Instructions: Test duration 40 minutes, closed book, 0.5 letter-sized formula sheet allowed

1. Answer all questions.
2. Write your answer on the same sheet as the question, or on the back if needed.
3. Put your name and student number on the top of each sheet.
4. Please write neatly and structure your answers in a clear way

Good luck

Question 1. (12 marks) (Show sufficient steps to justify your answers)
Heat flow along a thin insulated wire of radius $R(x) = R_0 e^{x/2}$, is governed by the following IBVP

$$u_t = \frac{\alpha}{[R(x)]^2} \frac{\partial}{\partial x}\left([R(x)]^2 \frac{\partial u}{\partial x}\right), \quad 0 < x < L$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

a) (3 marks) Use separation of variables $u(x,t) = X(x)T(t)$ to derive a Sturm-Liouville problem for $X(x)$, i.e. state the Sturm-Liouville problem and its boundary conditions.

$$\left[ T' = \frac{\alpha}{e^x} \frac{\partial}{\partial x} \left( e^x X' \right) \right] \Rightarrow \frac{1}{\alpha X T} = \frac{1}{R_0^2 \frac{\partial}{\partial x}} \left( \frac{\partial^2}{\partial x^2} \left( e^x \frac{\partial}{\partial x} \right) \right)$$

$$\frac{1}{\alpha} \frac{T'}{T} = X_e^x \left( e^x X' \right)' = -\lambda$$

function of $t$ function of $x$

$$\Rightarrow -X'' + \lambda X = 0$$

$$X(0) = X(L) = 0$$

This is a S.L. problem with $p(x) = e^x$, $q(x) = 0$, $r(x) = e^x$

$$\alpha_1 = 1, \quad \alpha_2 = 0$$

$$\beta_1 = 1, \quad \beta_2 = 0$$
b) (4 marks) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem.

We did this one in class. On differentiating we find
\[ e^x X'' + e^x X' + \lambda e^x X = 0 \]
which is a linear equation with constant coefficients. Therefore,
\[ r^2 + r + \lambda = 0 \]

Roots:
\[ r = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \]
Assume \( \lambda > \frac{1}{4} \) \Rightarrow \( r = -\frac{1}{2} \pm i\sqrt{\lambda - \frac{1}{4}} \)

\( X(x) = Ae^{-x/2} \cos(\sqrt{\lambda - \frac{1}{4}} x) + Be^{-x/2} \sin(\sqrt{\lambda - \frac{1}{4}} x) \)

\( X(0) = 0 \Rightarrow A = 0 \)
\[ \lambda_n = \frac{1}{4} + \left( \frac{n\pi}{L} \right)^2 \quad n = 1, 2, \ldots \]

\( X_n(x) = e^{-x/2} \sin \left( \frac{n\pi x}{L} \right) \) are eigenfunctions.

c) (3 marks) Using the results of a&b, write the series form of the solution to the IBVP and specify how you would compute the coefficients in terms of the initial condition: \( u(x,0) = f(x) \)

Now find \( \Gamma_n(t) = e^{-x \lambda_n t} = e^{-\frac{x^2}{4} t} + \left( \frac{n\pi}{L} \right)^2 t \)

\[ \Rightarrow u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{x^2}{4} t} \sin \left( \frac{n\pi x}{L} \right) \]

To compute \( C_n \) we use the initial condition
\[ f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n e^{-x/2} \sin \left( \frac{n\pi x}{L} \right) \]

To find \( C_j \) we multiply by \( [r(x) X_j(x)] \) and integrate over \([0,L]\):
\[ \int_0^L f(x) r(x) X_j(x) \, dx = \int_0^L f(x) e^{-x/2} \sin \left( \frac{n\pi x}{L} \right) \, dx \]

\[ \Rightarrow C_j = \frac{\int_0^L f(x) e^{-x/2} \sin \left( \frac{n\pi x}{L} \right) \, dx}{\int_0^L e^{-x/2} \sin \left( \frac{n\pi x}{L} \right) \, dx} \]

from orthogonality.
d) (2 marks) Suppose that: \( u(x,0) = f(x) = 2e^{-x^2/2} \sin \left( \frac{3\pi}{L} x \right) \). Find the solution to the IBVP

\[
\text{if } f(x) \text{ is given, we see that } f(x) = 2X_3(x) \\
\therefore \quad u(x,t) = 2e^{-x^2/2} \sin \left( \frac{3\pi}{L} x \right) e^{-\alpha t} \left[ \sum \frac{(3\pi/L)^n}{n!} \right] t
\]

Question 2. (8 marks) (Show sufficient steps to justify your answers)
Solve the following BVP for Laplace’s equation in a quarter annulus:

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \quad r \in (1,2), \theta \in (0, \pi/2)
\]

\[
u(1, \theta) = -2 \sin \theta \\
u(2, \theta) = \sin 3 \theta \\
u(r, 0) = 0 \\
u_\theta (r, \pi/2) = 0
\]

Look for \( u(r, \theta) = R(r) \Theta(\theta) \)

\[
\frac{r^2}{2R} \left[ R'' \Theta + \frac{1}{r} R' \Theta \right] = -\frac{1}{r^2} R \Theta'' \\
\frac{r^2 R''}{R} + \frac{r R'}{R} = -\Theta'' = \lambda = \text{const}
\]

2 conditions in \( \Theta \): \( \Theta(0) = 0 \quad \Theta(\pi/2) = 0 \)

\( \Theta'' + \lambda \Theta = 0 \quad \Rightarrow \quad \Theta = A \cos \sqrt{\lambda} \theta + B \sin \sqrt{\lambda} \theta \)

\( 0 = \Theta(0) = A \quad \Rightarrow \quad \Theta(\pi/2) = B \sqrt{\lambda} \cos \sqrt{\lambda} \pi/2 \)

\( 0 = \Theta(0) = A \quad \Rightarrow \quad \Theta(\pi/2) = B \sqrt{\lambda} \cos \sqrt{\lambda} \pi/2 \)

\( \Rightarrow \sqrt{\lambda} \pi/2 = (n-\frac{1}{2})\pi \quad \Rightarrow \lambda = (2n-1)^2 \quad n=1,2,3,\ldots \)
\[ e^n \Theta_n(\theta) = \sin (2n-1)\theta \quad n = 1, 2, 3, \ldots \]

\[ r^2 R''_n + r R'_n - \lambda R_n = 0 \quad \text{try} \quad R_n = r^k \]

\[ A \left[ r^2 (k(k-1)) r^{k-2} + rk r^{k-1} - \lambda r^k \right] = 0 \]

\[ \Rightarrow k^2 = \lambda_n = (2n-1)^2 \quad \Rightarrow k = \pm (2n-1) \]

\[ \therefore \text{in general} \quad R_n(r) = A r^{(2n-1)} + B r^{-(2n-1)} \]

\[ u(r, \theta) = \sum_{n=1}^{\infty} \left[ a_n r^{(2n-1)} + b_n r^{1-2n} \right] \sin (2n-1)\theta \]

\[ u(1, \theta) = -2 \sin \theta \Rightarrow a_1 + b_1 = -2 \quad a_1^2 + b_1^2 = 0 \]

\[ u(2, \theta) = \sin 3\theta \Rightarrow a_2 + b_2 = 0 \quad a_2^2 + b_2^2 = 1 \]

\[ a_k, b_k = 0 \quad \text{otherwise} \]

\[ \Rightarrow b_1 = -4a_1 \quad b_2 = -a_2 \quad a_2 \left( 8 - \frac{1}{2} \right) = 1 \]

\[ a_1 = \frac{2}{3} \quad b_1 = -\frac{8}{3} \]

\[ a_2 = 64/63 \]

\[ u(r, \theta) = \left( \frac{2}{3} - \frac{8}{3} \right) \sin \theta + \frac{64}{63} \left[ r^3 - r^{-3} \right] \sin 3\theta \]

The End