Continuity of Thomae's Function

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Let $\mathbb{N} = \{1, 2, \ldots, \}$, $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$, where $\mathbb{I}$ is the set of irrational numbers (uncountable), and $\mathbb{Q}$ is the set of rational numbers (countable). Further let $\mathbb{Q}_+$ be the set of positive rational numbers, in addition to 0.

Thomae's function $f : \mathbb{R} \to \mathbb{Q}_+$ is defined as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q}(\ n > 0, \text{ and } m, n \text{ have no common divisors}) \\ 0 & x \notin \mathbb{Q} \quad (x \in I) \end{cases}$$

We show that Thomae's function is continuous at every irrational point $x \in I$.

Our goal then is to show that for given $p \in \mathbb{I}$ and given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in \mathbb{R}$ for which $|x - p| < \delta$.

The Archimedean property of the reals states that, for x, y reals, $x > 0$, there exists $n_0 \in \mathbb{N}$ such that $nx > y$. Thus given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n_0\epsilon > 1$.

Next we show the following

Lemma 1. *Given $n_0 \in \mathbb{N}$ and reals a, b, $0 \leqslant a < b < \infty$, there are only finitely many rationals $x \in (a, b)$, where $x = m/n$, $\gcd(m, n) = 1$, such that $0 < n \leqslant n_0$.*

Proof. If rational $x = m/n$ with $n > 0$ is in (a, b), then $a \leqslant m/n$, so $an_0 \leqslant an < m$. Also $m/n \leqslant b$, so $m \leqslant nb \leqslant n_0 b$. Then $an_0 \leqslant m \leqslant bn_0$, so a finite number of integers m can exist in the interval (an_0, bn_0); but $(an_0, bn_0) \supset (a, b)$, so a finite number of integers m may exist in the interval (a, b). But we require that $0 < n \leqslant n_0$, so there are only finitely many $n > 0$ with this property (in particular, there are n_0 such ns). Since the maximum number of ms and ns is finite, we can form at most a finite number of distinct pairings of m and n (rational numbers) m/n in (a, b). □

Given the lemma above, and given $p \in \mathbb{I}$ and $\epsilon > 0$, the interval $(p-1, p+1)$ contains at most finitely many rationals m/n with $0 < n \leqslant n_0$. Then we can choose $0 < \delta < 1$ such that $(p - \delta, p + \delta) \subset (p - 1, p + 1)$ contains no rationals whose denominator n is at most n_0; i.e., for every $m/n \in (p - \delta, p + \delta)$ with $n > 0$ and $\gcd(m, n) = 1$, $n > n_0$.

Now for $x \in (p - \delta, p + \delta)$ (or $|x - p| < \delta$): if x is rational, $x = m/n$, $n > 0$ and $\gcd(m, n) = 1$, then by the choice of δ, $n > n_0$. Therefore $|f(x) -$

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$f(p)| = |1/n - 0| = 1/n < 1/n_0 < \epsilon$. If x is irrational, $|x - p| < \delta$ implies $|f(x) - f(p)| = |0 - 0| = 0 < \epsilon$.