Directional Derivatives and gradient vector

Example: A hill has altitude \( z = f(x, y) = e^{-x^2-y^2} \) and you drive with position \( x = t, y = 1 - t \).

Your height at time \( t \) is \( f(x(t), y(t)) \) and your rate of ascent is

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt}
\]

\[
= -2xe^{-x^2-y^2} \frac{dx}{dt} + 2ye^{-x^2-y^2} \frac{dy}{dt}
\]

\[
= -2xe^{-x^2-y^2} + 2ye^{-x^2-y^2}
\]

\[
= -2xe^{-x^2+y^2} + 2ye^{-x^2-y^2}
\]

\[
= 2e^{-x^2-y^2} (y-x)
\]

When is \( z(t) \) maximum? (When are we at the highest point of the road?)
\[ z = f(x, y) \]

\[ \Rightarrow \text{maximum height occurs when } \frac{dz}{dt} = 0, \text{ i.e. } y = x \]

which means \( 1 - t = t \) so \( t = \frac{1}{2} \)

the maximum height location is then \( (x_{max}, y_{max}) = \left( \frac{1}{2}, \frac{1}{2} \right) \)

and \( z_{max} = f(x_{max}, y_{max}) = e^{\frac{1}{2}} = \frac{1}{\sqrt{e}} \approx 0.60 \)

What we have done here is the idea of

the directional derivative
and we are at some location \((x_0, y_0)\). It doesn't make sense to ask how fast does \(f(x, y)\) change at \((x_0, y_0)\)? But it does make sense to ask how fast \(f(x, y)\) changes as we move in some direction \(\mathbf{u}\), (with \(|\mathbf{u}| = 1\)).

The change of elevation depends on the direction of the road.

\[
\left(\Delta_x f\right)(x_0, y_0) = \text{rate of change of } f(x, y) \text{ as we move in the direction of } \mathbf{u} \text{ at a constant speed of } 1.
\]

\[
\mathbf{u} = \langle a, b \rangle \quad |\mathbf{u}| = 1.
\]
\[ x = x_0 + at \] parametric equations for the line through \((x_0, y_0)\) in the direction of \(\vec{u}\).

\[ \vec{u} = \langle a, b, 0 \rangle \]

Then \((D_{\vec{u}} f)(x_0, y_0) = \frac{df}{dt} \bigg|_{t=0} = \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \bigg|_{t=0} \]

\[ \begin{align*}
\frac{\partial f(x_0, y_0)}{\partial x} \frac{dx(0)}{dt} + \frac{\partial f(x_0, y_0)}{\partial y} \frac{dy(0)}{dt} &= \frac{\partial f(x_0, y_0)}{\partial x} a + \frac{\partial f(x_0, y_0)}{\partial y} b \\
\frac{dx}{dt} &= \alpha \\
\frac{dy}{dt} &= \beta
\end{align*} \]

\[ (D_{\vec{u}} f)(x_0, y_0) = \langle \frac{\partial f(x_0, y_0)}{\partial x}, \frac{\partial f(x_0, y_0)}{\partial y} \rangle \cdot \langle a, b, 0 \rangle = \frac{1}{||\vec{u}||^2} \]

\[ \Rightarrow \text{We found that} \quad D_{\vec{u}} f = \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cdot \frac{1}{||\vec{u}||^2} \]

\[ \Delta \text{ Definition: The gradient of a function } f \text{ of } x, y \text{ is} \]

\[ \nabla f = \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \]
Thus, we have:

\[ \nabla^2 f = \Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \]

\( \Rightarrow \) we compute the gradient of \( f \) and from that we can easily compute any directional derivative. Special cases \( \nabla_1 f = \frac{\partial f}{\partial x}, \quad \nabla_2 f = \frac{\partial f}{\partial y} \)

**Example:** Find the gradient of \( f(x, y) = x^2 + y^2 \) (infinite paraboloid) and compute the directional derivative \( (\nabla f)(x_0, y_0) \) for \( x_0 = \sqrt{2} \) and \( y_0 = \sqrt{2} \)

\[ f(x_1, y_1) = \frac{1}{2} \begin{pmatrix} 2, 1 \end{pmatrix}, \quad \nabla_1 f = \frac{1}{\sqrt{2}} \begin{pmatrix} 2, 1 \end{pmatrix}, \quad \nabla_2 f = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, -1 \end{pmatrix}, \quad \nabla_3 f = \begin{pmatrix} 1, 0 \end{pmatrix} \]

\( \Rightarrow \) \( \frac{df}{dx} = 2x \quad \frac{df}{dy} = 2y \), so \( \nabla f(x_0, y_0) = \begin{pmatrix} 2x_0, 2y_0 \end{pmatrix} \)

\[ \nabla^2 f(x_0, y_0) = \begin{pmatrix} 2\sqrt{2}, 2\sqrt{2} \end{pmatrix} \]

\( (\nabla^2 f)(\sqrt{2}, \sqrt{2}) = \begin{pmatrix} 2\sqrt{2}, 2\sqrt{2} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1, 1 \end{pmatrix} \)

\[ = 4 \]

\( \Rightarrow \) The function increases in the direction of \( \nabla^2 f \) at a rate of 4.