\[
\Rightarrow \left( \frac{dz}{dr} \right)^2 + \frac{1}{r^2} \left( \frac{dz}{ds} \right)^2 = \left( \frac{dz}{dx} \right)^2 + \left( \frac{dz}{dy} \right)^2
\]

Chain rule and implicit differentiation

We can use the chain rule to better understand implicit differentiation. Suppose we have an equation \( F(x, y, z) = 0 \), if we can explicitly solve this equation for \( z \), we get \( z = f(x, y) \), some function of \((x, y)\). Maybe we can’t solve it explicitly but we can regard \( F(x, y, z) = 0 \) as \( z = f(x, y) \) such that \( F(x, y, f(x, y)) = 0 \).

Example: \( 2x + 3y - 4z - e^{xy}z^2 - 1 = 0 \)\n
\( \Rightarrow \) can’t solve for \( z \). We can get \( \frac{\partial z}{\partial x} = f_x \) and \( \frac{\partial z}{\partial y} \) by differentiating \( F(x, y, z) = 0 \) with respect to \( x \) or \( y \).
\[ \frac{\partial F(x, y, z)}{\partial x} = \frac{\partial (0)}{\partial x} = 0 \]

\[ \frac{\partial F(x, y, z)}{\partial y} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \]

because \( y \) is independent of \( x \)

\[ \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} = 0 \]

it comes that:

\[ \frac{\partial F}{\partial x} = -\frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \]

or

\[ \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial z} / \frac{\partial F}{\partial x} \]

similarly

\[ \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial z} / \frac{\partial F}{\partial y} \]

In the example \( F(x, y, z) = 2x + 3y - 4z(x, y)e^{x+y^2} - e^{xy z(x, y) - 1} \)

\[ \frac{\partial F(x, y, z)}{\partial x} = \frac{\partial (2x + 3y - 4z(x, y)e^{x+y^2} - e^{xy z(x, y) - 1})}{\partial x} = 0 \]
$\frac{\partial F(x,y,z)}{\partial x} = 2 - 4 \frac{dz}{dx} - \left( yze^{x} + xye^{x} \frac{dz}{dx} \right) = 0$

It comes that:

$\frac{dz}{dx} = \frac{yze^{x} - 2}{-4xye^{x} - xye^{x}}$

Similarly:

$\frac{dz}{dy} (2x + 3y - 4z - e^{x}) = 0$

It comes:

$3 - 4 \frac{dz}{dy} - e^{x} (x^{2} + xy \frac{dz}{dy}) = 0$

$\frac{dz}{dy} (4 - xye^{x}) = \frac{xy^{2} - 1}{x^{2} - 3}$

$\frac{dz}{dy} = \frac{xy^{2} - 1}{x^{2} - 3}$
Find now the equation of the plane tangent to the surface \( 2x + 3y - 4z - e^{xy^2-1} = 0 \) at the point \((x, y, z) = (1, 1, 1)\).

=> Check first that \((1, 1, 1)\) is on the surface:

\[
2 + 3 - 4 - 1 = 0 \quad \text{OK}
\]

=> The equation of the plane is:

\[
z_p = z_0 + \frac{\partial z}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y-y_0)
\]

=> \(z_p = 1 + \left(\frac{1-2}{-4-1}\right)(x-1) + \left(\frac{1-3}{-4-1}\right)(y-1)
\]

\[
= \left[1 + \frac{1}{5}(x-1) + \frac{2}{5}(y-1)\right]
\]

Find an approximate solution to the equation:

\[
\frac{5}{3} x^2 + \frac{7}{2} y^2 - 4z - e^{\frac{3}{36} z - 1} = 0
\]
= hint: consider $F(x, y, z)$ for $(x, y) = (\frac{5}{6}, \frac{7}{6})$ since $(x, y)$ is close to $(1, 1)$, the tangent plane is a good approximation

$$z_p(\frac{5}{6}, \frac{7}{6}) = 1 + \frac{1}{5}(\frac{5}{6} - 1) + \frac{2}{5}(\frac{7}{6} - 1)$$

$$= 1 + \frac{1}{5}(\frac{-1}{6}) + \frac{2}{5}(\frac{1}{6}) = 1 + \frac{1}{30} = \frac{31}{30}$$

$$2(\frac{5}{6}, \frac{7}{6}) \times z_p(\frac{5}{6}, \frac{7}{6}) = \frac{31}{30}$$