a bidder's (expected) utility is monotonically increasing with respect to its own valuation, v_i. Then, because a bidders' utility is linear in the participation probability, and they want to participate whenever there is positive expected utility to do so, bidders best respond by playing a cutoff strategy. The cutoff is defined by the valuation that gives zero expected utility for participating in the auction. When a bidder's valuation is equal to its cutoff, the bidder is indifferent to whether or not to participate in the auction. We break this indifference by assuming that bidders participate. The main consequence of the Lemma 1 is that every equilibria, if any exist, must be in cutoff strategies.

From now on, we abuse notation by denoting a cutoff strategy in terms of the cutoff itself. In addition, and without loss of generality, we order the bidders' identities according to their equilibrium cutoffs, with x_1 being the bidder with the lowest cutoff and x_n the bidder with the highest. For a given vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ define $\mathbf{x}^i = (x_1, x_2, \ldots, x_i)$ to be the vector of cutoffs up to bidder i. Let $A_i^n = \prod_{j>i}^n F_j(x_j)$ be the probability that bidders playing cutoffs *above* x_i do not participate in the auction; let $B_i(v) = \prod_{j<i} F_j(v)$ be the probability that bidders playing cutoffs *below* bidder i obtain valuations lower than v, and; let

$$r_i(\mathbf{x}^i) = x_i B_i(x_i) - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} s dB_{j+1}(s) \right), \tag{2}$$

be bidder i's expected *revenue* when bidder i plays the highest entry cutoff in a game with $n = i$ potential bidders and bidder i's valuation is equal to its cutoff.[6] The next lemma characterizes every equilibria in the participation game.

Lemma 2. *Let $x_1 \le x_2 \le \cdots \le x_n$ be cutoff strategies. They constitute an equilibrium if and only if the following condition holds for each bidder i:*

$$A_i^n r_i(\mathbf{x}^i) = c_i \tag{3}$$

[6]The following notation is being used throughout the article: $\sum \emptyset = 0$, $\prod \emptyset = 1$, and $x_0 = 0$.

To understand equation (3) recall that, in equilibrium, if a bidder's valuation is equal to its cutoff, it must be indifferent to participate in the auction. For any bidder i, if $v_i = x_i$, participation by any bidder with a higher cutoff would imply losing the object. This event occurs with probability $1 - A_i^n$ and leaves bidder i with zero revenue. As a consequence, bidder i only makes revenue with probability A_i^n. In this scenario, bidder i is the participating player with the highest entry cutoff, obtaining a revenue of $r_i(\mathbf{x}^i)$. The expected revenue of bidder i is the expected revenue conditional on winning times the probability of winning. In equilibrium, when a bidder's valuation is equal to its participation cutoff, the expected revenue of participating is equal to the participation cost. This indifference condition must hold for each bidder.

Lemma 2 characterizes all equilibria of the game but does not provide any information about whether equilibria exist or about which bidder plays which cutoff. Section 4 links bidders' public characteristics to equilibrium cutoffs. The next proposition, which follows from Brouwer's fixed-point theorem, establishes equilibrium existence.

Proposition 1. *For any game* $(F_i, c_i)_{i=1}^n$ *there exists an equilibrium.*

3.2 Welfare Analysis

To conclude this section, we discuss efficiency. As Stegeman (1996) pointed out, when participation is costly, *ex-ante* and *ex-post* efficiency are not equivalent. Moreover, when participation is costly, the revelation principle no longer applies because, in the equivalent direct mechanism, each bidder incurs a cost c_i of sending a message—i.e., submitting a bid—(Myerson, 1981). Thus, as there is no "cost free" way to elicit bidders' preferences, any optimal mechanism would trade off the direct cost of *ex-ante* soliciting more messages with the potential benefits from a better *ex-post* allocation when more messages are solicited. Although such a mechanism would be *ex-ante* optimal, any mechanism that does not solicit messages from all bidders would in general produce *ex-post* misallocation with positive probability.

To illustrate this point, consider Figure 1, which depicts an equilibrium with two potential bidders and equal participation costs ($c_i = c$), but different cutoff equilibrium strategies ($x_1 < x_2$). Note that for an allocation to be *ex-post* efficient, only the bidder with the highest valuation, which must be above the entry cost,

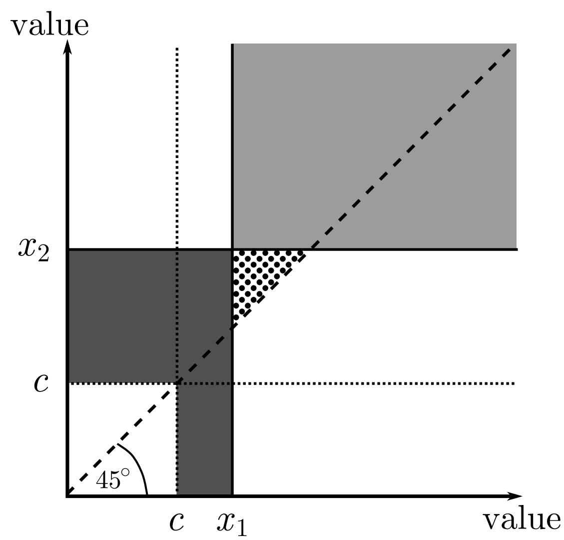

Figure 1: *Ex-post* inefficiency.

should participate in the auction. In general, three cases of inefficiencies arise: (i) The dark-shaded area represents realizations of $\mathbf{v} = (v_1, v_2)$ in which there is no participation but $v_i > c_i$; (ii) The lightly-shaded area represents situations in which both bidders enter the auction; finally, the dotted area (iii) represents realizations of $\mathbf{v}$ in which bidder 1 participates and has a lower valuation than the bidder 2, who does not participate, which means the the lowest valuation bidder gets the object.

Case (i) (Insufficient Participation) is inefficient because it is efficient that at least one bidder whose valuation is greater than c participates in the auction, pays the participation costs and obtains the good. Case (ii) (Excessive Participation) is inefficient because it is efficient that at most one bidder pays the participation costs. Case (iii) (Misallocation) is inefficient because, even if there is exactly one bidder who participates in the auction, it is the "wrong" bidder; i.e., the one with the lower valuation for the good. It is worth noticing that, conditional on participation, the bidder with the highest valuation wins the auction independent of the number of participants. Therefore, inefficiencies only arise due to miscoordinated participation.

From an *ex-ante* perspective, however, there is an efficient equilibrium. Consider the problem that a planner faces when choosing a strategy for each bidder conditional on the bidders' private information, i.e., the planner chooses a set of functions $\tau_i^* : \mathbb{R}_+ \to [0,1]$ determining the probability that bidder i participates given her valuation. Using similar arguments to those in Lemma 1 it can be shown that the planner will only consider cutoff functions.[7] Therefore, the planner chooses the vector of cutoffs $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that maximizes

$$W(\mathbf{x}) = \sum_{i=1}^{n} \left[\int_{x_i}^{\infty} (v_i \Omega_i(v_i, x_{-i}) - c_i) \, dF_i(v_i) \right] \tag{4}$$

where $\Omega_i(v_i, x_{-i}) = \prod_{k \neq i} F_k(\max\{v_i, x_k\})$ is the probability that bidder i obtains the object when her valuation is v_i. Notice that the transfer between the winning bidder and the seller is not relevant in terms of welfare. To explain (4) further, let's focus on the planner's payoffs coming from bidder i. With probability $dF_i(v_i)$, bidder i draws the valuation v_i and participates in the auction whenever $v_i \geq x_i$, in which case she pays the costs of participating c_i and wins the object with probability $\Omega_i(v_i, x_{-i})$. Total welfare is just the aggregation of all possible values for a given bidder (integral) aggregated across all bidders (summation).

Proposition 2. *There exists an equilibrium that is* ex-ante *efficient. Every equilibrium of the game corresponds to a critical point of the welfare function. That is, an equilibrium corresponds to either a (possibly local) maximum, minimum, or saddle point of* $W(\mathbf{x})$.

Proposition 2 shows that there is always an efficient equilibrium. It also says that every equilibrium of the game is a critical point of the welfare function as it satisfies the first order conditions of the planner's maximization problem. For the case of $n = 2$ potential bidders, the Hessian of the planner's problem, evaluated at a critical point and, without loss of generality, rearranging identities so that $x_1 < x_2$, is equal to:

$$H(\mathbf{x}) = -\begin{pmatrix} f_1(x_1)F_2(x_2) & x_1 f_1(x_1) f_2(x_2) \\ x_1 f_1(x_1) f_2(x_2) & f_2(x_2)F_1(x_2) \end{pmatrix}.$$

Under concavity of F_i (and using Lemma 4 in the Auxiliary results section in the Appendix), the second order condition for a maximum is satisfied at *every* critical point.[8] Therefore, only one critical point can exist and, because every equilibrium is a critical point, the game possess a unique equilibrium. This finding

[7] Notice that in this case the planner only solicits messages (bids) from bidders whose valuations are above the specified cutoff. That is, the planner does not solicit messages from all bidders with certainty.

[8] Under concavity, at every equilibrium $\mathbf{x}$, the first minor of $H(\mathbf{x})$ is always negative and

$$\det(H(\mathbf{x})) = f_1(x_1) f_2(x_2) \left(F_1(x_1)F_2(x_2) - (x_1)^2 f_1(x_1) f_2(x_2) \right),$$

suggests that concavity may be sufficient to guarantee uniqueness in general, which motivates part of our analysis below.

4 ***Strength*** and ***Herculean*** Equilibrium

In this section we further characterize the equilibria of the game. In particular, we relate the bidders' public characteristics $(F_i, c_i)_{i=1}^n$ to the game's equilibrium strategies and its efficiency properties. The next definition, which ranks bidders using the game fundamentals, helps in the characterization of the participation strategies.

Definition (*Strength*)**.** For a given game $(F_i, c_i)_{i=1}^n$, the *strength* of bidder i is the unique number $s_i \in \mathbb{R}_+$ that solves:

$$s_i \prod_{k \neq i} F_k(s_i) = c_i. \tag{5}$$

We say that bidder i is *stronger* than j if $s_i < s_j$.

Observe that the left hand side of (5) is strictly increasing in s_i, takes the value of 0 when $s_i = 0$, and is unbounded above. Therefore, *strength* is well defined, which means each bidder i has a unique scalar s_i and, consequently, ranking all the bidders of the game. *Strength* ranks bidders by their ability to endure competition. It does so by computing the *symmetric strategy* that makes bidder i indifferent to participating in the auction. Intuitively, bidder i is *stronger* than j $(s_i < s_j)$ when i is willing to enter under a lower valuation s_i and higher probability of facing competition, $1 - \prod_{k \neq i} F_k(s_i)$, than bidder j. The notion of *strength* naturally leads to our next definition.

Definition (*Herculean* Equilibrium)**.** An equilibrium is called *herculean* if the equilibrium cutoffs are ordered by *strength*, with *stronger* bidders playing *lower* cutoffs. That is, $x_i < x_j$ if and only if $s_i < s_j$.

Corollary 1. *In symmetric games, bidders'* strength *and the symmetric equilibrium cutoff coincide.*

With symmetric bidders, every potential bidder is equally *strong* and the definition of *strength*, and the notions of *herculean* equilibrium and *symmetric equilibrium* coincide. With *herculean* equilibrium, we construct an *asymmetric* analogue

which is positive whenever $x_1 < x_2$ and $F(x) \geq xf(x)$.

of the *symmetric* equilibrium for *asymmetric* entry games. We do so by ranking bidders according to *strength*, which summarizes each player's ability to endure competition by using the notion of symmetric strategies. Since *stronger* bidders are better able to face competition, they should be more inclined to participate in the auction than weaker bidders, resulting in lower participation cutoffs. The next section shows that this intuition is correct.

4.1 *Herculean* equilibrium under two bidders

From now on we order bidders by strength, with bidder 1 being the strongest bidder in the game. The following proposition is our main result in the two-bidder context.

Proposition 3. *There always exists an* herculean *equilibrium. Every* herculean *equilibria is characterized by cutoffs $x_1 \leq x_2$ that jointly solve*

$$x_1 F_2(x_2) = c_1 \tag{6}$$

$$x_2 F_1(x_2) - \int_{x_1}^{x_2} v dF_1(v) = c_2. \tag{7}$$

Moreover, if F_1 and F_2 are concave, an herculean *equilibrium is the unique equilibrium of the game and, therefore, efficient.*

Proposition 3 confirms the intuition that strong bidders; i.e., bidders who are better able to handle competition, should play lower participation cutoffs. In games where bidders have symmetric valuation distributions, this translates into the fact that *strong* bidders are more likely to participate in an *herculean* equilibrium. Furthermore, Proposition 3 shows that concavity of the distribution functions is sufficient to guarantee uniqueness of the equilibrium. Intuitively, concavity is a payoff-stability condition. Concavity guarantees that bidder i's expected revenue is increasing in their cutoff x_i, even when bidder j best responds to the increase in x_i by increasing competition (decreasing x_j). This implies that there is only one cutoff that makes bidder i indifferent between participating in the auction or not, which leads to a unique equilibrium.

Proposition 2 above shows that the participation game always has an *ex-ante* efficient equilibrium. Thus, when conditions for uniqueness hold, the equilibrium must be efficient and *herculean*. Concavity, therefore, is a guarantee of both unique equilibrium and efficient outcomes. Observe, however, that the proposition does not tell us that an *herculean* equilibrium is always *ex-ante* efficient. In the Online Appendix we show that, when conditions for multiplicity hold, the symmetric equilibrium in a symmetric game is not efficient, implying that the *herculean* equilibrium may not be efficient.

Proposition 3 generalizes existing results in Tan and Yilankaya (2006) and Cao and Tian (2013) in two ways. By introducing the notion of *strength*, we are able to associate bidders' public characteristics with equilibirum cutoff order in *any* game $(F_i, c_i)_{i=1}^2$. Also, our result shows that concavity is sufficient to achieve equilibrium uniqueness with no further restrictions. To prove uniqueness, Tan and Yilankaya (2006) assume that each bidder has the same participation costs (i.e., $c_i = c$ for all i), and that the stronger bidder First Order Stochastically Dominates (FOSD) the other. Similarly, Cao and Tian (2013) proves uniqueness under concavity and symmetric valuation distributions. Therefore, a *strong* bidder is defined as the player with lower participation costs. As shown in Lemma 3 below, our notion of *strength* encompasses both cases. Observe however, that the converse is not necessarily true. A *strong* bidder does not require a distribution that FOSD the *weak* bidder or an ordering by participation costs.

4.2 Equilibrium and Revenues

We highlight both the positive and normative implications of auction models with bidder asymmetry. Below, we argue that symmetric models of auction participation —and those models that impose a strong order among bidders—cannot fully capture the diverse outcomes that empirical work requires, because they limit the relationship between participation and observed outcomes. Let $\pi_i(\mathbf{x})$ be the *ex-ante* expected payoff of bidder i under the vector of cutoffs $\mathbf{x} = (x_1, x_2)$; i.e.,[9]

$$\pi_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} s dF_j(s) - c_i \right) dF_i(v). \qquad (8)$$

Proposition 4. *In a symmetric game, or in games where bidders play* herculean *equilibria and are ordered by cost or stochastic dominance, bidders' ex-ante payoffs and probability of participating can be ranked in terms of their cutoffs. Bidders playing the lowest cutoff obtain higher payoffs and are more likely to participate. In an asymmetric game, cutoff ranking does not correspond to payoff ranking.*

[9]For ease in exposition, the result is presented in the context of two bidders. The arguments can be easily extended to arbitrary number of bidders.

We construct an example of asymmetric bidders' whose payoff are not ranked according their entry cutoffs. Let $F_1(v) = v/(v+2)$, $F_2(v) = 1 - e^{-v/2.8854}$ and $c_i = 1$. Observe that both distributions are concave, which implies that there is a unique equilibrium. Because both bidders are equally strong ($s_i = 2$), the unique equilibrium is given by cutoffs equal to the bidders' strength; i.e., $x_i = 2$. However, the expected payoffs of bidder 1 are greater than the expected payoff of bidder 2. Because $F_1(v) < F_2(v)$ for every $v > 2$, bidder 1 gets a higher payoff that bidder 2. If we decrease bidder 2's cost by an infinitesimally small amount, we can construct an equilibrium in which bidder 2 gets a lower expected payoff than bidder 1 despite playing a lower entry cutoff. By continuity it is easy to see that if the participation cost of bidder 2 is slightly lower than the participation cost of bidder 1, then bidder 2 will be *stronger* than bidder 1 and will play a lower cutoff. In this case, bidder 2 will have a lower cutoff equilibrium strategy than bidder 1. If we make bidder 1's cost decrease infinitesimally small, however, we can construct an equilibrium in which bidder 2 gets a lower expected payoff than bidder 1, but is *stronger* and plays a lower cutoff than bidder 1. This kind of equilibria is not possible when the game is symmetric or in the cases studied by Tan and Yilankaya (2006) and Cao and Tian (2013).

4.3 *Herculean* equilibrium for three or more bidders

The existence of *herculean* equilibrium in games with three or more bidders is linked to the robustness of the ranking provided by *strength*. In particular, existence depends on whether the current *strength* order is maintained when a new potential bidder is added into the game. The notion of *strength* is local in the sense that the *strength* order among two bidders may depend on the existence of a third bidder. To illustrate this point Figure 2 depicts the *strength* of bidder 1 and 2 under symmetric costs with and without the presence of a third bidder. Using the definition of *strength* in equation (5), under symmetric participation costs, the *strength* of bidder i in a game with two bidders is given by $s_i F_{i-2}(s_i) = c$. In Figure 2 we depict a situation in which bidder 1 is *stronger* ($s_1 < s_2$). When a third bidder is added, the *strength* of bidder i is determined by $\bar{s}_i F_{i-2}(\bar{s}_i) = c/F_3(\bar{s}_i)$. As shown in the figure, when the functions cross, it is possible to have $\bar{s}_1 > \bar{s}_2$, thus reversing the *strength* ranking. When this rank is reversed, existence of an *herculean* equilibrium is not guaranteed. In Lemma Lemma 3 below we show that, under reasonable assumptions, the ranking is robust and the existence of *herculean*

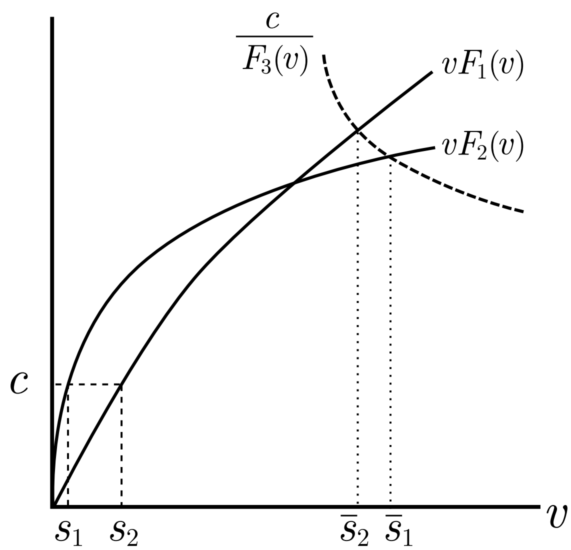

Figure 2: *Strength* is a local condition.

equilibrium is guaranteed. Concavity is a sufficient condition for uniqueness in all these cases.

4.4 *Herculean* equilibrium for two groups of bidders

Suppose there exist two groups of bidders, say groups 1 and 2. Each group is characterized by pairs (F_i, c_i). Let m_i be the number of bidders in group i. Without loss of generality, assume that bidders in group 1 are stronger than those in group 2 ($s_1 \leq s_2$).

Proposition 5. *There always exists a* herculean *equilibrium. Every* herculean *equilibrium is characterized by the cutoffs $x_1 \leq x_2$ that jointly solve*

$$x_1 F_1(x_1)^{m_1-1} F_2(x_2)^{m_2} = c_1 \tag{9}$$

$$F_2(x_2)^{m_2-1}\left[x_2 F_1(x_2)^{m_1} - \int_{x_1}^{x_2} v d\left(F_1(v)^{m_1}\right)\right] = c_2. \tag{10}$$

Moreover, if F_1 and F_2 are concave the herculean *equilibrium is the unique equilibrium of the game and, thus, efficient.*

Proposition 5 generalizes proposition 3 to the case in which there is more than one bidder in each group of bidders. The uniqueness proof has two steps which both use concavity. The first step shows that, under concavity, symmetric bidders—that is, bidders belonging to the same group—play symmetric strategies in equilibrium.

Thus, we can restrict our attention to group-symmetric strategies. Then, we show that, under concavity, the only group-symmetric equilibrium is the *herculean* one. By contrast, if the distribution of valuations were not concave, there could be equilibria in which bidders in the same group play different strategies, group-symmetric equilibria in which the cutoffs are not ordered by strength, or even multiple *herculean* equilibria. Notice that, by definition, any equilibria where bidders in the same group play a different strategy are not *herculean* equilibria.

Beyond the restriction that bidders belong to one of two groups, there is no restriction on the distribution of valuation or entry costs. The two-group model is especially useful for applied work when bidders are divided by exogenous factors such as incumbency (incumbent vs entrant) or size (small vs large). In the empirical auction literature, examples of two-group environments include Athey *et al.* (2011), Krasnokutskaya and Seim (2011) and Roberts and Sweeting (2013, 2016) among others.

4.5 Robust *strength*order among bidders

To conclude, we extend our results to an environment in which the *strength* ranking among two bidders is robust to the existence of other bidders. Lemma 3 shows that condition (11) is sufficient to guarantee that the *strength* ranking does not change when more potential bidders from different groups can participate in the auction. In particular, we assume that for any two bidders i and j, with $i < j$, the following condition holds:

$$F_i(v)c_i \leq F_j(v)c_j \text{ for all } v \geq \min\{c_i, c_j\}. \tag{11}$$

Lemma 3. *When condition* (11) *holds, bidders are ordered by* strength *with bidder* 1 *being the strongest bidder in the game.*

Condition (11) includes, as particular cases, bidders with symmetric costs ordered by FOSD (Tan and Yilankaya, 2006) and bidders with symmetric distributions of valuations ordered by participation costs (Cao and Tian, 2013). It further extends those enviroments in two ways. First, it does allow distribution functions to cross (as in Figure 2) so long as the crossing occurs for valuations under the entry costs. Second, the condition does not restrict bidders to belong to only one of two groups. In particular, if condition (11) holds with equality for some bidders, the condition allows for an arbitrary number of (strictly ordered) groups of bidders, with each group possessing an arbitrary number of bidders.[10]

Proposition 6. *Under condition* (11)*, an* herculean *equilibrium always exits. Furthermore, if F_i are concave, the* herculean *equilibrium is the unique equilibrium of the game and, thus, is efficient.*

5 Discussion

In this section we discuss potential applications for our framework. Also, we discuss the role of certain assumptions and potential extensions.

Participation Costs In the model, participation costs could represent entry fees charged by the auctioneer, the cost of preparing and submitting a bid, the opportunity cost of attending the auction or, travel costs to the auction site. In all these cases, our results are a starting point for auction design with costly participation and heterogeneous agents (c.f. Menezes and Monteiro, 2000; Celik and Yilankaya, 2009; Moreno and Wooders, 2011, who study the case of symmetric agents).

Information on Participation Costs We assumed that valuations are bidders' private information whereas participation costs are common or public knowledge. The opposite scenario in which valuations are common knowledge and participation costs are private satifies the monotonicity assumption in Espín-Sánchez and Parra (2018). When both the valuations and the participation costs are common knowledge, the auction resembles a coordination game with perfect information. These types of games generally involve multiple equillibria. Finally, the case where both valuations and participation costs are private is more complicated because it involves multi-dimentional types. Nonetheless, Cao *et al.* (2014) extended some of the results in Tan and Yilankaya (2006) to the case where both valuations and participation costs are private information. Because participation costs enter linearly in the utility function, and are not contingent on winning, we conjecture that the methodology developed in this article, combined with the results in Cao *et al.* (2014), could extend the uniqueness results to the private costs and valuations environment.

[10] Miralles (2008) presents an existence result in an scenario with n-bidders are ordered by FOSD. The proposition below generalizes existence for a broader class of games and also provide a condition for equilibrium uniqueness.

Information on own valuation In this article we study an environment in which bidders know their valuations. However, one can think of a more general setting where, before making the participation decision, bidders observe signals about their valuations. At one extreme there are LS models (Levin and Smith, 1994) with infinitely noisy signals; i.e., bidders have no private information when deciding whether to participate. At the other extreme are S models (Samuelson, 1985) in which bidders are perfectly (and privately) informed about their type. In between, are the *selective entry* models where bidders become partially informed before entry decisions (c.f. Gentry and Li, 2014; Sweeting and Bhattacharya, 2015). It is not hard to show that, if the private signal satisfies an information regularity condition, our results can be extended to the selective entry environment.

6 Concluding Remarks

In this article we generalized existing results about second-price auctions with participation costs by allowing heterogeneity both in distributions of valuations and in participation costs. We developed the concept of *strength*, which uses bidders' public characteristics—here, distributions of valuations and participation costs—to rank bidders in the game. We studied three environments that are of special interests for applied work: two arbitrarily heterogeneous bidders; any number of heterogeneous bidders that belong to one of two groups, and; any number of bidders belonging to any number of groups with group characteristics ordered according to a robustness condition for *strength*.

We showed that an equilibrium with cutoffs ordered by *strength*—called *herculean* equilibrium—always exists. Moreover, we showed that when the distribution of valuations are concave, the *herculean* equilibrium is the unique equilibrium of the game. Because there is always an *ex-ante* efficient equilibrium, when the conditions for uniqueness hold, the *herculean* equilibrium is also the *ex-ante* efficient equilibrium.

We believe that the methodology developed here can be extended to study second-price auctions with more general environments such as interdependent or affiliated values. The methodology can be extended to auction settings in which endogenous heterogeneity is created by the auction designer, including scenarios in which a bid handicap is imposed on a subset of bidders during the bidding stage (e.g., bid preference programs for entrants). Finally, our analysis would be useful when estimating an optimal participation fee. Moreover, because the revelation principle no longer applies, our tools would be useful to analyze revenue comparisons when using participation fees vs reserve price. Such models are a promising avenue for future research.

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Appendix

Omitted Proofs

Proof of Lemma 1. Pick any τ_{-i}. Since i's utility is linear in τ_i, it is a best response to participate with probability one whenever there is a positive payoff of doing so. Hence, it is sufficient to show that, conditional on bidder i participating in the auction ($\tau_i(v) = 1$), i's utility crosses zero at a singleton point and from below. Differentiating $u_i(\tau_i = 1, \tau_{-i}, v) = \left[vG_i(v) - \int_0^v x dG_i(x) - c_i\right]$ with respect to v we obtain that $du_i/dv = G_i(v) > 0$ for all $v > 0$, which implies that i's utility is strictly increasing in v. By the finite expectation assumption on F_i, u_i is unbounded above in v. Therefore, since $u_i(\tau_i = 1, \tau_{-i}, 0) < 0$, there exist a unique best response which is given by the unique value of v that solves $u_i(\tau_i = 1, \tau_{-i}, v) = 0$. ■

Proof of Lemma 2. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a profile of cutoff strategies. Denote bidder i's expected utility of participating in the auction when her valuation is v, and the opponents play the cutoffs $\mathbf{x}_{-i}$ by $u_i(0, \mathbf{x}_{-i}, v)$. Lemma 1 shows that bidder i's best response to $\mathbf{x}_{-i}$ is given by the unique valuation x_i satisfying $u_i(0, \mathbf{x}_{-i}, x_i) = 0$. In particular, using equation (1) when every opponent uses a cutoff strategy, $u_i(0, \mathbf{x}_{-i}, x_i) = 0$ is equivalent to (3) which proves the Lemma. ■

Proof of Proposition 1. We establish that the conditions of Brouwer Fixed Point Theorem are meet. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a collection of cutoffs. By Lemma 1, bidder i's best response to the profile of strategies $\mathbf{x}_{-i}$ is given by the unique valuation v that solves $u_i(0, \mathbf{x}_{-i}, v) = 0$. Since F_i is atomless and has full support, bidder i's best response is continuous in each of the opponent cutoffs. Moreover, since $u_i(0, \mathbf{x}_{-i}, v)$ is increasing in the opponents' cutoffs, the lowest utility for bidder i is achieved when each opponent participates with certainty (i.e., $\mathbf{x}_{-i} = \mathbf{0}_{-i}$). Let K_i be valuation of bidder i that satisfies $u_i(\mathbf{0}, K_i) = 0$. Hence, the vector of best responses is a continuous mapping from the compact and convex set $\times_{i=1}^n [0, K_i]$ to itself and all conditions of Brouwer Fixed Point Theorem are meet, proving existence of equilibrium. ■

Proof of Proposition 2. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where, without loss of generality, we order the bidders identities from the lowest cutoff chosen by the planer, x_1, to the highest, x_n. Differentiating (4) with respect to x_i we obtain

$$W_{x_i}(\mathbf{x}) = f_i(x_i)(c_i - x_i\Omega_i(\mathbf{x})) + \sum_{k \neq i} \int_{x_k}^{\infty} v_k \left(\frac{d\Omega_k(v_k, \mathbf{x}_{-k})}{dx_i}\right) dF_k(v_k).$$

Observing that $d\Omega_k(v_k, \mathbf{x}_{-k})/dx_i = f_i(x_i) \prod_{\ell \neq k,i} F_\ell(\max\{v_k, x_\ell\})$ if $v_k \leq x_i$ and zero otherwise, we can write

$$W_{x_i}(\mathbf{x}) = -f_i(x_i) \left(x_i\Omega_i(\mathbf{x}) - \sum_{k=1}^{i-1} \left\{ \int_{x_k}^{x_i} v_k \prod_{\ell \neq k,i} F_\ell(\max\{v_k, x_\ell\}) dF_k(v_k) \right\} - c_i \right). \quad (12)$$

Corner solutions are not welfare maximizing as, when we take $x_i = 0$, $W_{x_i}(0, \mathbf{x}_{-i}) > 0$ for all $\mathbf{x}_{-i}$; and $\lim_{x_i \to \infty} W_{x_i}(x_i, \mathbf{x}_{-i}) < 0$ due to the unboundedness of $x_i\Omega_i(\mathbf{x})$. Therefore, an interior maximum exists, which is characterized by a value of x_i satisfying $W_{x_i}(\mathbf{x}) = 0$.

It can be easily verified that the term inside the parenthesis in equation (12) is equal to zero whenever condition (3) holds. Therefore, we conclude that there exists a cutoff equilibrium that is *ex-ante* efficient. Moreover, since every equilibrium satisfies $W_{x_i} = 0$, they are a critical point of W. ■

Proof of Proposition 3. Corresponds to the case $m_1 = m_2 = 1$ in the proof of Proposition 5. See also the Online Appendix for a direct proof. ■

Proof of Proposition 4. Define the *ex-ante* expected payoff of bidder i under the vector of cutoffs $\mathbf{x} = (x_1, x_2)$ as:

$$\pi_i(\mathbf{x}) = \int_{x_i}^{\infty} \left(vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v,x_j\}} sdF_j(s) - c_i \right) dF_i(v)$$

Let $x_1 < x_2$, subtracting $\Delta_{1,2} = \pi_1(\mathbf{x}) - \pi_2(\mathbf{x})$ for the three different scenarios we obtain:

$$\Delta_{1,2} = \begin{cases} \int_{x_1}^{x_2} (v - c)dF(v) & \text{sym.} \\ \int_{x_1}^{x_2} (v - c_1)dF(v) + (c_2 - c_1)(1 - F(x_2)) & \text{cost} \\ \int_{x_1}^{x_2} (vF_2(x_2) - c)dF_1(v) + \int_{x_2}^{\infty} \Gamma_1(v, x_2)f_1(v) - \Gamma_2(v, x_1)f_2(v)dv & \text{FOSD} \end{cases}$$

where $\Gamma_i(v, x) = xF_{3-i}(x) + \int_x^v F_{3-i}(s)ds - c$ which is increasing in v and x and positive if the value of x corresponds to an equilibrium cutoff for bidder i.[11] The first case corresponds to symmetric bidders ($F_i(v) = F(v)$ and $c_i = c$ for all i). Since in equilibrium $x_1F(x_2) = c_1$, $\pi_1(\mathbf{x}) > \pi_2(\mathbf{x})$ whenever $x_1 < x_2$. Thus, in a symmetric game, cutoff order implies expected payoff order. Similarly, for the second case, in an *herculean* equilibrium in which bidders are ordered by costs ($c_2 > c_1$) and using the same argument above, bidders expected payoff are ordered. Lastly, in the third case, when bidders play a *herculean* equilibrium and bidders are ordered by first order stochastic dominance ($F_1(v) \leq F_2(v)$ for all v) we have that $\Gamma_1(v, x) \geq \Gamma_2(v, x)$ for any v and x. Then,

$$\int_{x_2}^{\infty} \Gamma_1(v, x_2)f_1(v)dv > \int_{x_2}^{\infty} \Gamma_2(v, x_1)f_1(v)dv \geq \int_{x_2}^{\infty} \Gamma_2(v, x_1)f_2(v)dv$$

where the first inequality follows from the change in identity and *herculean* cutoffs ($x_1 < x_2$), and the last inequality follows from integrating monotonic functions under stochastic dominance. Which proves that $\Delta_{1,2} > 0$ in the three cases. For the order in the participation probability notice that when the distribution are symmetric, cutoff order and probability order are equivalent. FOSD in a *herculean* equilibrium implies $x_1 < x_2$ iff F_1 FOSD F_2. Thus, $F_1(x_1) < F_1(x_2) \leq F_2(x_2)$ and the order follows. ■

Proof of Proposition 5. Start by observing that equations (9) and (10) define an equilibrium as they correspond to equation (3) for the case in which bidders play symmetrically within group.

Existence. By construction. If $s_1 = s_2 = s$ there is a *herculean* equilibrium with cutoffs

[11] Integration by parts was used to obtain $\Gamma_i(v, x)$

$x_1 = x_2 = s$. Assume $s_1 < s_2$, let $g(x)$ the function implicitly defined by

$$g(x)F_1(g(x))^{m_1-1}F_2(x)^{m_2} = c_1.$$

The function $g(x)$ represents the cutoff that bidders in class 1 had to play so that condition (9) is satisfied when everyone in class 2 plays the cutoff $x_2 = x$. Observe that $g(x)$ is strictly decreasing in x and satisfies $g(s_1) = s_1$. Define the function $h : [s_1, \infty) \to \mathbb{R}$ by

$$h(x) = F_2(x)^{m_2-1}\left[xF_1(x)^{m_1} - \int_{g(x)}^{x} yd(F_1(y)^{m_1})\right] - c_2.$$

The function $h(x)$ is continuous and corresponds to the payoffs that a member of class 2 obtains by playing the cutoff $x_2 = x$ when all other members of class 2 play x and all members of class 1 respond by playing $x_1 = g(x)$. A *herculean* equilibrium exists if there is x^* such that $h(x^*) = 0$ and $x^* > g(x^*)$. The next two claims prove the result.

Claim 1. $x^* \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$.

Proof. Because $g(x)$ is weakly decreasing in x and $g(s_1) = s_1$, $x_1 = g(x^*) < x^* = x_2$ if and only if $x_2 \in (s_1, \infty)$. □

Claim 2. $h(s_1) < 0$ and $h(x)$ is unbounded above.

Proof. Class 2 being the weak class implies $h(s_1) = s_1F_1(s_1)^{m_1}F_2(s_1)^{m_2-1} - c_2 < 0$. On the other hand, $h(x)$ is unbounded above as $xF_1(x)^{m_1-1}F_2(x)^{m_2}$ is unbounded and the finite expectation assumption. □

By the intermediate value theorem, Claim 2 plus continuity imply that there exists $x^* \in (s_1, \infty)$ such that $h(x^*) = 0$. On the other hand, $h(x^*) = 0$ holds if and only if equations (9) and (10) are satisfied. Therefore, by Claim 1, we have a *herculean* equilibrium with $x_1 = g(x^*)$ and $x_2 = x^*$.

Uniqueness. From Lemma 6 we know that, under concavity, symmetric bidders must play symmetric cutoffs. We need to show that there is no other *herculean* equilibrium, and that no non-*herculean* equilibria exists. Lemma 4.1 (concavity) is used in both steps.

Claim 3. There exists a unique *herculean* equilibrium.

Proof. In a *herculean* equilibrium bidders are ordered by strength, thus we have to show there is no other equilibrium such that $x_1 < x_2$ and equations (9) and (10) hold; i.e., there exists a unique $x^* > s_1$ such that $h(x^*) = 0$. It is sufficient to show that $h'(x) > 0$ for all $x \geq s_1$, so that $h(x)$ single-crosses zero from below. Differentiating

$$h'(x) = F_2(x)^{m_2-1}\Bigg\{(m_2-1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{m_1} - \int_{g(x)}^{x} yd\left(F_1(y)^{m_1}\right)\right] + F_1(x)^{m_1} + m_1g'(x)g(x)f_1(g(x))F_1(g(x))^{m_1-1}\Bigg\}.$$

Because $F_2(x)^{m_2-1} > 0$, it is sufficient to show that the term in braces is non-negative for all $x \geq s_1$. Implicitly differentiating $g(x)$

$$g'(x) = -\frac{m_2g(x)F_1(g(x))}{F_1(g(x)) + (m_1-1)g(x)f_1(g(x))}\frac{f_2(x)}{F_2(x)}$$

replacing into the expression in braces delivers

$$(m_2-1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{m_1}-\int_{g(x)}^{x} yd\left(F_1(y)^{m_1}\right)\right] + \left[F_1(x)^{m_1}-\frac{m_1 m_2 g(x)^2 f_1(g(x))F_1(g(x))^{m_1}}{F_1(g(x))+(m_1-1)g(x)f_1(g(x))}\frac{f_2(x)}{F_2(x)}\right]. \quad (13)$$

It is shown that a lower bound for the expression above is always positive. Maximize the subtracting term in the first square brackets by taking the upper bound $x\int_{g(x)}^{x} dF_1(y)^{m_1}$ in the integral. Using Lemma 4.1 in the auxiliary results section (concavity: $xf(x) \leq F(x)$), maximize the subtracting term in the second square brackets by substituting $g(x)f_1(g(x))$ for $F_1(g(x))$ in the denominator. Then, equation (13) becomes

$$F_1(x)^{m_1}+\left[(m_2-1)x-m_2 g(x)\right]F_1(g(x))^{m_1}\frac{f_2(x)}{F_2(x)} \geq F_1(x)^{m_1}\left(1-\frac{g(x)}{x}\right)$$

where $x \geq g(x)$ for $x \geq s_1$, and $f_2(x)/F_2(x) \leq x^{-1}$ (Lemma 4.1) were used to obtain the inequality. Hence the lower bound of (13) is non-negative iff $x \geq g(x)$, which is true as $x \geq s_1$. □

Claim 4. There is no equilibrium in which strong bidders play a higher cutoff than weak bidders.

Proof. To prove that the only equilibrium is the herculean, suppose we have a non-*herculean* equilibrium. By Lemma 6, in the auxiliary results section, symmetric bidders must play symmetric cutoffs under concave distribution functions. Thus, the only possibility is to have $x_1 > x_2$ but $s_1 < s_2$. Define $\bar{g}(x)$ to be the function that satisfies

$$\bar{g}(x)F_2(\bar{g}(x))^{m_2-1}F_1(x)^{m_1} = c_2.$$

Similarly, define

$$\bar{h}(x) = F_1(x)^{m_1-1}\left[xF_2(x)^{m_2}-\int_{\bar{g}(x)}^{x} yd(F_2(y)^{m_2})\right]-c_1.$$

The function $\bar{g}(x)$ is decreasing in x, satisfies $\bar{g}(s_2) = s_2$, and represents the cutoff that class 2 has to play so that condition (9) is satisfied when everyone in class 1 plays the cutoff $x_1 = x$. The continuous function $\bar{h}(x)$ corresponds to the payoffs that a member of class 1 obtains by playing the cutoff $x_1 = x$ when all other members of class 1 play x and all members of class 2 respond by playing $x_1 = g(x)$. We show that there is no x such that $x_1 = x > \bar{g}(x) = x_2$ and $\bar{h}(x) = 0$, which implies that condition (10) does not hold and no non-*herculean* equilibrium exists.

Observe that $x > \bar{g}(x)$ if and only if $x \in (s_2, \infty)$ and that $s_1 < s_2$ implies that

$$\bar{h}(s_2) = s_2 F_1(s_2)^{m_1-1}F_2(s_2)^{m_2}-c_1 > s_1 F_1(s_1)^{m_1-1}F_2(s_1)^{m_2}-c_1 = 0.$$

By an analogous argument given in Claim 3, concavity implies $\bar{h}'(x) > 0$, and $\bar{h}(x) > 0$ for all $x \in (s_2, \infty)$. Therefore, there is no $x > s_2$ such that $\bar{h}(x) = 0$ and by Lemma 2 no non-*herculean* equilibrium exists. □ ■

Proof of Lemma 3. By definition of i's *strength* $s_i \prod_{j\neq i} F_j(s_i) = c_i$. Equation (11) implies $c_{i+1}F_{i+1}(s_i)/F_i(s_i) > c_i$. Substituting for c_i on the RHS of i's *strength* and rearranging: $s_i \prod_{j\neq i+1} F_j(s_i) < c_{i+1}$. Since the LHS is increasing in s, $s_{i+1} > s_i$. ■

Proof of Proposition 6. *Existence.* For a given vector $\mathbf{v} = (v_1, \ldots, v_n)$, and following equation (2), define the family of functions $h_i^n(\mathbf{v}) = A_i^n r_i(v^i) - c_i$. This family of functions will be used in the proof of existence and uniqueness. Start by ordering bidders by strength, with 1 being the strongest and n the weakest. By Lemma 2 a *herculean* equilibrium $\mathbf{x} = (x_1, \ldots, x_n)$ exists if and only if $h_i^n(\mathbf{v}) = 0$ for all i. We construct $\mathbf{x}$ recursively. Let $\bar{v}^i = (v_i, \ldots, v_n)$ represent the elements of $\mathbf{v}$ in the ith and higher positions. For any vector $\bar{v}^2$, define $x_1(\bar{v}^2)$ to be the value of v_1 that solves $h_1^n(v_1, \bar{v}^2) = 0$; i.e., $x_1(\bar{v}^2) = c_1/A_1^n$. Using $x_1(\bar{v}^2)$ we can write $h_2^n(\mathbf{v})$ as a function of $\bar{v}^2$ only, that is $h_2^n(\bar{v}^2) = A_2^n r_2(\bar{v}^2) - c_2$ where:

$$r_2(\bar{v}^2) \equiv r_2(x_1(\bar{v}^2), v_2) = v_2 F_1(v_2) - \int_{c_1/A_1^n}^{v_2} x dF_1(x)$$

is the revenue function $r_2(v^2)$ after replacing the function $x_1(\bar{v}^2)$ for the value of v_1. The finite expectation assumption implies that $h_2^n(v_2, \bar{v}^3)$ is unbounded above in v_2. Evaluate $h_2^n(v_2, \bar{v}^3)$ at the value $\hat{v}_2$ that satisfies $\hat{v}_2 = x_1(\hat{v}_2, \bar{v}^3)$ (which always exists and is uniquely defined). Then, using that $x_1(\bar{v}^2) = c_1/A_1^n$ and $\hat{v}_2 = x_1(\hat{v}_2, \bar{v}^3)$, $h_2^n(\hat{v}_2, \bar{v}^3) = c_1 F_1(\hat{v}_2)/F_2(\hat{v}_2) - c_2$. Therefore, by (11) , $h_2^n(x_1(\hat{v}_2, \bar{v}^3), \bar{v}^3) \leq 0$ (strict if $s_1 < s_2$) and, by the intermediate value theorem, in general there exists $x_2(\bar{v}^3) \geq x_1(x_2(\bar{v}^3), \bar{v}^3)$ (strict if $s_1 < s_2$) such that $h_2^n(x_2(\bar{v}^3), \bar{v}^3) = 0$. Observe that by replacing $v_2 = x_2(\bar{v}^3)$ into $x_1(\bar{v}^2)$, we can write both $(x_1,\, x_2)$ as implicit functions of $\bar{v}^3$. Also, by the argument above, the order between x_1 and x_2 is robust to any $\bar{v}^3$, implying that the order will not reverse when constructing $\bar{x}^3$ (though, the actual values of x_1 and x_2 do change).

Suppose we have shown that, for any vector $\bar{v}^i$, $x_1(\bar{v}^i) \leq x_2(\bar{v}^i) \leq \cdots \leq x_{i-1}(\bar{v}^i)$ (strict whenever $s_{j-1} < s_j$) where for each $k \leq i$, $x_k(\bar{v}^i)$ has been recursively defined as: a value v_k solving $h_k^n(v_k, \bar{v}^{k+1}) = 0$, and by replacing the solution $x_k(\bar{v}^{k+1})$ into each $x_j(\bar{v}^{j+1})$ for $j \leq k$. We show that there exists $x_i(\bar{v}^{i+1}) \geq x_{i-1}(x_i(\bar{v}^{i+1}), \bar{v}^{i+1})$ (strict if $s_{i-1} < s_i$) that solves $h_i(x_i(\bar{v}^{i+1}), \bar{v}^{i+1}) = 0$. Notice that $h_{i-1}^n(x_{i-1}, \bar{v}^i) = 0$ implies $r_{i-1}(x_{i-1}, \bar{v}^i) = c_{i-1}/A_{i-1}^n$. Substituting the vector of solutions x^{i-1} we can write $h_i^n(\mathbf{v})$ as $h_i^n(\bar{v}^i) = A_i^n r_i(\bar{v}^i) - c_i$. Because of the finite expectation assumption, $h_i^n(\bar{v}^i)$ is unbounded above in v_i. Take v_i to be value of that satisfies $\hat{v}_i = x_{i-1}(\hat{v}_i, \bar{v}^{i+1})$ and notice that Lemma 5.2 (see the Auxiliary Results section) implies

$$r_i(x_{i-1}(\hat{v}_i, \bar{v}^{i+1}), \bar{v}^{i+1}) = F_{i-1}(x_{i-1}(\hat{v}_i, \bar{v}^{i+1})) r_{i-1}(x_{i-1}(\hat{v}_i, \bar{v}^{i+1}), x_{i-1}(\hat{v}_i, \bar{v}^{i+1}), \bar{v}^{i+1}).$$

Then, using $r_{i-1}(x_{i-1}(\bar{v}^i), \bar{v}^i) = c_{i-1}/A_{i-1}^n$, we can write

$$h_i^n(\hat{v}_i, \bar{v}^{i+1}) = c_{i-1} F_{i-1}(\hat{v}_i)/F_i(x_{i-1}(\hat{v}_i, \bar{v}^{i+1})) - c_i.$$

$\hat{v}_i = x_{i-1}(\hat{v}_i, \bar{v}^{i+1})$ and condition (11), the expression above is non positive and the existence of $x_i(\bar{v}^{i+1}) \geq x_{i-1}(x_i(\bar{v}^{i+1}), \bar{v}^{i+1})$ follows by the intermediate value theorem. Once again, the proof is independent of above cutoffs so that $x_{i-1} \leq x_i$ regardless of the construction of $\bar{x}^{i+1}$. □

Uniqueness: Preliminaries. This proof uses induction. We start by outlining the main argument. We order bidders from weakest to strongest. Define $H_k^n : \mathbb{R}^n \to \mathbb{R}^k$ to be the function equal to $h_i^n(\mathbf{v})$ (defined in the existence proof above) in the $i \leq k$ dimension. Fix a value k, by the existence proof we know there exists recursively defined functions $x^k : \mathbb{R}^{n-k} \to \mathbb{R}^k$ satisfying $H_k^n(x^k(\bar{v}^{k+1}), \bar{v}^{k+1}) = 0$. For any $i \leq k$, the total differential of $h_i^n(x^k(\bar{v}^{k+1}), \bar{v}^{k+1})$ with respect to v_j, $j > k$, is:

$$A_i^n \left[\sum_{s=1}^{i-1} A_s^{i-1} r_s(x^s) f_s(x_s) \frac{dx_s}{dv_j} + B_i(x_i) \frac{dx_i}{dv_j} + r_i(\mathbf{x}^i) \left(\sum_{s>i}^{k} \frac{f_s(x_s)}{F_s(x_s)} \frac{dx_s}{dv_j} + \frac{f_j(v_j)}{F_j(v_j)} \right) \right]. \quad (14)$$

Using this equation and the implicit function theorem, we can write the vector of derivatives $dx^k(\bar{v}^{k+1})/dv_{k+1}$ as the solution to the following system of linear equations:

$$A_i^n M_k D_k + A_i^n R_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} = 0, \quad (15)$$

where (T denotes transpose):

$$D_k = \left(\frac{dx_1}{dx_{k+1}}, \frac{dx_2}{dx_{k+1}}, \ldots, \frac{dx_k}{dx_{k+1}} \right)^T, \quad R_k = (r_1(x^1), r_2(x^2), \ldots, r_k(x^k))^T$$

and

$$M_k = \begin{pmatrix} B_1(x_1) & r_1(x^1)\frac{f_2(x_2)}{F_2(x_2)} & r_1(x^1)\frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_1(x^1)\frac{f_k(x_k)}{F_k(x_k)} \\ A_1^1 r_1(x^1) f_1(x_1) & B_2(x_2) & r_2(x^2)\frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_2(x^2)\frac{f_k(x_k)}{F_k(x_k)} \\ A_1^2 r_1(x^1) f_1(x_1) & A_2^2 r_2(x^2) f_2(x_2) & B_3(x_3) & \cdots & r_3(x^3)\frac{f_k(x_k)}{F_k(x_k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{k-1} r_1(x^1) f_1(x_1) & A_2^{k-1} r_2(x^2) f_2(x_2) & A_3^{k-1} r_3(x^3) f_3(x_2) & \cdots & B_k(x_k) \end{pmatrix}.$$

If M_k is invertible, the the solution to (15) is given by:

$$D_k = -M_k^{-1} R_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1}). \quad (16)$$

Using this last property we show that $dh_i^i(v)/dv > 0$. Furthermore, we show that this implies $dh_i^n(v_i, \bar{v}^{i+1})/dv_i > 0$ for any $n \geq i$. In words, in a game with $n = i$ bidders in which every cutoff $x^{i-1}(v_i)$ has been recursively constructed as a function of v_i, bidder i has a unique best response. Moreover, this implies that in every game with $n \geq i$ bidders, for any $\bar{v}^{i+1}$, player i has a unique best response.[12] Then, by the induction argument, each step of the construction $x_k(\bar{v}^k)$ is uniquely defined and the *herculean* equilibrium is unique.

Claim 5. There exists a unique *herculean* equilibrium.

Proof. Fix a step k in the recursion constructing x and let $x^k(\bar{v}^{k+1})$ be the vector of functions defined above. Fix any positive vector $\bar{v}^{k+2}$, we need to show that there is a unique value of x_{k+1} that solves $h_{k+1}^n(x^{k+1}, \bar{v}^{k+2}) = 0$. In particular, we show

[12]Notice that this does not imply that for each $n > i$, bidder i's best response is unique and the same across all $n > i$.

$dh^n_{k+1}(x^{k+1}, \bar{v}^{k+2})/dv_{k+1} > 0$, so that $h^n_{k+1}(v_{k+1}, \bar{v}^{k+2})$ single crosses zero from below. Using (14),

$$\frac{dh^n_{k+1}(v_{k+1}, \bar{v}^{k+2})}{dv_{k+1}} = A^n_{k+1}(d_k D_k + B_{k+1}(v_{k+1}))$$

where $d_k = (A^k_1 r_1(x^1) f_1(x_1),\, A^k_2 r_2(x^2) f_2(x_2),\, \ldots,\, A^k_k r_k(x^k) f_k(x_k))$. Using (16), if M_k is invertible we can write $D_k 1 = -M_k^{-1} R_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1})$ and

$$\frac{dh^n_{k+1}(v_{k+1}, \bar{v}^{k+2})}{dv_{k+1}} = A^n_{k+1}\left(B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})}\right)$$

where $q_k = d_k M_k^{-1} R_k$. Because $A^n_{k+1} > 0$, it is sufficient to show that the parenthesis (which corresponds to $dh^{k+1}_{k+1}(v_{k+1})/dv_{k+1}$) is positive for all relevant values of v_{k+1}. We show this and the invertibility of M_k by induction.

Observe $h^n_1(v) = A^n_1 v$, thus $dh^n_1(v)/dv > 0$ bidder one has a unique best response for any $n \geq 1$ (given by $x_1 = c_1/A^n_1$). For bidder two, observe $M_1 = B_1(x_1) = 1$ is invertible and $q_1 = (x_1)^2 f_1(x_1)$ is well defined. Then, $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) = F_1(v_2) - (x_1)^2 f_1(x_1) f_2(v_2)/F_2(v_2)$. Using Lemma 4.1 (concavity) twice $x_1 F_1(x_1)/v_2$ is an upper bound for the subtracting term. Since, by construction, we are interested in $v_2 \geq x_1$, $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) > 0$.

Suppose we have shown that M_{j-1} is invertible and $B_j(x_j) - q_{j-1} f_j(x_j)/F_j(x_j) > 0$ for all $j \leq k$. Let $l_k = (B_k(x_k) - q_{k-1} f_k(x_k)/F_k(x_k))^{-1}$ and observe that $l_k > 0$ by induction hypothesis; then, by the definition of M_k and using blockwise inversion,

$$M_k = \begin{pmatrix} M_{k-1} & R_{k-1}\frac{f_k(x_k)}{F_k(x_k)} \\ d_{k-1} & B_k(x_k) \end{pmatrix} \text{ and } M_k^{-1} = \begin{pmatrix} O & -\frac{f_k(x_k)}{F_k(x_k)} p_k(M_{k-1}^{-1} R_{k-1}) \\ -l_k(d_{k-1} M_{k-1}^{-1}) & l_k \end{pmatrix}$$

where $O = M_{k-1}^{-1} + \frac{f_k(x_k)}{F_k(x_k)} l_k (M_{k-1}^{-1} R_{k-1} d_{k-1} M_{k-1}^{-1})$, and the inverse of M_k is well defined. We need to show $B_{k+1}(v_{k+1}) - q_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1}) > 0$. Observing that $R_k = (R_{k-1}, r_k(x^k))^T$, $d_k = (d_{k-1} F_k(x_k), r_k(x^k) f_k(x_k))$, and using the definition of M_k^{-1} and l_k we can write:

$$q_k = F_k(x_k) q_{k-1} + f_k(x_k)\,(r_k(x^k) - q_{k-1})^2 \Big/ (B_k(x_k) - q_{k-1} f_k(x_k)/F_k(x_k))\,, \qquad (17)$$

Thus, $B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} > 0$ is equivalent to show:

$$\left(B_k(v_{k+1}) \frac{F_k(v_{k+1}) F_{k+1}(v_{k+1})}{f_k(x_k) f_{k+1}(v_{k+1})} - q_{k-1} \frac{F_k(x_k)}{f_k(x_k)}\right)\left(B_k(x_k) - q_{k-1} \frac{f_k(x_k)}{F_k(x_k)}\right) > (r_k - q_{k-1})^2$$

where $B_{k+1}(v_{k+1}) = B_k(v_{k+1}) F_k(v_{k+1})$ was used. By the existence proof we are only interested in $v_{k+1} \geq x_i$; using this condition, that $B_k(v)$ is decreasing in v, and Lemma 4.1 in the auxiliary result section, we find that $(B_k(x_k) x_k - q_{k-1})^2$ is a lower bound for the LHS of the expression above. Lemma 5.1 shows $B_i(x_k) x_k \geq r_k(x^k)$. Thus we just need to show that $B_k(x_k) x_k - q_{k-1} \geq 0$, which is done by proving $r_k(x^k) - q_{k-1} \geq 0$. We do this by induction. Since q_0 is not defined, we start with $i = 2$. Using integration by parts $r_2(x^2) - q_1$ is equal to

$$x_1 F_1(x_1) + \int_{x_1}^{x_2} F_1(s)ds - (x_1)^2 f_1(x_1) > \int_{x_1}^{x_2} F_1(s)ds \geq 0$$

where Lemma 4.1 was used in the last step. Suppose we have shown $r_j(x^j) \geq q_{j-1}$ for $j \leq i$. We show $r_{i+1}(x^{i+1}) \geq q_i$. Using equation (17), this is equivalent to:

$$r_{i+1}(x^{i+1})/F_i(x_i) - q_{i-1} - \left(r_i(x^i) - q_{i-1}\right)^2 \bigg/ \left(B_i(x_i)\frac{F_i(x_i)}{f_i(x_i)} - q_{i-1}\right) \geq 0.$$

Lemma 5.2 shows $r_{i+1}(x^{i+1})/F_i(x_i) \geq r_i(x^i)$. By induction hypothesis $r_i(x^i) \geq q_{i-1}$ and we can rewrite the condition as

$$1 \geq \left(r_i(x^i) - q_{i-1}\right) \bigg/ \left(B_i(x_i)\frac{F_i(x_i)}{f_i(x_i)} - q_{i-1}\right).$$

The result follows from Lemma 4.1 and Lemma 5.1. Thus $r_{i+1}(x^{i+1}) \geq q_i$, which proves $dh_{k+1}^{k+1}(v_{k+1})/dv_{k+1} > 0$ for all $v_{k+1} \geq x_k$, which implies $dh_{k+1}^n(v_{k+1}, \bar{v}^{k+2})/dv_{k+1} > 0$ for all $\bar{v}^{k+2}$ and a unique *herculean* equilibrium exists. □

Claim 6. There is no non-*herculean* equilibria.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an ordered vector of equilibrium cutoffs. Starting from the lower cutoff, let i be the first bidder to play a smaller cutoffs than a stronger player $i+1$; i.e., $x_i < x_{i+1}$ but $s_i > s_{i+1}$. In other words, every bidder $k \leq i$ have their cutoffs in the same order as they strength. Because of this, we can use our recursive construction in the existence proof and our induction argument in the uniqueness proof up to bidder i, so that best responses are uniquely defined for any vector $\bar{x}^{i+1}$ that bidders may play.

Let's analyze $h_{i+1}^n(\mathbf{v})$. Because $h_i^n(x_i, \bar{v}^{i+1}) = 0$ we know $r_i(x_i, \bar{v}^{i+1}) = c_i/A_i^n$. Substituting the vector of solutions x^i we can write $h_{i+1}^n(\mathbf{v})$ as $h_{i+1}^n(\bar{v}^{i+1}) = A_{i+1}^n r_{i+1}(\bar{v}^{i+1}) - c_{i+1}$. Take v_{i+1} to be value of that satisfies $v_{i+1} = x_i(v_{i+1}, \bar{v}^{i+2})$ and notice that Lemma 5.2 implies $r_{i+1}(x_i, \bar{v}^{i+2}) = F_i(x_i) r_i(x_i, x_i, \bar{v}^{i+2})$. Then, using $r_i(x_i, \bar{v}^{i+1}) = c_i/A_i^n$, we can write $h_{i+1}^n(x_i, \bar{v}^{i+2}) = c_i F_i(x_i)/F_{i+1}(x_i) - c_{i+1}$ which is *positive* under (11) and the condition that bidder $i+1$ is stronger than bidder i. We need to show that there is no $v_{i+1}^* > x_i$ such that $h_{i+1}^n(v_{i+1}^*, \bar{v}^{i+2}) = 0$. This follows from the proof of uniqueness as concavity implies $dh_{i+1}^n(v_{i+1}, \bar{v}^{i+2})/dv_{i+1} > 0$ for $v_{i+1}^* > x_i$, which implies the result. □ ■

Auxiliary Results

Lemma 4. *A concave and differentiable distribution function $F(x)$ with $F(0) = 0$ satisfies the following properties: For every $x > 0$,*

1. *$xf(x) \leq F(x)$.*
2. *$F(x)/x$ is decreasing in x.*

Proof. A concave differentiable function is bounded above by its first-order Taylor approximation, i.e., for every x and y such that $x > y$: $F(x) - F(y) \geq (x - y)f(x)$.

Taking $y = 0$ and using $F(0) = 0$, the first claim follows. For the second claim, define $\phi(x) = F(x)/x$. Then, $\phi'(x) = (f(x)x - F(x))/x^2$ is non-positive by the first claim. $\square$

Lemma 5. *Let $(x_1, x_2, \ldots, x_n)$ be an ordered vector from smallest, x_1, to largest, x_n. Then, the following properties hold.*

1. $x_i B_i(x_i) \geq r_i(\mathbf{x}^i)$ *and strict if exists $j < i$ such that $x_j < x_{j+1}$.*
2. $r_i(\mathbf{x}^i) > F_{i-1}(x_{i-1}) r_{i-1}(x^{i-1})$ *and with equality if $x_i = x_{i-1}$.*

Proof. Recall the definition of $r_i(\mathbf{x}^i)$ in equation (2). For the first claim simply observe, $x_i B_i(x_i) - r_i(\mathbf{x}^i) = \sum_{k=1}^{i-1} \left(A_k^{i-1} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right)$ which is strictly positive if there exists a bidder $j < i$ such that $x_j < x_i$ or zero otherwise. For the second claim we show that $r_i(\mathbf{x}^i) = F_{i-1}(x_{i-1}) r_{i-1}(x^{i-1}) + \int_{x_{i-1}}^{x_i} B_i(s) ds$, which proofs the claim. Rewriting (2):

$$r_{i-1}(x^{i-1}) = x_i B_i(x_i) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left(A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) - \int_{x_{i-1}}^{x_i} s dB_i(s).$$

Integrating by parts the last term we obtain:

$$r_{i-1}(x^{i-1}) = x_{i-1} B_i(x_{i-1}) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left(A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) + \int_{x_{i-1}}^{x_i} B_i(s) ds.$$

Since, by definition, $B_i(x_{i-1}) = B_{i-1}(x_{i-1}) F_{i-1}(x_{i-1})$, the result follows. $\square$

Lemma 6. *Symmetric bidders with concave CDF's must play symmetric equilibrium cutoffs.*

Proof. By contradiction. W.l.o.g. order bidders identities in terms of their cutoff order, with bidder one being the bidder with the lower cutoff. Suppose there exists an equilibrium such that bidders $q < p$ are symmetric; i.e., $F_q = F_p = G$ and $c_q = c_p = c$, but play $x_q < x_p$. Integrating (3) by parts we obtain (see derivation in the Online Appendix):[13]

$$\sum_{j=1}^{i} \left\{ \prod_{k \geq j, k \neq i} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < j} F_\ell(y) \right) dy \right\} = c_i. \tag{18}$$

Subtracting (18) of q to that of p delivers

$$\begin{aligned} 0 = \sum_{j=q+1}^{p} &\left\{ \prod_{k \geq j, k \neq p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < q} F_\ell(y) \right) G(y) \left(\prod_{\ell = q+1}^{j-1} F_\ell(y) \right) dy \right\} \\ &- (G(x_p) - G(x_q)) \sum_{j=1}^{q} \left\{ \prod_{k \geq j, k \neq q,p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left(\prod_{\ell < j} F_\ell(y) \right) dy \right\} \end{aligned} \tag{19}$$

We show that a strict lower bound for the right-hand side of (19) is non-negative, a contradiction to (19). The first summation is strictly positive, we take a lower bound of

[13] Recall the notation conventions: $\sum_\emptyset = 0$, $\prod_\emptyset = 1$ and $x_0 = 0$.

this summation by taking a lower bound of its integrals in three steps: (i) for the terms in the first product ($\ell < q$), replace $F_\ell(y)$ by $F_\ell(x_q)$; (ii) substitute $G(y)$ by $G(x_q)$ and; (iii) for the terms in the second product (ranging from $q+1$ to $j-1$), replace $F_\ell(y)$ by $F_\ell(x_\ell)$. Hence, the following *strict* lower bound for the first summation is obtained[14]

$$(x_p - x_q)G(x_q)\prod_{\ell<q} F_\ell(x_q)\prod_{k>q,k\neq p} F_k(x_k)$$

Now we construct an upper bound to the subtracting term in (19) by substituting in the integral $F_\ell(x_j)$ for $F_\ell(y)$. Then, the second summation in equation (19) becomes

$$\prod_{k>q,k\neq p} F_k(x_k)\left(\sum_{j=1}^{q-1}\left\{x_j\prod_{k=j+1}^{q-1}F_k(x_k)\left(\prod_{\ell\leq j}F_\ell(x_j)-\prod_{\ell\leq j}F_\ell(x_{j+1})\right)\right\}+x_q\prod_{\ell<q}F_\ell(x_q)\right)$$

Since $x_j \leq x_{j+1}$, the summation in the previous expression is over non-positive terms. We can obtain an upper bound by replacing the summation with zero. Then, our strict lower bound for the right-hand side of (19) is

$$(x_pG(x_q) - x_qG(x_p))\prod_{\ell<q} F_\ell(x_q)\prod_{k>q,k\neq p} F_k(x_k).$$

Because the products are positive, the previous expression is non-negative if and only if $G(x_q)/x_q > G(x_p)/x_p$. The result follows from Lemma 4.2 and $x_q < x_p$. □

[14]The strict inequality is guaranteed by taking $G(x_q)$ as lower bound of $G(y)$ over the range of integration x_q to x_p with $x_q < x_p$.

Online Appendix

Second Price Auctions with Participation Costs

by José-Antonio Espín-Sánchez, Álvaro Parra and Yuzhou Wang

Supplemental Material –Not for Publication

Proof of Proposition 3. We start by proving existence. Suppose that both bidders are equally strong ($s_1 = s_2 = s$). Set $v_1 = v_2 = s$ and observe that (6) and (7) hold and a equilibrium exists by Lemma 2.

Suppose w.l.o.g. that $s_1 < s_2$; i.e., bidder 1 is the strong bidder of the game. We construct an equilibrium where $x_1 < x_2$. Define $g(v) = c_1/F_2(v)$ to be the equilibrium cutoff played by bidder one when bidder two plays $x_2 = v$; observe $g'(v) = -g(v)f_2(v)/F_2(v) < 0$. Define the function $h : [s_1, \infty) \to \mathbb{R}$ by

$$h\left(v\right) = vF_1\left(v\right) - \int_{g(v)}^{v} xf_1\left(x\right)dx - c_2$$

which is a continuous function of v. The function $h(v)$ represents bidder 2's revenue of drawing valuation v when she plays a cutoff $x_2 = v$ and 1 plays the cutoff $x_1 = g(x_2)$. To have an *herculean* equilibrium we need a value x_2 satisfying $h(x_2) = 0$ and $x_2 > g(x_2)$. The next two claims prove the result.

Claim 7. $v \in [s_1, \infty)$ is necessary and sufficient to have *herculean* cutoffs.

Proof. Observe that $g(v)$ is weakly decreasing in v and $g(s_1) = s_1$. Therefore, $x_2 \geq g(x_2)$ if and only if $x_2 \in [s_1, \infty)$. □

Claim 8. $h\left(s_1\right) < 0$ and $h\left(v\right)$ is unbounded above.

Proof. bidder 2 being weak ($s_1 < s_2$) implies

$$h(s_1) = s_1F_1(s_1) - c_2 < s_2F_1(s_2) - c_2 = 0.$$

On the other hand, $h(v)$ is unbounded above as $vF_1(v)$ is unbounded and the finite expectation assumption. □

Claim 8 plus continuity imply that there exists $x^* > s_1$ such that $h(x^*) = 0$. On the other hand, $h(x^*) = 0$ holds if and only if equations (6) and (7) are satisfied. Therefore, by Lemma 2 and Claim 6, we have a *herculean* equilibrium with $x_1 = g(x^*)$ and $x_2 = x^*$.

Now we prove uniqueness. We start by showing that among the *herculean* class the equilibrium is unique. Then we extend the uniqueness result among all equilibria. In order to have a unique equilibrium in the *herculean* class it is sufficient to show that $h'(v) > 0$, so that $h(v)$ single crosses zero at v^* from below;

$$h'(v) = F_1(v) + g'(v)g(v)f_1(g(v)) = F_1(v) - g(v)^2\frac{f_2(v)}{F_2(v)}f_1(g(v)).$$

Using Lemma 4.1 (concavity) of the main article twice, we can write $h'(v)$ as

$$h'(v) > F_1(v) - \frac{g(v)}{v}F_1\left(g(v)\right).$$

Claim 7 says we are only interested in $v \geq s_1$, which implies $v > g(v)$. Thus $h'(v) > 0$ proving uniqueness withing the *herculean* class.

To prove that the only equilibrium is the *herculean*, suppose we have a non-*herculean* equilibrium; i.e., $x_1 \geq x_2$. Define $\bar{g}(v) = c_2/F_1(v)$ to be the equilibrium cutoff played by bidder 2 when bidder one plays $x_1 = v$, and let

$$\bar{h}\left(v\right) = vF_2\left(v\right) - \int_{g(v)}^{v} xf_2\left(x\right)dx - c_1$$

represent bidder 1's revenue of drawing valuation v when she plays a cutoff $x_1 = v$ and 2 plays the cutoff $x_2 = \bar{g}(x_1)$. As before, because $\bar{g}(s_2) = s_2$ and $\bar{g}(v)$ being decreasing, in order to have an non-*herculean* equilibrium $\bar{h}$ has to be defined on $[s_2, \infty)$. Now observe that $\bar{h}\left(s_2\right) = s_2F_2\left(s_2\right) - c_1 > 0$. By repeating the argument above $\bar{h}'(v) > 0$ and $\bar{h}(v) > 0$ for all $v \in (s_2, \infty)$, so there is no x^* such that $\bar{h}\left(x^*\right) = 0$ and no non-*herculean* equilibrium exists. ■

Lemma 7. *For any given bidder i, equilibrium condition $A_i^n r_i(\mathbf{x}^i) = c_i$ is equivalent to*

$$\sum_{j=1}^{i}\left\{\prod_{k\geq j,k\neq i}F_k(x_k)\int_{x_{j-1}}^{x_j}\left(\prod_{\ell<j}F_\ell(y)\right)dy\right\} = c_i. \tag{OA 1}$$

Proof. Recall

$$r_i(\mathbf{x}^i) = x_iB_i(x_i) - \sum_{j=1}^{i-1}\left(A_j^{i-1}\int_{x_j}^{x_{j+1}} sdB_{j+1}(s)\right),$$

where $A_i^n = \prod_{k=i+1}^{n} F_k(x_k)$ and $B_i(v) = \prod_{k<i} F_k(v)$. Integrating $r_i(\mathbf{x}^i)$ by parts

$$\begin{aligned} r_i(\mathbf{x}^i) &= x_iB_i(x_i) - \sum_{j=1}^{i-1}\left\{A_j^{i-1}\left(x_{j+1}B_{j+1}(x_{j+1}) - x_jB_{j+1}(x_j) - \int_{x_j}^{x_{j+1}} B_{j+1}(s)ds\right)\right\} \\ &= x_iA_1^{i-1}F_1(x_1) + \sum_{j=1}^{i-1} A_j^{i-1}\int_{x_j}^{x_{j+1}} B_{j+1}(s)ds, \end{aligned}$$

where $A_j^{i-1}B_{j+1}(x_{j+1}) = A_{j+1}^{i-1}B_{j+2}(x_{j+1})$ was used. Observing that $x_1 = \int_{x_0}^{x_1} B_1(s)ds$, and multipliying $r_i(\mathbf{x}^i)$ by A_i^n, we obtain (OA 1). ■

Proposition 7. *In two-bidder symmetric game with multiple equilibria and convex distributions of valuations, the symmetric equilibrium is never efficient.*

Proof. Under symmetric bidder playing a symmetric equilibrium, the Hessian (and its determinant) of the welfare function becomes

$$H(\mathbf{x}) = -\begin{pmatrix} f(x)F(x) & xf(x)^2 \\ xf(x)^2 & f(x)F(x) \end{pmatrix} \qquad \det(H) = f(x)^2(F(x) - xf(x))(F(x) + xf(x)).$$

Convexity of the distribution function implies $xf(x) > F(x)$. Thus, the determinant is negative, which implies that the symmetric equilibrium is a saddle point. ■