Amplitude-Dependent Wave Devices Based on Nonlinear Periodic Materials

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Motivate study of nonlinear periodic structures

Detail a perturbation approach for a set of infinite nonlinear difference equations
  • First order dispersion correction

Present results for 1D and 2D lattices, to include potential devices based on nonlinear response

Discuss wave-wave interactions and further device implications

Present nonlinear string experiment

Conclude with final thoughts on needed research
Nonlinear Periodic Structures

• Nonlinear periodic structures exhibit additional unique wave properties
  - Existence of highly stable localized solutions\(^1\) even without defects
  - Solitary waves and solitons\(^2,3\)
  - Variations in wave speeds and propagation direction related to wave amplitude and nonlinearity

• Our interest is in tunable phononic devices (frequency isolators, filters, logic ports, resonators, etc…)

• Most nonlinear analysis of discrete systems begins with a long wavelength approximation and then posing of an equivalent continuous system

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Perturbation Approach

• Analytical treatment for weakly nonlinear media

• Treats the infinite, discrete system without reverting to the long wavelength limit

• Amounts to a Lindstedt Poincare’ approach combined with Bloch Analysis

• A Multiple Scales perturbation approach is employed for wave-wave interactions
General Approach

\[ \mathbf{b}_1 = \frac{1}{a} (\mathbf{a}_2 \times \mathbf{e}); \quad \mathbf{b}_2 = \frac{1}{a} (\mathbf{e} \times \mathbf{a}_1) \]

where, \[ \mathbf{e} = \frac{1}{a} (\mathbf{a}_1 \times \mathbf{a}_2) \]

\[ \mathbf{r}_{n_1,n_2} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 \]

\[ \mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}, \quad i,j = 1,2, \]
Dynamic behavior is governed by,

\[ \mathbf{u}_{n_1,n_2} = [u_1 \ u_2 \ u_3 \ ... \ u_{N-1} \ u_N]^T \]

\( N \) – denotes number of degrees of freedom for a unit cell

For the 9 cell assembly,

\[ \mathbf{\hat{M}} \frac{d^2 \mathbf{\hat{u}}}{dt^2} + \mathbf{\hat{K}} \mathbf{\hat{u}} + \varepsilon \mathbf{f}_{NL} (\mathbf{\hat{u}}) = \mathbf{\hat{f}}_{int} + \mathbf{\hat{f}}_{ext} \]
• Equations of motion for the unit cell are extracted from the previous equation expressed for 9-cell assembly.

\[ \omega^2 \sum_{p,q=-1,0,1} M^{(p,q)} \frac{d^2 u_{n_1+p,n_2+q}}{d \tau^2} + \sum_{p,q=-1,0,1} K^{(p,q)} u_{n_1+p,n_2+q} \]

\[ + \varepsilon f_{NL}(u_{n_1 \pm p,n_2 \pm q}) = f_{ext}^{n_1,n_2}(\tau) \]

• Weakly nonlinear model is governed by,

\[ \omega^2 M \frac{d^2 u_{n_1,n_2}}{d \tau^2} + \left[ \sum_{p,q=-1,0,1} K^{(p,q)} u_{n_1+p,n_2+q} \right] + \varepsilon f_{NL}(u_{n_1 \pm p,n_2 \pm q}) = f_{ext}^{n_1,n_2}(\tau) \]

• Free wave propagation is analyzed by setting the external forcing to zero.
Asymptotic Expansions

• Asymptotic expansion of frequency and displacement,

\[ u_{n_1,n_2} = u_{n_1,n_2}^{(0)} + \varepsilon u_{n_1,n_2}^{(1)} + O(\varepsilon^2) \]

\[ \omega = \omega_0 + \varepsilon \omega_1 + O(\varepsilon^2) \]

• Substituting the above expansions leads to ordered equations,

\[ \varepsilon^0: \quad \omega_0^2 M \frac{d^2 u_{n_1,n_2}^{(0)}}{d\tau^2} + \sum_{p,q=-1}^{+1} K^{(p,q)} u_{n_1+p,n_2+q}^{(0)} = 0 \]

\[ \varepsilon^1: \quad \omega_0^2 M \frac{d^2 u_{n_1,n_2}^{(1)}}{d\tau^2} + \sum_{p,q=-1}^{+1} K^{(p,q)} u_{n_1+p,n_2+q}^{(1)} = -2 \omega_0 \omega_1 M \frac{d^2 u_{n_1,n_2}^{(0)}}{d\tau^2} - f_{NL}(u_{n_1,n_2}^{(0)}, u_{n_1\pm p, n_2\pm q}^{(0)}) \]

• 0th order equation can be solved for Bloch waves
Solution Approach

- Zero$^\text{th}$ order solution is obtained using Bloch wave assumption

\[ \varepsilon^0: \quad \omega_0^2 M \frac{d^2 u^{(0)}_{n_1,n_2}}{d\tau^2} + \sum_{p,q=-1}^{+1} K(p,q) u^{(0)}_{n_1+p,n_2+q} = 0 \]

- Bloch wave theorem is imposed by assuming the following displacement expression,

\[ u_{n_1,n_2}(\tau) = u_0 e^{i k \cdot r_{n_1,n_2}} e^{i \tau} \]

\[ u_{n_1+p,n_2+q}(\tau) = u_{n_1,n_2}(\tau) e^{i (\pm p \mu_1 \pm q \mu_2) - \mu_2 n_2} \]

- Substituting above into the zero$^\text{th}$ order equation,

\[ \omega_0^2 M \frac{d^2 u^{(0)}_{n_1,n_2}(\tau)}{d\tau^2} + \left[ \sum_{p,q=-1}^{+1} K(p,q) e^{i (\pm p \mu_1 \pm q \mu_2)} \right] u^{(0)}_{n_1,n_2}(\tau) = 0 \]

leads to,

\[ [-\omega_0^2 M + \tilde{K}(k)] u_0(k) = 0 \]

Eigenvalue problem
Therefore, the RHS of $\varepsilon^1$ order equation with $e^{i\tau}$ dependence is

$$\omega_0^2 M \frac{d^2 u^{(1)}_{n_1, n_2}}{d\tau^2} + \sum_{p,q=-1}^{1} K^{(p,q)} u^{(1)}_{n_1+p, n_2+q} = -2\omega_0 \omega_1 M \frac{d^2 u^{(0)}_{n_1, n_2}}{d\tau^2} - f_{NL} \left( u^{(0)}_{n_1, n_2}, u^{(0)}_{n_1+p, n_2+q} \right)$$

Reference unit cell, $(n_1, n_2) = (0, 0)$

$$u^{(0)}(\tau) = \frac{A_0}{2} u_{0,j}(k) e^{i\tau} + c.c.$$

Can be easily seen that nonlinear force is periodic in $\tau$

$$f_{NL} \left( u^{(0)}(\tau), u^{(0)}_{p,q}(\tau) \right) = f_{NL} \left( u^{(0)}(\tau + 2\pi), u^{(0)}_{p,q}(\tau + 2\pi) \right)$$

Therefore, the RHS of $\varepsilon^1$ order equation with $e^{i\tau}$ dependence is

$$\omega_0^2 M \frac{d^2 u^{(1)}_{n_1, n_2}}{d\tau^2} + \sum_{p,q=-1}^{1} K^{(p,q)} u^{(1)}_{n_1+p, n_2+q} = \left[ \omega_0 j \omega_1 A_0 M u_{0,j}(k) - c_1(A_0) \right] e^{i\tau}$$

For $j^{th}$ mode,

$$\omega_0^2 M \frac{d^2 u^{(1)}_{n_1, n_2}}{d\tau^2} + \sum_{p,q=-1}^{1} K^{(p,q)} u^{(1)}_{n_1+p, n_2+q} = f_j e^{i\tau}$$
Solvability condition for the $j$th mode

$$u_0^{H} f_j = 0$$

Finally, the first order correction to frequency for any $j$th mode:

$$\omega_{1,j}(A_0, k) = \frac{u_0^{H}(k)c_1(A_0)}{\omega_{0,j}A_0u_0^{H}(k)Mu_{0,j}(k)}$$

$$\omega_j = \omega_{0,j} + \varepsilon \omega_{1,j}(A_0, k) + O(\varepsilon^2)$$
Nonlinear force interaction can be described by:

\[ f = k\delta + \Gamma \delta^3 \]

\[ \omega_{0,1}(k) = \sqrt{2k(1 - \cos(\mu))/m} \]

\[ \omega = \omega_{0,1} + \varepsilon \left(3|A_0|^2(\Gamma \cos(2\mu) - 4\Gamma \cos(\mu) + 3\Gamma)/4m\omega_{0,1}\right) + O(\varepsilon^2). \]
Nonlinear Device

- Dispersion in one-dimensional nonlinear periodic chains
Each mass is connected to 4 surrounding masses.

Assumed force interaction, where 

\[ f = k\delta + \Gamma \delta^3 \]

\[ k \] is the stiffness parameter, \( \delta \) is the relative displacement between two masses.

Stiffness parameters \( k_1 = 1.0 \text{ N/m} \), \( k_2 = 1.5 \text{ N/m} \), \( A_0 = 2.0 \).

\[ \Gamma_1 = \Gamma_2 = +1.0, \quad \text{Linear (} \Gamma_1 = \Gamma_2 = 0), \quad \Gamma_1 = \Gamma_2 = -1.0 \]
\[ \beta = m_1 / m_2 = 2, \text{ Stiffness parameters: } k_1 = 1.0 \text{ Nm}^{-1}, \; k_2 = 1.5 \text{ Nm}^{-1}, \; A_0 = 2.0 \]

- - - \( \Gamma_1 = \Gamma_2 = +1.0 \) (hard), \quad ---- Linear ( \( \Gamma_1 = \Gamma_2 = 0 \)),
- - - - - \( \Gamma_1 = \Gamma_2 = -1.0 \) (soft)
Group velocity defines energy flow as wave propagates

\[ c_g = \nabla \omega (\mathbf{k}) \]

From nonlinear dispersion, we know that

\[ \omega_j = \omega_{0,j} + \varepsilon \omega_{1,j} (|A_{0,j}|, \mathbf{k}) + \mathcal{O}(\varepsilon^2) \]

Hence,

\[ c_{g,j}(\mathbf{k}, |A_{0,j}|) = \nabla \omega_{0,j}(\mathbf{k}) + \varepsilon \nabla \omega_{1,j}(\mathbf{k}, |A_{0,j}|) + \mathcal{O}(\varepsilon^2) \]

• Group velocity contours are also amplitude dependent

• Useful for predicting energy flow in nonlinear structures
Amplitude-Dependent $c_g$

$$\omega = f(A, k)$$

$$c_g = \nabla \omega(k)$$

$$\omega = 1.75 \text{ rads}^{-1}$$

$$A_0 = 0.10$$

$$A_0 = 0.50$$

$$A_0 = 0.75$$

$$A_0 = 1.00$$

$$A_0 = 1.25$$

$$A_0 = 1.50$$

$$A_0 = 1.75$$

$$A_0 = 1.90$$

$$A_0 = 2.00$$

$$\mu_2$$

$$\mu_1$$

$$k = \mu_1 b_1 + \mu_2 b_2$$

Wave is impeded along $a_1$ axis
• Imposing the displacement on the left boundary at frequency $\omega_0$ and phase shift

• The phase shift determines the angle at which wave is injected $\theta$ and also the wavenumber along $u_x$ axis

• Numerical integration of equations of motion

• From the response, the propagation constants are computed using FFTs in space

• $\theta$ is varied from $0$ to $\pi/2$ to determine iso-frequency contour in one quadrant

A plane wave is injected into a finite spring-mass lattice at incident angle $\theta$
Monatomic Lattice Results

\[ \omega = 1.6 \text{ rads}^{-1} \]

\[ m = 1, \ k_1 = 1.5 \text{ Nm}^{-1}, \ k_2 = 1.0 \text{ Nm}^{-1}, \]
\[ \Gamma_1 = +1.0 \text{ Nm}^{-3}, \ \Gamma_2 = -1.0 \text{ Nm}^{-3}, \]

\( k \) – Linear Stiffness
\( \Gamma \) – Nonlinear Stiffness

- \( A_0 = 0.1 \) (Perturbation Analysis), \( \bullet \ A_0 = 0.1 \) (Numerical Estimation),
- \( A_0 = 2.0 \) (Perturbation Analysis), \( \blacksquare \ A_0 = 2.0 \) (Numerical Estimation)
\[ m = 1, \quad k_1 = 1.5 \text{ Nm}^{-1}, \quad k_2 = 1.0 \text{ Nm}^{-1}, \]
\[ \Gamma_1 = +1.0 \text{ Nm}^{-3}, \quad \Gamma_2 = -1.0 \text{ Nm}^{-3} \]

\( k \) – Linear Stiffness
\( \Gamma \) – Nonlinear Stiffness

\[ A_0 = 0.1 \text{ (Perturbation Analysis)}, \]
\[ A_0 = 0.1 \text{ (Numerical Estimation)}, \]
\[ A_0 = 2.0 \text{ (Perturbation Analysis)}, \]
\[ A_0 = 2.0 \text{ (Numerical Estimation)} \]

Outliers indicate evanescent waves
• Point harmonic forcing in mono-atomic lattice generates spherical wave front
• Quasi-symmetric linear stiffness but asymmetric in nonlinear stiffness
• Asymmetric nonlinear stiffness generates “dead zone” along $a_1$ axis with amplitude increase
Nonlinearly Activated Waveguide

- “Low” Amplitude vs. “High” Amplitude

Low-Amplitude Excitation

High-Amplitude Excitation
Wave-Wave Interactions

- Two waves (A and B) introduced
- Results in additional term due to wave-wave interaction (Method of Mult. Scales)
  - Similar relation holds for $\omega_B$ with indices $A$ and $B$ switched

\[
\omega_A = \sqrt{2 - 2 \cos(\kappa_A a)} + \varepsilon \cdot \frac{3}{8} A^2 (2 - 2 \cos(\kappa_A a))^{3/2} + \varepsilon \cdot \frac{3}{4} B^2 (2 - 2 \cos(\kappa_A a))^{1/2} (2 - 2 \cos(\kappa_B a))
\]

- Additional waves result in the same wave-wave interaction term (with appropriate indices)

\[
\omega_A = \sqrt{2 - 2 \cos(\kappa_A a)} + \varepsilon \cdot \frac{3}{8} A^2 (2 - 2 \cos(\kappa_A a))^{3/2} + \varepsilon \sum_i \left( \frac{3}{4} B_i^2 (2 - 2 \cos(\kappa_A a))^{1/2} (2 - 2 \cos(\kappa_i a)) \right)
\]

Manktelow et. al., 2010, *Nonlinear Dynamics*
Wave Interaction Significance

- **Tunable** dispersion relation by introducing a second wave
- Display both dispersion relations on the same plot:
  - Let $\omega_B > \omega_A$ and $\omega_B = r \omega_A$
- Wave interactions provide additional latitude in device design

![Diagram]

$$(K, \omega) \text{ no wave interaction}$$

$$(K_B, \omega_B) \text{ with wave interactions}$$

Potentially significant shift in the band gap that could be utilized in metamaterial design

Parameters:
- $r = \sqrt{2}$
- $A = 2$
- $B = 2$
**Example:** $B$ wave $2 \cdot \cos (k_B a_j - \omega_B t)$ may be shifted by 10% using nonlinear wave interactions:

- Simulation 1: $r=3$, $A=4.36$
- Simulation 2: $r=5$, $A=8.00$
Application: Beaming Control

- Numerical simulations validate the expected direction shift
  - Control wave field in horizontal direction
  - (Image filtering to remove control wave from view)
Application: Tunable Focusing

- Device schematic: two sources at a wave-beaming frequency produces a high-intensity region.

- Numerical simulation of monoatomic lattice
  - Control wave field introduces dynamic anisotropy
  - Increased stiffness from control wave alters the beam direction.
The classical Duffing oscillator exhibits a well-known frequency shift and models many physical resonators.

\[ m\ddot{u} + k_1 u + \varepsilon k_3 u^3 = \varepsilon f(t) \]

\[ \omega = \omega_n + \varepsilon \sigma \]

What about a chain of oscillators?

\[ m\ddot{u}_p + k_1(2u_p - u_{p+1} - u_{p-1}) + \varepsilon k_3(u_p - u_{p+1})^3 + \varepsilon k_3(u_p - u_{p-1})^3 = 0 \]

\[ \omega(\mu) = \omega_n \sqrt{2 - 2 \cos(\mu)} + \frac{3 k_3 A^2}{8 m \omega_n} (2 - 2 \cos(\mu))^{3/2} + O(\varepsilon^2) \]

Backbone curve looks remarkably like the dispersion frequency shift in the monoatomic chain.
Observe that for $\mu = \pi/3$ the dispersion shift is identical to the Duffing backbone curve.

- Dispersion shifts associated with free-wave propagation are analogous to backbone curves in finite systems.
- Provides a means for experimentally measuring dispersion shifts.
Wire/mass system approximates a monoatomic chain

Measure resonances at large amplitudes to determine dispersion shifts

\[ m \ddot{v}_p + \frac{T_0}{a} (v_p - v_{p+1}) + \frac{T_0}{a} (v_p - v_{p-1}) + \frac{E A c}{2a^3} (v_p - v_{p+1})^3 + \frac{E A c}{2a^3} (v_p - v_{p-1})^3 = 0 \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( D [\text{mm}] )</th>
<th>( a [\text{mm}] )</th>
<th>( E [\text{GPa}] )</th>
<th>( m [\text{g}] )</th>
<th>( \rho_v [\text{kg/m}^3] )</th>
<th>( T_0 [\text{N}] )</th>
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<tr>
<td>Value</td>
<td>0.254</td>
<td>32.5</td>
<td>205</td>
<td>1.57</td>
<td>7850</td>
<td>21.8</td>
</tr>
</tbody>
</table>

(a) Mass element  
(b) Apparatus  
(c) Scanning head
Experimental Verification

Resonant peaks in a finite periodic system fall on dispersion branches [10].

- Propagating wave (240 Hz, pass band)
- Evanescent wave (300 Hz, stop band)
Experimental Verification

- Slow time-domain frequency sweeps over natural frequencies illustrates Duffing nonlinearity
  - Despite large amplitudes near resonance, signal is essentially monochromatic
  - Hilbert transform converts time-domain signal into an analytic signal
Conclusions

- Resonance backbone curves are related to free-wave propagation
- Resonances in finite periodic systems can be analyzed via the dispersion relation of a unit cell

\[
\omega_1 = \frac{3}{16} \frac{EA_c A^2}{m \omega_n d^3} (2 - 2 \cos(\mu))^{3/2}
\]
Follow-On Research

- Experimental verification
  - 1D string is very limited
  - 2D offers opportunity to study wave-wave interactions (shifting focus, etc.) and amplitude-dependent group velocity
Follow-On Research

- Device construction
  - Perhaps RF devices?

- Strongly nonlinear periodic materials/structures
  - Stability of plane waves
  - Reconfigurability
  - Solitons


