CPSC 320 Little-o/Little-ω Overview

Big O, Θ, and Ω are **roughly** equivalent to asymptotic $\leq$, $=$, and $\geq$ comparisons on functions. That naturally leaves analogues of $<$ and $>$ to define.

1 Formal Definitions via Logic

A function f is little-o of another function g if f grows *strictly slower* than g. That is, $f \in o(g)$ exactly when for every positive real numbers c, there is a positive integer n_0 such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$. Or, stated symbolically:

$$f \in o(g) \equiv \forall c \in \mathbf{R}^+ \exists n_0 \in \mathbf{Z}^+ \forall n \geq n_0, f(n) \leq c \cdot g(n)$$

This is almost exactly like the big-O definition: the difference is that the quantifier in front of c in the definition of o is universal, whereas it is existential in the definition of O. So for **every** possible scaling factor c (including very small ones like $\frac{1}{10000}$), once n is large enough, $g(n)$ is **still** bigger than $f(n)$.

Little-ω is exactly the converse definition: a function f is little-ω of another function g if f grows *strictly faster* than g. That is:

$$f \in \omega(g) \equiv \forall c \in \mathbf{R}^+ \exists n_0 \in \mathbf{Z}^+ \forall n \geq n_0, f(n) \geq c \cdot g(n)$$

Note that $f(n) \in \omega(g(n))$ exactly when $g(n) \in o(f(n))$.

2 Formal Definitions via Limits

When we want to know how two functions compare asymptotically, a **very** handy tool is to compare what happens to $f(n)/g(n)$ when n is very large. In particular, in the cases where the limit is well-defined, we can apply the following theorem:

1. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $g(n) \in o(f(n))$ and $f(n) \in \omega(g(n))$.
2. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant real number $c > 0$, then $f(n) \in \Theta(g(n))$ (and so $g(n) \in \Theta(f(n))$).
3. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n) \in o(g(n))$ and $g(n) \in \omega(f(n))$. (equivalently, this means $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$.)

It turns out we can prove that the limit definitions are equivalent to the logical definitions above (since limits also have quantifier-based definitions!). With a bit of calculus (remind yourself of "L'Hôpital's Rule"), using the limits technique is often **much** easier than using the logical definitions.

Try these out to compare: $n + 3$, $3n$, $n^2 - 1$, and 2^n.

Note that if the limit does not exist, then it does not mean we can not use one of our asymptotic notations; it simply means we will have to use the logic definition to determine whether or not they are comparable. For instance, if $f(n) = n$, and $g(n)$ oscillates between $n/2$ and $2n$, then $lim_{n\to\infty} f(n)/g(n)$ does not exist (the value oscillates between $1/2$ and 2 without ever settling down near one or the other extreme). However $f \in \Theta(g)$.

3 Little-o is not really Big-O minus Θ

A common misconception is to assume that if $f \in O(g)$, and $f \notin \Theta(g)$, then $f \in o(g)$. This is not in fact correct: consider the function $n|\sin n|$.

- Because $|\sin n|$ oscilllates between 0 and 1, $n|\sin n|$ oscillates between 0 and n. If we compare that to n asymptotically, we find that $n|\sin n| \in O(n)$ (with the constant scaling factor $c = 1$, in fact!)
- However $n|\sin n| \notin \Theta(n)$ and $n|\sin n| \notin o(n)$. (In the case of the limit, the ratio of these two functions is just $|\sin n|$ which oscillates between 0 and 1 and so does not approach either value or anything in between!)

So the analogy of comparing o, O, Θ, Ω and ω to $<$, $\leq$, $=$, $\geq$, and $>$ respectively is useful but not exact.