Example: \( f(x, y) = \frac{xy}{x^2 + y^2} \) is not continuous at \((0,0)\) and is continuous for all \((a, b) \neq (0,0)\) 

Example

\[ f(a, b) \]

\[ z = f(x) \]

This function is not continuous at \((a, b)\)

Partial derivatives:

**Lecture 9**

For a function \( z = f(x, y) \) of 2 variables, if we fix one of variables, then we get a function of a single variable (the one not fixed), and we can take its derivative:

- For \( y \) fixed, we get \[ \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x \not\in \text{function of } (x, y) \]

- For \( x \) fixed, we get \[ \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y \not\in \text{function of } (x, y) \]

\[ \frac{\partial f}{\partial x} (x, y) = f_{x}(x, y) = \text{derivative with respect to } x, \text{ treating } y \text{ as a constant} \]
Example: \( f(x, y) = x^4 + 2x^2y^2 + y + e^{xy} \)

\[
\frac{\partial f}{\partial x} = 4x^3 + 4xy^2 + ye^{xy}
\]

\[
\frac{\partial f}{\partial y} = 4x^2y + 1 + xe^{xy}
\]

\[
f_x(1, 2) = \frac{\partial f}{\partial x} (1, 2) = 4(1)^3 + 4(1)(2)^2 + 2e^{1.2}
\]

\[
= 20 + e^2
\]

\[
f_y(1, 2) = \frac{\partial f}{\partial y} (1, 2) = 4 \cdot 2 + 1 + e^2 = 9 + e^2
\]

\[\Rightarrow \text{in terms of limits,}\]

\[
f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]

\[
f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\]
Geometric interpretation: slope of the trace curves

Trace curve: intersection with plane \( y = b \) (i.e. \( y \) is fixed)

Slope of tangent = \( \frac{df(a, b)}{dx} \)

Similar picture for \( \frac{df(a, b)}{dy} \)

Slope of tangent to trace curve is \( \frac{df(a, b)}{dy} \).

Trace curve: intersection with plane \( z = a \) (\( z \) is fixed)
Just as with one variable we have implicit differentiation.

So if \( z = \sqrt{1 - x^2 - y^2} \) \( \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - y^2}} = \frac{-x}{z} \)

If \( z \) is implicitly a function of \( x \) and \( y \):

\[
1 = x^2 + y^2 + z^2 \quad \text{(as } z = z(x,y)) \text{ implicitly)}
\]

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + z^2) \\
\frac{\partial f}{\partial x} = 2x + 2y \frac{\partial z}{\partial x} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-x}{z} \quad \text{(as before)}
\]
Compare geometry

\[ x^2 + y^2 + z^2 = 1 \]
\[
\Rightarrow z = \sqrt{1 - x^2 - y^2}
\]
\[
\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}
\]

Look for example at trace curve corresponding to \( y = 0 \)

\[
\frac{\partial z}{\partial x} (1,0) = \frac{-1}{\sqrt{1-(1)^2}} = \frac{-1}{0} = \pm \infty
\]

\[
\frac{\partial z}{\partial x} (0,0) = \frac{0}{\sqrt{1-(0)^2}} = 0
\]

\[
\frac{\partial z}{\partial x} (-1,0) = \frac{-(-1)}{\sqrt{1-(-1)^2}} = \frac{1}{0} = \pm \infty
\]

\[
\frac{\partial z}{\partial x} \left( \frac{1}{2},0 \right) = \frac{-\left( \frac{1}{2} \right)}{\sqrt{1-\left( \frac{1}{2} \right)^2}} = \frac{-\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{-1}{\frac{1}{2}} = \frac{-1}{\sqrt{3}}
\]
Example: \( f(x,y) = \int_{y}^{x} g(t) \, dt \)

**Fundamental Theorem of Calculus**

\[ \Rightarrow \ \frac{\partial f}{\partial x} = g(x) \quad \text{and} \quad \frac{\partial f}{\partial y} = -g(y) \]

**Higher derivatives**

\[
\begin{align*}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y}
\end{align*}
\]

**Theorem**: \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \) order doesn't matter.

Recall from single-variable calculus:

- \( f'' > 0 \) concave downwards
- \( f'' < 0 \) concave upwards

\( f'' > 0 \): \( f' \) at \( x \):

\( f''(x) > 0 \): \( f' \) at \( x \):

\( f'' > 0 \) concave upwards