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The diagrams of the Solow-Swan growth model

Robert M. Solow (1956) and Trevor W. Swan (1956, 2002) independently developed the neoclassical growth model. Swan (1956) was published ten months later than Solow (1956), but involved a more complete analysis of technical progress, which Solow treated separately in Solow (1957). The distinguishing feature of the neoclassical growth model is the assumption that inputs are substitutable in production. The earlier growth models of Harrod (1939) and Domar (1946) were interpreted by Solow (1956, p. 65) as assuming fixed-coefficient production technologies that gave their models “knife-edge” equilibria, with the implausible implication that any deviation at all from equilibrium would cause the model to diverge further and further away from equilibrium. However, Swan (1956, p. 343) regretted that “some of his [Harrod’s] readers seem to have been misled into the belief that, in Harrod’s model, equality between the warranted and the natural rates of growth can occur only ‘by a fluke’” and insisted that his own model only made explicit the implications of a mechanism that Harrod (1948) had “stated very clearly”.[1] Swan goes on to quote Harrod (1948, p. 96) on how a progressive decline in the interest rate (and hence of the marginal product of capital as capital accumulated) would equate the natural and warranted growth rates.

Although the models of Solow (1956) and Swan (1956) are fundamentally the same, there are some significant differences, including differences in the diagrams that illustrate the model. The Solow diagram highlights the substitutability of labour for capital by measuring the

[1] Harrod's "warranted" rate of growth arising from savings and investment behaviour is represented by the rate of growth of capital in the Solow (1956) and Swan (1956) models. In the absence of technical progress, Harrod's "natural" rate of growth is the rate of growth of the labour supply.

capital/labour ratio along the horizontal axis. The levels of output and investment per head are shown on the vertical axis. The Swan diagram relates the output/capital ratio along the horizontal axis to the rates of growth of output, capital and labour along the vertical axis. This focus on rates of growth is particularly useful for illustrating the effects of changes in the rate of technical progress (see Dixon, 2003). Swan (1956) also considers effects on growth arising from a fixed supply of a third factor, land, which creates diminishing returns. We present the Swan diagram in Section 1 because we believe it provides an easier introduction to the model. We then follow in Section 2 with the Solow diagram explaining the more intricate effects of capital/labour substitutability. Section 3 contains a brief history of the development of the Solow-Swan model.

1. The Swan Diagram

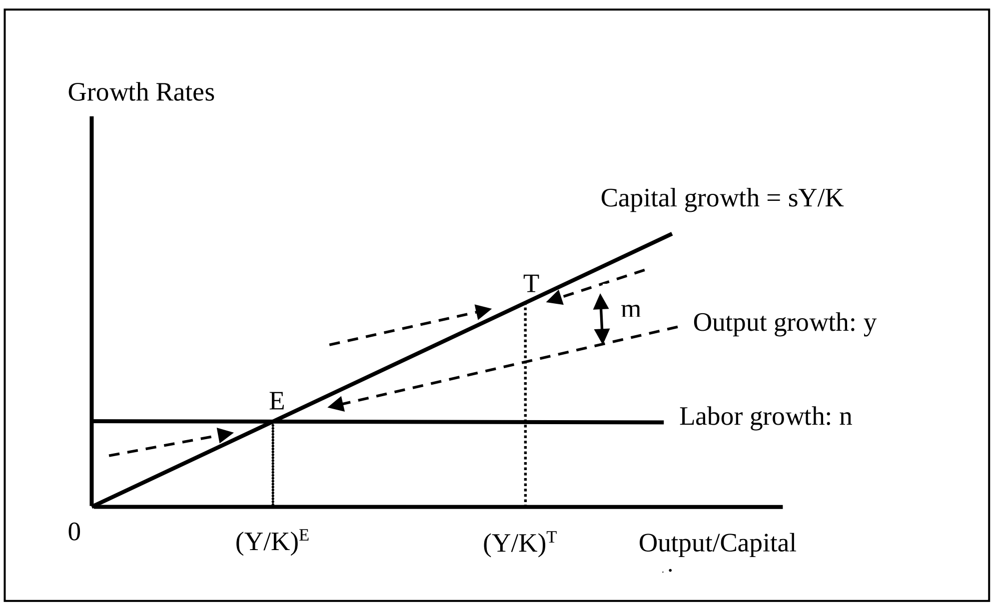

Figure 1: The Swan Diagram

Figure 1 presents the basic Swan diagram. Output, denoted Y, is produced under constant returns to scale by labour, L, and capital, K. In the simplest version of the model, labour is assumed to grow at a constant (and exogenous) rate, n ≡ (dL/dt)/L, which is represent by a horizontal line in

Figure 1. All labour is fully employed. Assuming that saving is a fixed proportion, s, of income, investment, dK/dt, is equal to saving, sY, and the rate of growth of capital is given by (dK/dt)/K = sY/K. Since Y/K varies along the x axis in Figure 1, the rate of growth of capital is simply a straight line through the origin with slope s. Assuming no technical progress, the rate of growth of output, denoted $y \equiv (dY/dt)/Y$, is intermediate between the rates of growth of labour and capital as shown by the lower dotted arrows.[2] Equilibrium is at E, where the rates of growth of capital, labour and output all coincide. To the left of point E, output is growing faster than capital, so Y/K rises towards $(Y/K)^E$. To the right of point E, output is growing more slowly than capital so Y/K falls.

The model suggests the surprising result that an increase in capital formation through a rise in the willingness to save, s, is not a route to increased long-run growth. An increase in s would increase the slope of the capital growth line causing both it and the growth line for output to swing upwards. Initially capital grows faster and there is a rise in the rate of growth of output per head (since y is above n), but in equilibrium, the growth rates for capital and output must return to their original level equal to the growth rate of labour which is unchanged at n. However, equilibrium involves a higher output per head since the increase in investment raises the marginal productivity of labour. Correspondingly, the output/capital ratio is reduced below the original $(Y/K)^E$ in Figure 1 due to the lower marginal productivity of capital.

By contrast, an increase in the rate of technical progress raises long-run growth. As shown by the higher dotted arrows, an exogenous rate, m, of "neutral" technical progress shifts

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[2] If $Y = F(K,L)$ is linearly homogeneous, then $y = \alpha sY/K + \beta n$ where $\alpha \equiv F_K K/Y$, $\beta \equiv F_L L/Y$ and $\alpha + \beta = 1$. Swan (1956) assumed a Cobb-Douglas production function in which α and β are constants, but Swan's earlier model (Swan 2002), published posthumously, was more general (see Pitchford 2002).

up the growth rate of output by m.[3] The outcome is a new equilibrium at T with a higher output/capital ratio, $(Y/K)^T$. A useful insight is that once equilibrium is achieved, a steady rate of technical progress increases both Y and K at the same rate so as to maintain Y/K constant. As explained by Dixon (2003), a greater productivity of labour directly increases Y/L, but the higher output also induces more saving and capital formation so as to maintain a constant ratio of capital to labour measured in efficiency units.

2. The Solow Diagram

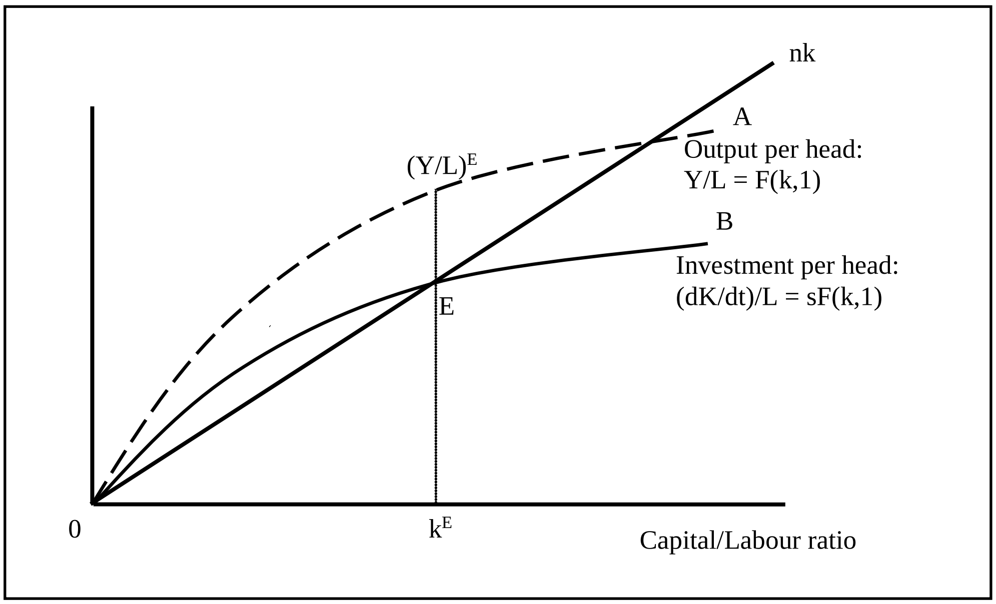

Figure 2: The Solow Diagram

Figure 2 represents the basic Solow diagram. To examine the equilibrating role of the capital/labour ratio, Solow (1956) uses a general linearly homogeneous (constant returns to scale) production function to express the output/labour ratio or output per head as a function, Y/L = F(k, 1) where $k \equiv K/L$. As shown by the dotted line, 0A, in Figure 2, an increase in k increases output per head at a decreasing rate. Investment per head, given by (dK/dt)/L = sY/L, is a fixed

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[3] For $Y = e^{mt}K^{\alpha}L^{\beta}$ with $\alpha + \beta = 1$, Swan (1956) obtains $y = \alpha sY/K + \beta n + m$. As shown by Swan (1964), Harrod neutral (labour saving) technical change is required for the existence of steady state growth, but Hicks neutral and Harrod neutral coincide for a Cobb-Douglas production function.

proportion, s, of output per head as shown by the solid line, 0B. The vertical distance between lines 0A and 0B represents consumption per head. With no technical progress, equilibrium requires that the rate of growth of capital equal the rate of growth of the labour supply: that is $(dK/dt)/K = n$, which implies $(dK/dt)/L = nk$. As illustrated in figure 2, nk is straight line from the origin with slope n. Hence the equilibrium is at point E, corresponding to a capital/labour ratio of k^E and an output/labour ratio of $(Y/L)^E$. If k is initially to the left of k^E, then capital is growing faster than labour, which leads k to rise. Similarly, if k is to the right of k^E, then k falls.[4]

In Figure 2, an increase in the propensity to save, s, increases the slope of the line 0B causing the equilibrium capital/labour to rise above the original k^E. Output per head increases (from the movement to the right along 0A), but consumption per head, represented by the vertical distance between lines 0A and 0B, falls. Technical progress causes both an upward shift in 0A and a movement along OA due to the rise in the capital to labour ratio. A constant rate of technical progress would cause 0A to continuously swing upwards. The seminal contribution of Solow (1957) was to address how technical progress represented by upwards shifts in OA could be estimated separately from movements along OA. Figure 2 can be adjusted to allow for neutral technical progress by defining the capital/labour ratio in efficiency units, but this obscures comparative static effects of changes in technical progress.

Solow (1956) points out the possibility of multiple equilibria. For example, if there is a region in which investment per head increases at an increasing rate with k, it is possible that the line OB (representing investment per head) cuts the line nr in three places. There are then two stable equilibria, one at a low value of k and one at a high value of k, with an unstable equilibrium between the two stable equilibria.

[4] From Solow (1956), $dk/dt = sF(k,1) – nk$.

3. History

Robert Solow's 1956 use of a variable capital/output ratio to eliminate what he saw as an empirically implausible knife-edge equilibrium in the Harrod-Domar one-sector growth model followed from his earlier work generalising John von Neumann's fixed-coefficient, multi-sector growth model by introducing capital/labour substitution and variable coefficients (e.g. Solow and Samuelson 1953). Trevor Swan first presented his growth model in July 1956 in an interdisciplinary seminar at Australian National University discussing W. Arthur Lewis's *Theory of Economic Growth* (1955). Swan had declined to talk about Lewis's chapter on capital but, when the economic historian Noel Butlin concluded his discussion of the chapter, presented his own model as a comment on Butlin's talk (Pitchford 2002, Dixon 2003). Swan used a general production function in his remarks in the seminar and in his post-seminar notes published posthumously as Swan (2002). But in Swan (1956), he followed suggestions by Conrad Leser and Geoffrey Sawer at the seminar to simplify the exposition by assuming a Cobb-Douglas production function and by putting the percentage rate of growth instead of units of output on the vertical axis of his diagrams. The cost of this change was a later misperception that Swan's analysis depended on the Cobb-Douglas functional form.

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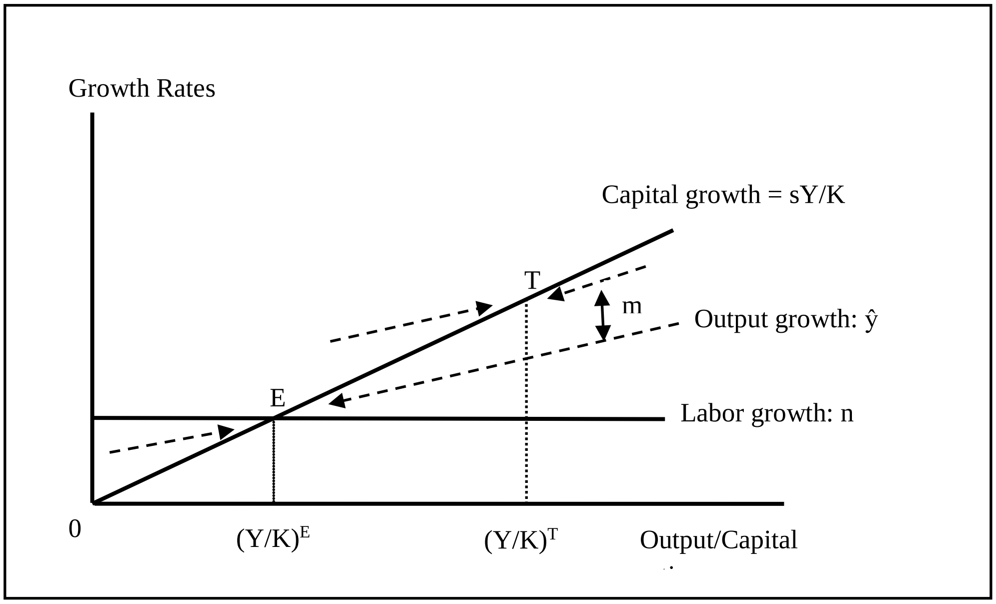

Figure 1: The Swan Diagram

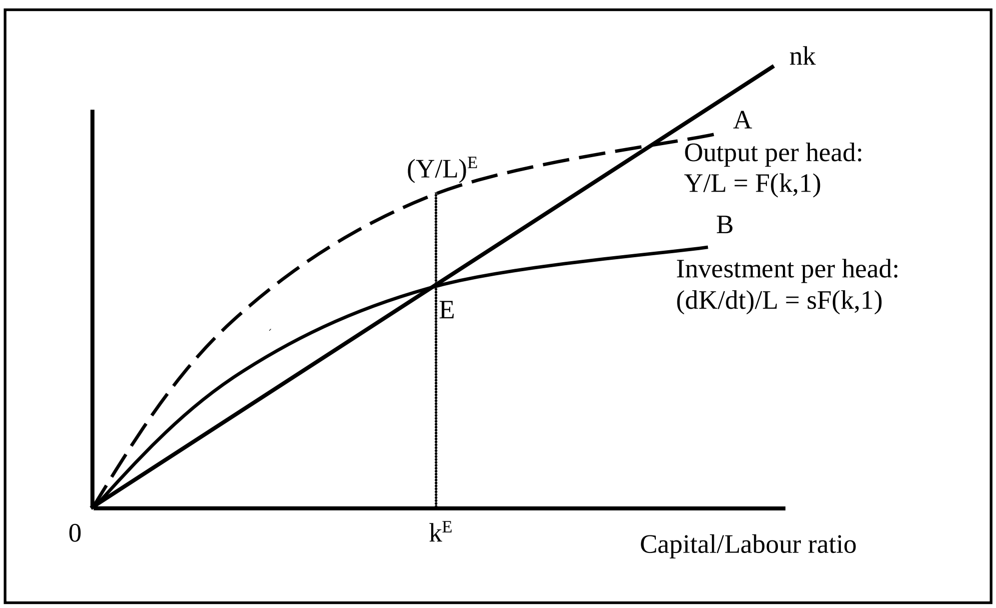

Figure 2: The Solow Diagram