Solutions to Review of Mathematical Proof Techniques

- **Direct proof:** start with the assumptions or definitions and apply some reasoning to obtain the conclusion. May involve breaking the proof into cases, or proving a logically equivalent statement.

  - **Exercise:** Prove that the square of any odd number is odd.

  SOLUTION: We make use of the property that any odd integer can be written in the form $2k+1$ where k is an integer. If x is odd, then we can write $x = 2k+1$ for some integer k. This means that

  $$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

  Because $2k^2 + 2k$ is an integer, we conclude that x^2 is also odd.

- **Proof by contradiction:** assume that the conclusion is false and obtain a contradiction with an assumption or some known facts. In CPSC 320, this is frequently (but not always!) the easiest way to prove a statement.

  - **Exercise:** Let a and b be two real numbers. Prove that if a is rational and ab is irrational, then b is irrational.

  SOLUTION: Assume to the contrary that b is rational. Any rational number can be written in the form $\frac{p}{q}$, where p and q are both integers. Therefore, we can write

  $$a = \frac{p}{q} \text{ and } b = \frac{r}{s},$$

  for some integers p, q, r, s. But this implies that

  $$ab = \frac{pr}{qs}.$$

  Because p, q, r, s, pr is an integer, as is qs. But this implies that ab is rational, which is a contradiction. Therefore, b must be irrational.

- **Proof by contrapositive:** Instead of proving the statement "if A, then B", we prove the logically equivalent *contrapositive* statement "if not B, then not A." In CPSC 320, this is often useful in proving the correctness of a reduction (as we saw in the RHP example).

  - **Exercise:** Prove that if x^2 is even, then x must be even.

  SOLUTION: Proving this statement is equivalent to proving the contrapositive, which is: if x is odd, then x^2 is odd. And we already proved that in the direct proof exercise – QED!

- **Proof by induction:** used to prove that a statement holds for every natural number ($n = 0, 1, \ldots$). There are three components: first, the *base case*, in which you establish that the statement holds for some small, trivial case $n = n_0$ (usually $n_0 = 0$ or $n_0 = 1$). Then, the *inductive hypothesis* **assumes** that the statement holds for $n = k$ (weak induction) or for all $n \leq k$ (strong induction) and we show in the *inductive step* that, given the inductive hypothesis, the statement must hold for $n = k+1$. We can then conclude that it's true for all natural numbers n.

– **Exercise:** Prove that the first n odd integers sum to n^2.

SOLUTION: We prove the statement by induction (surprising, we know). We denote the sum of the first n odd integers by $S(n)$ and we wish to prove that $S(n) = n^2$.

Base case: $S(1) = 1 = 1^2$; hence, the statement holds for the base case $n = 1$.

Inductive hypothesis: assume that $S(k) = k^2$ (i.e., the statement holds for $n = k$).[1]

Inductive step: we will now prove that $S(k+1) = k + 1^2$. The $(k+1)^{\text{th}}$ odd number is $2(k+1) - 1 = 2k + 1$. We can then write

$$\begin{aligned} S(k+1) &= S(k) + 2k + 1 \\ &= k^2 + 2k + 1 \qquad \text{(by I.H.)} \\ &= (k+1)^2. \end{aligned}$$

This shows that $S(n) = n$ for all $n \geq 1$, which completes the proof.

[1]We have used weak induction here. Using strong induction – and assuming that the statement holds for all $n \leq k$ – would have also worked.