Double integrals in Polar Coordinates

\[ r^2 = x^2 + y^2 \]
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

Often the region and/or the function is easier to write in polar coords. Especially true in scientific applications where problems often have a rotational symmetry, e.g.

\[ R = \{(x,y) : 1 \leq x^2 + y^2 \leq 2, x \geq 0, y \geq 0\} \]
\[ R = \{ (r, \theta) : 1 \leq r \leq \sqrt{2}, 0 \leq \theta \leq \frac{\pi}{2} \} \]

R is awkward to do as type I or type II.
Definition: A "polar rectangle" is \( R = \{(r, \theta) | a \leq r \leq b, a \leq \theta \leq \beta\} \)

\[ r = b \quad \text{we want to consider} \]
\[ \iint_R f(x,y) \, dA \quad \text{when } R \text{ is a polar rectangle.} \]

\( \Rightarrow \) Recall: for ordinary rectangle \( R_0 \), we defined
\[ \iint_R f(x,y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \, \Delta x \Delta y \]

For polar rectangle \( R \), we subdivide \( R \) into polar subrectangles.

\( \Rightarrow \) area of a slice of pie \( \frac{(\theta) \pi r^2}{2} = \frac{1}{2} \theta r^2 \)

divide up \([a, b] \) into \( n \) pieces of size \( \Delta r = b - a \)

divide up \([a, \beta] \) into \( m \) pieces or size \( \Delta \theta = \frac{\beta - a}{m} \)
Zoom on a polar sub-rectangle.

\[ r_j^* = \frac{1}{2} (r_i + r_{i-1}) \]
\[ \theta_j^* = \frac{1}{2} (\theta_i + \theta_{i-1}) \]

In rectangular words, centre is \((r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*)\)

as area of slice of pie is \(\int_{2\pi}^0 \frac{(r)^2 \pi r^2}{2} = \frac{\pi r^2}{2}\)

The area of \(\Delta A_i \) is \(\Delta \theta \frac{r_i^2 - r_{i-1}^2}{2} - \Delta \theta \frac{r_{i-1}^2}{2}\)

\[ \Rightarrow \Delta A_i = \frac{1}{2} (r_i^2 - r_{i-1}^2) \Delta \theta \]
\[ \Delta A_i = \frac{1}{2} \left( \frac{r_i + r_{i-1}}{r_i^*} \right) \frac{r_i^*}{r_i^* - r_{i-1}^*} \Delta \theta \]
\[ \Delta A_i = r_i^* \Delta r \Delta \theta \]

Polar Riemann sum: \(\sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) \Delta A_i\)

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) r_i^* \Delta r \Delta \theta \]

\text{-call this } g(r_i^*, \theta_j^*)
Let \( m, n \to \infty \). Riemann sum converges to \( \int_{a}^{b} \int_{a}^{b} g(r, \theta) \, dr \, d\theta \) and we found that:

\[
\int_{R} \int_{R} f(x, y) \, dx \, dy = \text{limit of polar sum}
\]

\[
= \int_{0}^{\pi/2} \int_{0}^{2} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

Think of \( dA = r \, dr \, d\theta \).

**Example**

Evaluate \( I = \int_{R} \int_{R} x^2 \, dA \)

\[
I = \int_{0}^{\pi/2} \int_{0}^{2} (r \cos \theta)^2 \, r \, dr \, d\theta
\]

\[
= \frac{1}{2} \left[ \int_{0}^{\pi/2} \cos^2 \theta \, d\theta \right]
\]

\[
= \frac{1}{2} \left[ \int_{0}^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta \right]
\]

\[
= \frac{1}{4} \left[ \int_{0}^{\pi/2} (1 + \cos 2\theta) \, d\theta \right]
\]

\[
= \frac{15}{8}
\]

Use \( \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \).