for: \((x_0, y_0) = (0, 0)\) \(z_0 = e^0 = 1\)  \[\text{Lecture 12}\]

\[
\frac{\partial z}{\partial x} = -2xe^{-x^2-y^2} \Rightarrow \frac{\partial z}{\partial x}(0,0) = 0
\]

\[
\frac{\partial z}{\partial y}(0,0) = 0
\]

\[z = z_0 = 1\] is the equation of the tangent plane at \((x_0, y_0) = (0, 0)\)

**Def.** The "Linear approximation" or "tangent plane approximation" to \(f\) at \((a, b)\) is

\[f(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)\]

def: Linear approximation

**Example:** Find the linear approximation to \(f(x, y) = \sqrt{x^2+y^2}\) at \((3, 4)\) and use it to approximate

\[f(3.1, 3.9)\]

\[
f_x = \frac{x}{\sqrt{x^2+y^2}} \quad f_y = \frac{y}{\sqrt{x^2+y^2}}
\]

\[
f_x(3, 4) = \frac{3}{5} \quad f_y(3, 4) = \frac{4}{5}
\]

\[
\Rightarrow f(3.1, 3.9) \approx 5 + \frac{3}{5}(3.1-3) + \frac{4}{5}(3.9-4)
\]

\[
= 5 + \frac{3}{50} - \frac{4}{5} = 4.9800 \quad \text{exact value is} \quad 4.98197\]
Notation: \( \Delta x = x - a \), \( \Delta y = y - b \), \( \Delta z = f(x,y) - f(a,b) \)

The linear approximation becomes:

\[ \Delta z \approx f_x(a,b) \Delta x + f_y(a,b) \Delta y \]

Approximation gets better as \( \Delta x, \Delta y \to 0 \) (because \( \Delta z \to 0 \))

Differentials: In single variable calculus, for \( y = f(x) \) we have \( dy = f'(x) \, dx \)

\( y \) (the true change in \( y \), i.e. \( y(a+dx) - y(a) \))

So \( f(a+dx) = f(a) + dy \) but \( dy \) is unknown, so we use \( dy \approx f'(a) \, dx \) with \( du = f'(a) \, dx \).
In two variables case, for \( z = f(x,y) \) we define:

\[
dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

or similarly

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

Example: A rectangular box has dimensions 80 x 100 x 50 cm, each measured to within 1 mm. Estimate the maximum error in the volume.

Solution: Volume \( V = xyz \)

\[
dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = yz dx + xz dy + xy dz
\]

\[
dV = (100 \times 50)(0,1) + (80 \times 50)(0,1) + (80 \times 100)(0,1) \text{ cm}^3
\]

\[
= 500 + 400 + 800 = 1700 \text{ cm}^3
\]

\[
\Rightarrow \text{Max error is } \pm 1700 \text{ cm}^3 \Rightarrow \text{relative max error } \frac{1700}{80 \times 100 \times 50} = 9.4\%.
\]
Example (chapter 14.1 no 34 Apex)

Estimate the amount of metal in a closed cylindrical can, 10cm high, 4 cm in diameter, if top and bottom are 0.1 cm thick and sides 0.05 cm thick.

\[
\text{Volume } V = \pi r^2 h
\]

\[
dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh
\]

\[
dV = 2\pi rhdr + \frac{\pi r^2 dh}{L}, \text{ where } L \text{ is surface area of top (or bottom)}
\]

\[
dV = 2\pi \cdot 1\cdot 10 \cdot 0.1 + \pi \cdot 4 \cdot 0.1 = 2 \pi \cdot 0.8 + 4 \pi \cdot 0.1
\]

\[
dV = \frac{40\pi}{80} + 4 \pi = 2\pi \approx 6.28 \text{ cm}^3
\]

Material for midterm 1 ends here \(\Box\)