Gradient and Directional derivatives of Function of 3-variables

Similar ideas apply for 3 or more variables.

For \( w = F(x, y, z) \) we define \( \vec{\nabla} F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \) and the directional derivative is \( D_{\vec{u}} F = \nabla F \cdot \vec{u} \) with \( \vec{u} = \alpha \vec{i} + \beta \vec{j} + \gamma \vec{k} \) and \( ||\vec{u}|| = 1 \).

A contour (or level) surface of \( F \) is given by the points \( (x, y, z) \) \( F(x, y, z) = k \) (constant). A similar argument to the one above shows that \( \nabla F(x_0, y_0, z_0) \) is orthogonal to the contour surface containing \( (x_0, y_0, z_0) \).

Thus, we define the tangent plane to the contour surface to be the plane with normal \( \nabla F(x_0, y_0, z_0) \) and containing \( (x_0, y_0, z_0) \), namely:

\[
\frac{\partial F(x_0, y_0, z_0)}{\partial x} (x-x_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial y} (y-y_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial z} (z-z_0) = 0
\]
special case: surface $z = f(x, y)$

regard as contour-surface $F(x, y, z) = -z + f(x, y) = 0$

In this case,

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$\frac{\partial F}{\partial y}(x_0, y_0, z_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$\frac{\partial F}{\partial z}(x_0, y_0, z_0) = -1$$

and the tangent plane is

$$\frac{\partial f}{\partial x}(x_0, y_0) (x - x_0) + \frac{\partial f}{\partial y}(y - y_0) - (z - z_0) = 0$$,

as before.

Example: Show that every tangent plane to the surface $x^2 + y^2 = z^2$ passes through the origin.

Find tangent plane at $(x_0, y_0, z_0)$: Let $F(x, y, z) = x^2 + y^2 - z^2$

Surface is $F(x, y, z) = 0$. 
Then \( \frac{\partial F}{\partial x} = 2x, \frac{\partial F}{\partial y} = 2y, \frac{\partial F}{\partial z} = -2z \).

Tangent plane:

\[ 2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0 \]

Contains \((0, 0, 0)\)? Yes if

\[ 2x_0(-x_0) + 2y_0(-y_0) - 2z_0(-z_0) = 0 \]

\[ -2(x_0^2 + y_0^2 + z_0^2) = 0 \quad \text{so Yes!} \]

\[ \text{[because the equation of the surface is} \quad x^2 + y^2 = z^2\text{]} \]
Max/Min of functions of 2 variables

One of the most powerful ways to use calculus of one variable was to find maximum or minimum values.

To maximize or minimize $f(x)$ on an interval $[a, b]$, we look at:

1. Critical points $f'(x) = 0$
2. Boundary points

Critical points can be:

1. Local max $f'' < 0$
2. Local min $f'' > 0$
3. $f'' = 0$ could be either or neither.
For $f(x, y)$ on a closed and bounded domain $\mathbb{R}^n$, we will have the same strategy: critical points are the zeros of the partial derivatives.

Critical points:

$(x_0, y_0)$ is a critical point if $\nabla^2 f(x_0, y_0) = 0$; i.e., the tangent plane at $(x_0, y_0)$ is horizontal.

Ordinary critical points come in 3 kinds:

- **local max.**

- **local min.**

- **saddle point**

Ordinary critical points come in 3 kinds:

- **hill**

  $z = -x^2 - y^2$

- **hole**

  $z = x^2 + y^2$

- **pass or saddle**

  $z = -x^2 + y^2$
Second derivative test (Proof is p.930 in the Stewart book; have a look, it is interesting)

Let \( D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 \)

If \((x_0, y_0)\) is a critical point then:

\( D(x_0, y_0) < 0 \) \(\Rightarrow\) \((x_0, y_0)\) is a saddle.

\( D(x_0, y_0) > 0 \) \(\Rightarrow\) \((x_0, y_0)\) is a local min or local max.

\( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) < 0 \): local max.

\( D(x_0, y_0) < 0 \) and \( f_{xx}(x_0, y_0) > 0 \): local min.

\( D(x_0, y_0) = 0 \) and \( f_{xx}(x_0, y_0) = 0 \): is impossible, because \(-f_{yy}^2\) cannot be positive.

Example: Find and classify the critical points of \( f(x, y) = (2x-x^2)(2y-y^2) \)

\[ \nabla f = \nabla \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} = (2y-y^2)(2-2x) = 2(1-x)(2-y)y \\ \frac{\partial f}{\partial y} = (2x-x^2)(y-2y) = 2(1-y)(2-x)x \end{cases} \]