This constitutive equation contains three parameters: \( \eta_0 \), the zero-shear-rate viscosity; \( \lambda_1 \), the relaxation time; and \( \lambda_2 \), the retardation time. The kinematic tensors are those defined in §6.1 and used in the retarded-motion expansion. The convected Jeffreys model contains several other models as special cases:

a. If \( \lambda_2 = 0 \), the model reduces to the "convected Maxwell model."\(^3\) The convected Maxwell model has been widely used for viscoelastic flow calculations, because of its simplicity.

b. If \( \lambda_1 = 0 \), the model simplifies to a second-order fluid with a vanishing second normal stress coefficient (see Eq. 6.2-1 with only the constants \( b_1 \) and \( b_2 \) nonzero).

c. If \( \lambda_1 = \lambda_2 \), the model reduces to a Newtonian fluid with viscosity \( \eta_0 \).

Although we know that the convected Jeffreys model is admissible, we do not know a priori whether or not it describes the kinds of material functions shown in Chapter 3 for typical polymeric liquids.\(^4\) Hence it is very important (as it is for any model) to test it in as many different rheometric experiments as possible.

**EXAMPLE 7.2-1**  Time-Dependent Shearing Flows of the Convected Jeffreys Model

(a) Show how the convected Jeffreys model simplifies for a general time-dependent, simple shearing flow with velocity gradient \( \mathbf{\hat{v}} = \frac{\partial \mathbf{u}}{\partial t} \). Then use this result to evaluate the material functions for (b) steady shear flow, (c) small-amplitude oscillatory shearing flow, (d) start-up of steady shear flow, and (e) stress relaxation following steady shear flow.

**SOLUTION**  (a) We start by writing out the various tensors that enter into the constitutive equation. Appendix C gives the matrix representations for \( \tau_{11} \), \( \tau_{12} \), and \( \tau_{12} \), for an unsteady shearing flow. When these are substituted into the constitutive equation for the convected Jeffreys model we obtain the following matrix equation:

\[
\begin{pmatrix}
\tau_{xx} & \tau_{yx} & 0 \\
\tau_{yx} & \tau_{yy} & 0 \\
0 & 0 & \tau_{zz}
\end{pmatrix} + \lambda_1 \frac{d}{dt} \begin{pmatrix}
\tau_{xx} & \tau_{yx} & 0 \\
\tau_{yx} & \tau_{yy} & 0 \\
0 & 0 & \tau_{zz}
\end{pmatrix} - \begin{pmatrix}
2\tau_{xx} & \tau_{yy} & 0 \\
\tau_{yy} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \lambda_1 \dot{\tau}_{xy}
\]

\[
= -\eta_0 \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \ddot{\gamma}_{xy} + \lambda_2 \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \frac{d^2 \gamma_{xy}}{dt^2} - 2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \lambda_1 \ddot{\gamma}_{xx}
\]

(7.2-2)

From this matrix equation we can now obtain a set of coupled differential equations for the stress tensor components:

\[
(1 + \lambda_1 \frac{d}{dt}) \tau_{xx} - 2\tau_{xx} \lambda_1 \gamma_{xy}(t) = 2\eta_0 \lambda_2 \gamma_{xy}(t)
\]

(7.2-3)

\[
(1 + \lambda_1 \frac{d}{dt}) \tau_{yy} = 0
\]

(7.2-4)

\(^3\) The convected Maxwell model was first obtained from a molecular model by M. S. Green and A. V. Tobolsky, *J. Chem. Phys.* **14**, 80-92 (1946).

\(^4\) The convected Jeffreys model is derived in Chapter 13 from the kinetic theory of dilute solutions of elastic dumbbells.
Differential Constitutive Equations \[ 347 \]

\[(1 + \lambda_1 \frac{d}{dt}) \tau_{xx} = 0 \] \hspace{1cm} (7.2-5)

\[(1 + \lambda_1 \frac{d}{dt}) \tau_{xx} - \tau_{xx} \lambda_1 \beta \gamma_x(t) = -\eta_0 \left(1 + \lambda_2 \frac{d}{dt}\right) \gamma_x(t) \] \hspace{1cm} (7.2-6)

From these equations it can be seen that the normal stresses \(\tau_{yy}\) and \(\tau_{zz}\) are zero for all simple, time-dependent shearing flows. Thus we can drop the dashed underlined term in Eq. 7.2-6. We now use Eqs. 7.2-3 through 6 to calculate specific material functions.

(b) For steady shear flow the differential equations in Eqs. 7.2-3 and 6 simplify to algebraic equations. The two equations for \(\tau_{xx}\) and \(\tau_{yx}\) are easily solved to give the viscometric functions:

\[\tau_{xx} = -\eta_0 \dot{\gamma}_x \quad \text{or} \quad \eta = \eta_0 \]

\[\tau_{yy} = -2\eta_0(\lambda_1 - \lambda_2) \dot{\gamma}_x \quad \text{or} \quad \Psi_1 = 2\eta_0(\lambda_1 - \lambda_2) \]

\[\tau_{yz} = 0 \quad \text{or} \quad \Psi_2 = 0 \] \hspace{1cm} (7.2-7)

The convected Jeffreys model thus gives a constant viscosity and first normal stress coefficient. The second normal stress coefficient is zero.

(c) To get the small amplitude oscillatory shearing properties, we take the shear strain to be \(\gamma_x(0, t) = \gamma_0 \cos \omega t\), in which \(\gamma_0 = \dot{\gamma}_0 / \omega\) is the amplitude of the shear strain, and seek solutions to Eqs. 7.2-3 and 6 in the limit of small \(\gamma_0\). For this flow the differential equation for the shear stress is

\[
(1 + \lambda_1 \frac{d}{dt}) \tau_{xx} = -\eta_0 \gamma_0 \omega (\cos \omega t - \lambda_2 \omega \sin \omega t) \] \hspace{1cm} (7.2-8)

Since we seek a steady periodic solution, the nonhomogeneous part of the first-order, linear, ordinary differential equation suggests we try a solution for \(\tau_{yx}\) of the form

\[\tau_{yx} = A \cos \omega t + B \sin \omega t \] \hspace{1cm} (7.2-9)

By substituting Eq. 7.2-9 into Eq. 7.2-8 we can see that \(A\) and \(B\) must be given by (cf. Eqs. 3.4-3b)

\[A = -\eta_0 \left(1 + \lambda_1 \lambda_2 \omega^2\right) \gamma_0 \omega = -\eta'(\omega) \gamma_0 \omega \] \hspace{1cm} (7.2-10)

\[B = -\eta_0 \left(\lambda_1 - \lambda_2 \omega^2 \right) = -\eta''(\omega) \gamma_0 \omega \] \hspace{1cm} (7.2-11)

Note that \(\eta'\) and \(\eta''\) are the same as for the linear viscoelastic Jeffreys model.

To show that the convected Jeffreys model gives identical predictions to the Jeffreys model in the linear viscoelastic limit is a little more complicated than the calculation of \(\eta'\) and \(\eta''\). It is also necessary to demonstrate that as \(\gamma_0 \to 0\) the normal stresses become negligible compared with the shear stress (see Problem 7B.1). Notice that \(\dot{\gamma}_0\) can be large in the linear viscoelastic limit so that the demand above on \(\gamma_0 \to 0\) is more restrictive than simply requiring \(\tau_{xx}\) to be small compared with \(\tau_{yx}\), as \(\dot{\gamma}_0 \to 0\). The latter is easily seen to hold by inspection of the governing differential equations, Eqs. 7.2-3 and 6.
(d) For start-up of steady shear flow we let $\dot{\gamma}_s = \dot{\gamma}_0 H(t)$, where $H$ is the Heaviside unit step function. The stresses $\tau_{xx}$ and $\tau_{yx}$ are then given by the differential equations

$$
\left(1 + \lambda_1 \frac{d}{dt}\right) \tau_{xx} - 2 \lambda_1 \dot{\gamma}_0 \tau_{yx} = + 2 \eta_0 \lambda_2 \dot{\gamma}_0^2 
$$
(7.2-12)

$$
\left(1 + \lambda_1 \frac{d}{dt}\right) \tau_{yx} = - \eta_0 \dot{\gamma}_0 (1 + \lambda_2 \delta(t))
$$
(7.2-13)

where the Dirac delta function has been introduced as the derivative of the step function $dH/dt = \delta(t)$ (see Eqs. 5.2-10a through c). Equation 7.2-13 is then multiplied by the integrating factor $e^{\lambda_2 t}$ and integrated from $t = 0^-$ (where $\tau_{yx} = 0$) to an arbitrary time $t > 0$ to give (cf. Table 3.4-1)

$$
\tau_{xx} = -\eta_0 \dot{\gamma}_0 \left[ \frac{\lambda_2}{\lambda_1} - \frac{\lambda_2}{\lambda_1} \frac{1}{1 - e^{-\left(\lambda_1 \right) t}} \right]
$$
(7.2-14)

When this last result is combined with Eq. 7.2-12 we obtain for $\tau_{xx}$

$$
\tau_{xx} = -2 \eta_0 (\lambda_1 - \lambda_2) \dot{\gamma}_0^2 \left[ 1 - \frac{t}{\lambda_1} e^{-\left(\lambda_1 \right) t} \right]
$$
(7.2-15)

From Eqs. 7.2-14 and 15, it can be seen that the shear stress and first normal stress growth functions do not depend on shear rate as is found for the polymer solution and melt data in Chapter 3. Moreover, the stresses grow monotonically to the steady-state values, so that the convected Jeffreys model does not predict the stress overshoot observed experimentally in polymeric liquids. Finally, there is a jump in the shear stress at $t = 0$, following which the shear stress grows monotonically to steady state. This sudden jump is associated with the retardation time $\lambda_2$ in the model, but does not seem to be observed experimentally. However, as a result of the jump, the shear stress is predicted to grow more rapidly than the normal stress, in qualitative agreement with data.

(e) For stress relaxation following steady shear flow we take the shear rate to be $\dot{\gamma}(t) = \dot{\gamma}_0 (1 - H(t))$. When this is inserted into Eq. 7.2-6 we obtain the following differential equation for the shear stress:

$$
\left(1 + \lambda_1 \frac{d}{dt}\right) \tau_{xy} = \eta_0 \lambda_2 \dot{\gamma}_0 \delta(t)
$$
(7.2-16)

This equation is integrated from $t = 0^-$ (where the shear stress is the steady-state value $-\eta_0 \dot{\gamma}_0$) to time $t > 0$ to give the shear stress relaxation function (cf. Table 3.4-1)

$$
\tau_{xx} = -\eta_0 \dot{\gamma}_0 \left[ 1 - \frac{\lambda_2}{\lambda_1} \right] e^{-\left(\lambda_1 \right) t}
$$
(7.2-17)

To find the normal stress relaxation we note from Eq. 7.2-3 that for $t \geq 0$ when the shear rate is zero, the normal stress $\tau_{xx}$ relaxes like $e^{-\lambda_1 t}$ from its steady-state shear flow value. Thus,

$$
\tau_{xx} = -2 \eta_0 (\lambda_1 - \lambda_2) \dot{\gamma}_0^2 e^{-\left(\lambda_1 \right) t}
$$
(7.2-18)
As we found for the stress growth material functions, the rate of stress relaxation does not depend on the shear rate. In addition, there is again a discontinuity in the shear stress at \( t = 0 \), whereas the normal stresses are continuous. The jump in \( \tau_{xx} \) results in the fact that the shear stress relaxes to zero more rapidly than the normal stress, in qualitative agreement with experiments.

**EXAMPLE 7.2-2**  
Time-Dependent Shearfree Flows of the Convedt Jeffreys Model

(a) Show how the convected Jeffreys model simplifies for an arbitrary, time-dependent shearfree flow. (b) Then use this result to find the model response to start-up of steady planar and elongational flow. (c) What are the steady-state material functions for the convected Jeffreys model in these flows?

**SOLUTION**  
(a) We again turn to Appendix C where the various tensors in the convected Jeffreys model are displayed in matrix form for shearfree flows. When these are inserted into the constitutive equation, Eq. 7.2-1, we obtain the following uncoupled, ordinary differential equations for the stress components:

\[
\tau_{xx} + \lambda_1 \frac{d\tau_{xx}}{dt} + \lambda_1 (1 + b) \tau_{xx} \delta(t) = +\eta_0 \left[ (1 + b)\dot{\varepsilon}[1 + (1 + b)\lambda_2 \dot{\varepsilon}] + \lambda_2 (1 + b) \frac{d\dot{\varepsilon}}{dt} \right] \tag{7.2-19}
\]

\[
\tau_{yy} + \lambda_1 \frac{d\tau_{yy}}{dt} - 2\lambda_1 \tau_{xx} \delta(t) = -\eta_0 \left[ 2\dot{\varepsilon}[1 - 2\lambda_2 \dot{\varepsilon}] + 2\lambda_2 \frac{d\dot{\varepsilon}}{dt} \right] \tag{7.2-20}
\]

The equation for \( \tau_{yy} \) is the same as for \( \tau_{xx} \) except that \( +b \) is replaced by \( -b \).

(b) In start-up of steady shearfree flow, the elongation rate is given by the Heaviside unit step function \( \delta(t) = \delta_0 H(t) \) where \( \delta_0 \) is constant. When the differential equation for \( \tau_{xx} \), for example, is specialized to this strain rate function we obtain

\[
\left( 1 + (1 + b)\lambda_1 \delta_0 + \lambda_1 \frac{d}{dt} \right) \tau_{xx} = \eta_0 (1 + b)\delta_0 \left[ 1 + (1 + b)\lambda_2 \delta_0 + \lambda_2 \delta(t) \right] \tag{7.2-21}
\]

which is to be solved subject to the initial condition that \( \tau_{xx}(0^-) = 0 \). Integrating Eq. 7.2-21 from \( t = 0^- \) to \( t \) gives

\[
\tau_{xx} = \eta_0 (1 + b)\delta_0 \frac{1 + (1 + b)\lambda_2 \delta_0}{1 + (1 + b)\lambda_1 \delta_0} - \eta_0 \left( 1 - \frac{\lambda_2}{\lambda_1} \right) \frac{(1 + b)\delta_0 e^{-\eta_1 (1 + b)\lambda_1 \delta_0}}{1 + (1 + b)\lambda_1 \delta_0} \tag{7.2-22}
\]

Similarly we find that \( \tau_{yy} \) is identical to \( \tau_{xx} \) with \( b \) replaced by \( -b \) and \( \tau_{zz} \) is given by

\[
\tau_{zz} = -2\eta_0 \delta_0 \frac{1 - 2\lambda_2 \delta_0}{1 - 2\lambda_1 \delta_0} + 2\eta_0 \left( 1 - \frac{\lambda_2}{\lambda_1} \right) \frac{\delta_0 e^{-\eta_1 (1 + 2\lambda_1 \delta_0)}}{1 - 2\lambda_1 \delta_0} \tag{7.2-23}
\]

An interesting result of the above formulas is that no steady state is attained if

\[
\lambda_1 \delta_0 > \frac{1}{2} \quad \text{or} \quad \lambda_1 \delta_0 < -\frac{1}{2} \quad \text{or} \quad \lambda_1 \delta_0 < -\frac{1}{2} \tag{7.2-24}
\]

Since we specify shearfree flows with \( b \) in the range \( 0 \leq b \leq 1 \), the first two of these conditions determine which shearfree flows of the convected Jeffreys model will approach a steady state in the start-up experiment.
For $-1/(1 + b) < \dot{\lambda}_1 \dot{\varepsilon}_0 < 1/2$ we find the stress growth material functions (Table 3.5-1):

\[
\eta^1 = -\frac{\tau_{yy} - \tau_{xx}}{\dot{\varepsilon}_0} = (3 + b)\eta_0 \frac{\dot{\lambda}_2}{\dot{\lambda}_1} + \frac{(3 + b)\eta_0(1 - (\dot{\lambda}_2/\dot{\lambda}_1))}{(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)(1 - 2\dot{\lambda}_1 \dot{\varepsilon}_0)} \nonumber
\]

\[
- \eta_0(1 + b)\left(1 - \frac{\dot{\lambda}_2}{\dot{\lambda}_1}\right) e^{-n(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)/\dot{\lambda}_1} \nonumber
\]

\[
- 2\eta_0\left(1 - \frac{\dot{\lambda}_2}{\dot{\lambda}_1}\right) e^{-n(1 - 2\dot{\lambda}_1 \dot{\varepsilon}_0)/\dot{\lambda}_1} \tag{7.2-25}
\]

\[
\eta^2 = -\frac{\tau_{yy} - \tau_{xx}}{\dot{\varepsilon}_0} = 2b\eta_0 \frac{\dot{\lambda}_2}{\dot{\lambda}_1} + \frac{2b\eta_0(1 - (\dot{\lambda}_2/\dot{\lambda}_1))}{(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)(1 + (1 - b)\dot{\lambda}_1 \dot{\varepsilon}_0)} \nonumber
\]

\[
- \eta_0(1 + b)\left(1 - \frac{\dot{\lambda}_2}{\dot{\lambda}_1}\right) e^{-n(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)/\dot{\lambda}_1} \nonumber
\]

\[
+ \eta_0(1 - b)\left(1 - \frac{\dot{\lambda}_2}{\dot{\lambda}_1}\right) e^{-n(1 - (1 - b)\dot{\lambda}_1 \dot{\varepsilon}_0)/\dot{\lambda}_1} \tag{7.2-26}
\]

(c) For steady-state shearfree flows, the governing differential equations, Eqs. 7.2-19 and 20, reduce to algebraic equations that are easily solved for the stresses. These are then combined with the material function definitions to give for any $\dot{\varepsilon}_0$:

\[
\bar{\eta}_1 = (3 + b)\eta_0 \frac{\dot{\lambda}_2}{\dot{\lambda}_1} + \frac{(3 + b)\eta_0(1 - (\dot{\lambda}_2/\dot{\lambda}_1))}{(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)(1 - 2\dot{\lambda}_1 \dot{\varepsilon}_0)} \tag{7.2-27}
\]

\[
\bar{\eta}_2 = 2b\eta_0 \frac{\dot{\lambda}_2}{\dot{\lambda}_1} + \frac{2b\eta_0(1 - (\dot{\lambda}_2/\dot{\lambda}_1))}{(1 + (1 + b)\dot{\lambda}_1 \dot{\varepsilon}_0)(1 + (1 - b)\dot{\lambda}_1 \dot{\varepsilon}_0)} \tag{7.2-28}
\]

Note that the steady material functions become infinite at the critical strain rates given by Eq. 7.2-24.

§7.3 NONLINEAR DIFFERENTIAL MODELS

In the preceding section we wrote down an admissible constitutive equation that is capable of describing time-dependent flows. However, the model is not able to portray well the rheological properties that are observed in typical polymer solutions and melts. For example, we noted previously the deficiencies of constant viscosity and normal stress coefficients in steady shear flow and the infinite elongational viscosity at finite elongation rates. In this section we illustrate some of the methods that have been used to generate constitutive equations that more accurately represent real material data. In each of the approaches, it is necessary to give up the starting point of a linear model that we used in §7.2.