Problem: What is the angle between the faces of a regular tetrahedron?

The apex is \((1,1,1)\), it is \(\sqrt{2}\) from the other 3 vertices.

\(\vec{a} = \langle 1, 0, 1 \rangle\)
\(\vec{b} = \langle 0, 1, 1 \rangle\)
\(\vec{c} = \langle -1, 1, 0 \rangle\)

The normal to face determined by \(\vec{a}, \vec{b}\) is
\(\vec{n}_1 = \vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1, -1, 1 \rangle\)

The normal to face determined by \(\vec{b}, \vec{c}\)
\(\vec{n}_2 = \vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1, -1, 1 \rangle\)

The angle between the faces is the same as the angle between normal vectors \(\vec{n}_1, \vec{n}_2\).
\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}
\]

\[|\vec{n}_1| = |\vec{n}_2| = \sqrt{3} \quad \text{and} \quad \vec{n}_1 \cdot \vec{n}_2 = -1 + 1 - 1 = -1 \quad \text{so}
\]

\[\cos \theta = \frac{1}{\sqrt{3}} = 1 \quad \Rightarrow \quad \theta = \cos^{-1} \left( \frac{1}{3} \right) \approx 70^\circ
\]

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**equations of lines and planes**

A line in 3D is determined by a direction vector and a point it is going through.

\[(x_0, y_0, z_0)\]

\[\vec{n} = \langle a, b, c \rangle\]

\[\vec{r} = \vec{r}_0 + t \vec{n}\]

There is a parameter. Any vector \(\vec{r}\) from the origin is of the form: \(\vec{r} = \vec{r}_0 + t \vec{n}\) for some \(t\).
So if \((x, y, z)\) is a point on the line, there is some \(t\) such that

\[
\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle
\]

\[
= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle
\]

\(a, b, c\) are the components of \(\vec{w}\) \((\vec{w} = \langle a, b, c \rangle)\).

**Parametric equation of the line:**

\[
\begin{align*}
x &= x_0 + ta \\
y &= y_0 + tb \\
z &= z_0 + tc
\end{align*}
\]

\(\text{This is not unique, we could have chosen another } \langle x_0, y_0, z_0 \rangle \text{ or any other vector } \vec{w} \text{ parallel to } \vec{w}^2.\)

\(\Rightarrow\) We can also eliminate \(t\) from the above equations to get 2 equations involving only \(x, y, z\) as variables (no \(t\))

\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}
\]

Some times called set of symmetric equations.
Example: Find the equations of the line passing through the points \((2, 0, 3)\) and \((3, 4, 0)\).

Let \((x_0, y_0, z_0) = (2, 0, 3)\)

\[
\vec{v} = \vec{AB} = (3-2)i + 4j - 3k
\]

\[
\Rightarrow a = 1, \quad b = 4, \quad c = -3
\]

\[
\Rightarrow \frac{x - 2}{1} = \frac{y - 0}{4} = \frac{z - 3}{-3}
\]

The first equation yields:

\[
y = 4x - 8
\]

The second:

\[
z = -\frac{3}{4}y + 3 = -\frac{3}{4}(4x-8) + 3
\]

\[
z = -3x + 9
\]

Our line is the intersection of these two planes.
where does this line intersects the $y$-$z$ plane?

$\Rightarrow$ $y$-$z$ plane is defined by $x = 0$

$\Rightarrow \begin{cases} y = -8 \\ z = 9 \end{cases}$

The point where the line intersects the $y$-$z$ plane is $(0, -8, 9)$.

$\Rightarrow$ is there a similar way to determine a plane (with vectors) $\Rightarrow$ A plane is determined by a point on it $(x_0, y_0, z_0)$ and a normal to this point $\vec{n} = \langle a, b, c \rangle$

This vector is $\langle x-x_0, y-y_0, z-z_0 \rangle$
$(x, y, z)$ is on the plane determined by $\vec{n}$ and $(x_0, y_0, z_0)$ if and only if the vector $\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$

\[ \iff ax + yb + cz + d = 0 \]

with $d = -ax_0 - by_0 - cz_0$

Example: find the equation of the plane passing through the given points $(0,0,3), (0,2,0)$ and $(1,0,0)$

\[ \vec{a} = (-1, 2, 0) \quad \vec{b} = (-1, 0, 3) \]

Compute $\vec{n} = \vec{a} \times \vec{b} = \det \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\vec{i} + 3\vec{j} + 2\vec{k}$
any point \((x, y, z)\) on the plane satisfies

\[6(x-x_0) + 3(y-y_0) + 2(z-z_0) = 0\]

pick \(x_0 = 0\) or \(x_0 = 0\) or \(x_0 = 1\)
\(y_0 = 0\) \hspace{1cm} y_0 = 2\) \hspace{1cm} y_0 = 0\)
\(z_0 = 3\) \hspace{1cm} z_0 = 0\) \hspace{1cm} z_0 = 0\)

\[\Rightarrow 6x - 6 + 3y + 2z = 0\]