Yield Stress Fluid Models
What is a yield stress fluid

Examples of fluids with a yield stress

- Liquid chocolate
- Fruit puree
- Some paints
- Fuel gels
- Toothpaste
- Dairy products
- Sewage
- Polymer gels
- Hagfish mucus
- Basaltic lavas
- Hair gel
- Drilling fluids
- Mud
- Diaper cream
Characteristics

• Continuum mechanics: a fluid is a continuum that cannot resist a shear stress while at rest
  – In generalised Newtonian fluids, shear stresses are defined by the flow curve ⇒ effective viscosity $\eta(\dot{\gamma})$
    \[\sigma = -pI + \tau, \quad \tau = \eta(\dot{\gamma})\dot{\gamma}\]
  – At rest means $\dot{\gamma} = 0$, so unless $\eta(\dot{\gamma}) \to \infty$ the fluid cannot generate finite shear stresses at rest

• Yield stress fluid can resist a shear stress while at rest: characterised by limiting yield stress $\tau_0$
  – In other words, at zero strain rate the shear stress can be finite, and this is achieved mathematically by: $\eta(\dot{\gamma}) \to \infty$
  – Are these then really fluids, according to the usual definition?
  – Should they be classified as generalised Newtonian fluids
How do we model:

Real life

http://www.aubingroup.com/products/drilling-fluids

Idealised

- Only rigid body motion below yield stress $\tau_0$
- Shear-thinning viscous behaviour above $\tau_0$

Common yield stress models

- Features to fit:
  - Yield stress
  - High shear viscosity
  - Degree of shear-thinning
  - “Location” of transition

- Bingham:
  \[
  \tau = \tau_0 + \dot{\gamma}, \quad \eta = \eta_\infty + \frac{\tau_0}{\dot{\gamma}}
  \]

- Casson:
  \[
  \tau = \eta_\infty \dot{\gamma} + 2\sqrt{\eta_\infty \tau_0 + \frac{\tau_0}{\dot{\gamma}}}, \quad \eta = \eta_\infty + 2\sqrt{\eta_\infty \dot{\gamma} \tau_0} + \tau_0
  \]

- Herschel-Bulkley:
  \[
  \tau = \kappa \dot{\gamma}^{n-1} + \tau_0, \quad \eta = \kappa \dot{\gamma}^n
  \]

- Roberts (?):
  \[
  \tau = \frac{\eta_0 \dot{\gamma}}{1 + \lambda \dot{\gamma}} + \kappa \dot{\gamma}^n, \quad \eta = \frac{\eta_0 \dot{\gamma}}{1 + \lambda \dot{\gamma}^{n-1}}
  \]
Above relationships are for: $\tau > \tau_0$

- General tensorial constitutive laws:
  \[
  \tau_{ij} = \eta(\dot{\gamma})\dot{\gamma}_{ij} \quad \Leftrightarrow \quad \tau > \tau_0 \\
  \dot{\gamma}_{ij} = 0 \quad \Leftrightarrow \quad \tau \leq \tau_0
  \]

- E.g. Bingham
  \[
  \tau_{ij} = \left(\eta_\infty + \frac{\tau_0}{\dot{\gamma}}\right)\dot{\gamma}_{ij} \quad \Leftrightarrow \quad \tau > \tau_0 \\
  \dot{\gamma}_{ij} = 0 \quad \Leftrightarrow \quad \tau \leq \tau_0
  \]

- For $\tau \leq \tau_0$ the deviatoric stress is indeterminate
  - What does this mean?
  - Why is it indeterminate?

- Is this a problem?
  - Strain rate is determinate, e.g.
  - If your primary variables are $(u,p)$, this is fine in principle
Plane Couette flow (simple shear)

- Fluid element at the time $t + dt$
- Shearing surfaces

$Irgens$ 2013
Cylindrical coordinates

\[ \nabla \cdot \sigma = \begin{pmatrix} -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \tau_{rr} \right] + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{rz}}{\partial r} - \frac{\tau_{\theta\theta}}{r} \\ -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \tau_{\theta r} \right] + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial r} \\ -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \tau_{z r} \right] + \frac{1}{r} \frac{\partial \tau_{z\theta}}{\partial r} + \frac{\partial \tau_{zz}}{\partial r} \end{pmatrix} \]

\[ u = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \dot{\gamma}_{rr} = 2 \frac{\partial u}{\partial r}, \quad \dot{\gamma}_{\theta\theta} = 2 \frac{\partial v}{\partial r} + \frac{2u}{r}, \quad \dot{\gamma}_{zz} = 2 \frac{\partial w}{\partial z} \]

\[ \dot{\gamma}_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} = \dot{\gamma}_{\theta r}, \quad \dot{\gamma}_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial r} = \dot{\gamma}_{z\theta}, \quad \dot{\gamma}_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = \dot{\gamma}_{rz}, \]
Pipe flow
Steady torsion flow

(a) Rotating disk
(b) Line of shear
(c) Shearing surface

Irgens 2013
Cylindrical Couette flow
Comments

• Many simple viscometric flows can be solved
  – Not all easy, e.g. try Poiseuille flow along an annulus for a Bingham fluid

• General advice, for any 1D/axisymmetric problem:
  – Integrate the momentum equation(s) to define the stress components & use general physical insight into the flow
  – Then input the constitutive law & integrate (maybe numerically) for the velocity
    ▪ Note that the yield stress is the key phenomenon here, not the different forms of shear-thinning
Eugene Bingham: 1878-1945

- 1916-1945 Professor of Chemistry at Lafayette College, US
  - Earlier career?
  - Bureau of standards
- Influential book 1922: “Fluidity and Plasticity”
- Credited with inventing term “rheology”
- Bingham fluid
- Bingham Medal
  - Premier award of the American Society of Rheology
AN INVESTIGATION OF THE LAWS OF PLASTIC FLOW

By Eugene C. Bingham

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FIG. 1.—The viscometer used for very viscous or plastic materials
Unlike viscous fluids Bingham records a "friction constant" (a stress) that must be exceeded and thereafter a linear relation.

**Fig. 8.**—The results of experiments at $25^\circ$ C with Capillaries No. 1, No. 2, No. 3, and No. 6.1 with 50 per cent by weight of clay suspended in pure water are here plotted. For medium pressures the curves are linear and intersect each other at a point on the pressure axis which is widely removed from the origin. Comparing this with Fig. 2, we see that this affords a sharp distinction between viscous and plastic flow. Capillary No. 1 shows that at high pressure the flow suddenly increases, which is believed to be due to slipping. Capillary No. 2, shows that there may be an increased flow at low pressures, which is probably due to seepages.

**Fig. 9.**—The results of experiments at $25^\circ$ C with Capillaries No. 1, No. 6.1, No. 6.2, and No. 6.3 with 50 per cent by weight of clay suspended in a 0.1 per cent solution of potassium carbonate, and with Capillary No. 1 at $40^\circ$ C, are here plotted. The friction constant is less and the mobility far greater than in the neutral suspension. Rise in temperature increases the mobility. With Capillary No. 6.3 the results are irregular at high pressures.
Yield stress phenomena 1:

- Things get stuck, e.g. in corners
  - Easiest to observe in antiplane shear flows
  - Seminal work by Mosolov & Miasnikov 1965-1967

- Also have “plugs” in the interior

Huilgol (2006), JNNFM

Treskatis et al. 2015

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Fig. 5. Determination of the yield surface for a pipe of equilateral triangular cross-section.
Yield stress phenomena 2:

Things get stuck in the fluid

Chocolate chips do not sink to bottom of the cookies!

Static bubbles in blue hair gel (Carbopol)

Shapes not unique

Yield stress phenomena 3:

Critical stress needed to mobilize any flow

Carbopol solution, stationary in my office for >2 weeks

Simplest estimate is balance between $\Delta \rho g D$ & $\tau_0$

Simulations of flow onset through a 2D porous media filled with yield stress fluid, on increasing $\Delta P$

Talon & Bauer 2013

Fig. 5. Flow field example inside the porous media (solid sites are darker) for different applied pressure drop $\Delta P$. The numerical parameters are $\phi = 0.75$, $L_x = 1024 \delta x$, $\lambda = 6 \delta x$, $m = 10^9$, $A = 0.2$, $\nu_0 = 10^{-3}$ and $\tau_0 = 10^{-5}$. 
Yield stress phenomena 4:

Transient flows stop in a finite time

- Navier Stokes + Energy

\[
\frac{Ra}{Pr} \frac{\partial u_i}{\partial t} + \frac{Ra}{Pr} u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + T \delta_{i2},
\]

\[
\frac{\partial u_i}{\partial x_i} = 0,
\]

\[
Ra \frac{\partial T}{\partial t} + Ra u_i \frac{\partial T}{\partial x_i} = \frac{\partial T}{\partial x_i \partial x_i},
\]

- 4 dimensionless groups

\[
Ra = \frac{\hat{g} \hat{\beta} \Delta \hat{T} \hat{L}^3}{\hat{\nu} \hat{\kappa}} \quad B = \frac{\hat{\tau}_y}{\hat{\rho_0} \hat{g} \hat{\beta} \Delta \hat{T} \hat{L}}
\]

Transient flows stop in a finite time:

Flow control via yield stress, $B$

- $Pr = 1, Ra = 10^6$
- $t=0$: Newtonian convection ($B=0$)
- $t>0$: Increase yield stress: $B > B_{cr}$
  - Flow stops
- $t>t_1$: Decrease yield stress to:
  $B = 0.015 < B_{cr} = 1/32$
  - Flow restarts

The yield stress debate

- There is no yield stress
  - Everything flows if we wait long enough
  - Practical engineering timescales

- Very viscous vs yield?
  - Very viscous is certainly convenient for (lazy) computational engineers
    - Are the answers correct?!

- Issues about rheometry
  - Small strain rate ranges
  - Measuring the yield stress
    - how?

- Is there a mathematical problem with e.g. Bingham
  - Singularity?
  - Well-posedness, etc
  - Computing

- Complex fluids where yield stress is inherent in microstructure
  - Colloidal suspensions
  - Dense granular suspensions
  - Foams, some emulsions

- Static & dynamic yield stress. Avalanche phenomena
  - Is yield stress a dynamic property?
  - Is it a consequence of thixotropy?

- The elastic connection
  - Yielding and elasto-plastic transition
  - Models & rheometry

- The psychology of physicists
  - “There are only X types of YSF”
  - G.U.T. issues...
When should GNF’s be used

- Empirical fits, so only within limits of the fitting
- One interpretation of GNF’s is in the context of a viscoelastic fluid, i.e. only the viscous part of the deviatoric stress is captured in

\[ \mathbf{\sigma} = -p \mathbf{I} + \mathbf{\tau}, \quad \mathbf{\tau} = \eta(\dot{\gamma}) \dot{\gamma} \]

- How big are elastic effects?
- How do we know – many simple flows e.g. pipe flow & pressure drop are not directly affected by elasticity

- If GNF properties are measured in steady simple shear, can these be used in flows that are multi-dimensional, with imposed timescales and internal microstructural timescales?
  - Need to compare timescale of fluid with timescale of the flow
Viscoelastic fluids I
Approaches

Phenomenology/rheometry
• Key viscoelastic phenomena, e.g.
  – Rod climbing, die swell, free surface deformations, hole pressure effect, elastic recoil, tubeless siphon, relaxation after a transient, secondary flows (negative wakes etc)
• Design simple experiment to observe
  – Understand elastic contributions by reference to inelastic flows (e.g. GNFs)
  – Use simple experiment and measurements as a form of rheometry

Constitutive modelling
• Macroscopic methods
  – Derived from mechanical analogues
  – Extended with increasing degrees of complexity
• Continuum mechanics
• Micro-mechanical
  – With/without meso/macro upscaling
• Deterministic vs statistical
• Is the underlying issue a closure problem?
  – Models should be able to represent observed phenomena
  – Additional mathematical issues with some constitutive models
Announcement of the grand opening of

RHEOLOGY DRUGSTORE

Our motto: “Fit The Data”
Proprietor: Daniel D. Joseph

“To make your experiment agree with your theory you should have the right fluids.”

We carry many different fluids, corresponding to the thirty or forty models currently considered most realistic.

Standard brandname Fluids (well advertised):

Maxwell        Curtiss-Bird        Johnson-Segalman
Jeffreys       White-Metzner       Lodge’s
BKZ           Phan Thien-Tanner      Green-Tobolsky
KBKZ         Newtonian             Oldroyd
Dool-Edward        Reiner-Rivlin     Giesekus

Graded Fluids:                           Composite Fluids:

Single integral                         With Springs and
Multiple order integral                    Dumbbells
1st, 2nd, 3rd order, etc.                With Beads and Chains
Fluids of complexity 1, 2, 3, etc.        With Reptating Snakes

Retarded fluids with a strong backbone and fading memory

Mathematician’s Delight:

Models with 1, 2, or 3 Fréchet derivatives
Less good fluids with only 1, 2, or 3 Gateaux derivatives

Less expensive fluids:

Liquid gold
Milky Way dust
Water with c=1 cm/sec
Rod climbing – mechanism?

- Flow is observed to be steady & axisymmetric with velocity $v(r)$ in azimuthal direction

  - Consider a deep container, to neglect $z$-gradients

$$\frac{-\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \tau_{rr} \right] + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} - \tau_{\theta\theta} \approx 0$$

$$0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \tau_{\theta r} \right] + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \approx 0$$

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \tau_{zr} \right] + \frac{1}{r} \frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \approx 0$$
Newtonian fluids?

- Consider distribution of $p$ along a horizontal plane immersed in fluid
  - Solve $\theta$ equation with boundary conditions to give azimuthal $v(r)$
  - Integrate r equations:
    $$p(r,z) = \int_{r_i}^r \frac{\rho v(\tilde{r})^2}{\tilde{r}} d\tilde{r} + k(z)$$

- Pressure increases with $r$
  - Fluid rises near wall to compensate via static pressure
  - Constant pressure at surface, might be used to find shape

$$- \frac{\rho v^2}{r} = - \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r^2 \tau_{rr}) - \frac{\tau_{\theta\theta}}{r}$$

$$0 = \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \tau_{\theta r}]$$

$$0 = - \frac{\partial p}{\partial z} - \rho g$$
Viscoelastic fluid

• Consider distribution of $p-\tau_{zz}$ along a horizontal plane immersed in fluid
  
  — Use radial momentum equation:
  
  $$
  \frac{\partial}{\partial r} [p - \tau_{zz}] = -\frac{\partial \tau_{zz}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} [r \tau_{rr}] - \frac{\tau_{\theta\theta}}{r} + \frac{\rho v^2}{r}
  $$
  
  $$
  = -\frac{[\tau_{\theta\theta} - \tau_{rr}]}{r} + \frac{\partial}{\partial r} [\tau_{rr} - \tau_{zz}] + \frac{\rho v^2}{r}
  $$

  — Distribution of $p-\tau_{zz}$ is due to varying normal stress differences with $r$

  — Change of variables (exercise)

  $$
  \frac{\partial}{\partial \ln r} [p - \tau_{zz}] = \rho v^2 - [\tau_{\theta\theta} - \tau_{rr}] - 2\tau_{\theta r} \frac{\partial}{\partial \tau_{\theta r}} [\tau_{rr} - \tau_{zz}]
  $$

  $$
  - \frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} [r \tau_{rr}] - \frac{\tau_{\theta\theta}}{r}
  $$

  $$
  0 = \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \tau_{\theta r}]
  $$

  $$
  0 = -\frac{\partial p}{\partial z} - \rho g
  $$

(a) Newtonian liquid (b) Polymeric fluid
\[
\frac{\partial}{\partial \ln r} [p - \tau_{zz}] = \rho v^2 - [\tau_{\theta\theta} - \tau_{rr}] - 2\tau_{\theta r} \frac{\partial}{\partial \tau_{\theta r}} [\tau_{rr} - \tau_{zz}]
\]

- \([\tau_{\theta\theta} - \tau_{rr}]\) is called \(1^{\text{st}}\) normal stress difference \(N_1\)
  - \(1^{\text{st}}\) implies \(\tau_{\theta\theta}\) in direction of flow minus \(\tau_{rr}\) in direction of shear

- \([\tau_{rr} - \tau_{zz}]\) is called \(2^{\text{nd}}\) normal stress difference \(N_2\)
  - \(2^{\text{nd}}\) implies \(\tau_{rr}\) in direction of shear minus \(\tau_{zz}\) in direction of orthogonal to shear plane

- If \(N_1 > 0\) it can counteract effects of inertia so that \(p - \tau_{zz}\) decreases with \(r\)
  - Fluid is pushed up near the centre
  - Effects of \(N_2\) are more complex
  - Suggestion that \(N_2\) depending on shear stress

(a) Newtonian liquid
(b) Polymeric fluid
2\textsuperscript{nd} Problem: free surface deformation

- Inclined open channel
  - Flow in x-direction
  - Surface bulges upwards

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \cos \beta \\
0 &= -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\
0 &= -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \sin \beta
\end{align*}
\]
Assumptions and analysis
- $\tau_{ij}$ depend only on $y$, if channel is deep
- $|\tau_{zy}| \ll |\tau_{xy}|$
- $p$ is independent of $x$

\[
\tau_{xy} = -y \rho g \cos \beta
\]

\[
\begin{align*}
\frac{\partial}{\partial y} [p - \tau_{zz}] &= \frac{\partial}{\partial y} [\tau_{yy} - \tau_{zz}] \\
\frac{\partial}{\partial z} [p - \tau_{zz}] &\approx \frac{\partial p}{\partial z} = -\rho g \sin \beta
\end{align*}
\]

- At free surface $z = z_i(y)$, $p - \tau_{zz}$ balances ambient pressure:

\[
p_a = [\tau_{yy} - \tau_{zz}](y) - \rho g z_i(y) \sin \beta + C
\]

\[
z_i(y) = \frac{[\tau_{yy} - \tau_{zz}](y) + C - p_a}{\rho g \sin \beta}
\]
2nd Problem: free surface deformation

- Again see that normal stress difference is responsible for free surface variation
  - \( \tau_{yy} - \tau_{zz} = N_2(y) \) = 2nd normal stress difference
  - At centre, expect \( \tau_{yy} - \tau_{zz} = 0 \), as velocity gradients vanish
  - At wall, \( z_i = 0 \)
- Positive bulge implies negative \( N_{2,wall} \)
  - Note that with no shear, there is no bulging
  - Amount of shear at the wall:
    \[
    \tau_{xy} = -y \rho g \cos \beta
    \]
  - Measuring \( z_i(0) \) provides a way of estimating \( N_2(\tau_{xy}) \)

\[
p_a = \left[ \tau_{yy} - \tau_{zz} \right](y) - \rho g z_i(y) \sin \beta + C
\]
\[
z_i(y) = \frac{N_2(y) + C - p_a}{\rho g \sin \beta}
\]
\[
N_{2,wall} = p_a - C
\]
\[
z_i(0) = -\frac{N_{2,wall}}{\rho g \sin \beta}
\]
Viscometric functions

- First viscometric function is the effective viscosity $\eta$
- Two other viscometric functions are used to characterize normal stress differences:
  - Normal stress differences generated by shear:
    \[ \tau_{11} - \tau_{22} = N_1(\dot{\gamma}) = \psi_1(\dot{\gamma})\dot{\gamma}^2 \]
    \[ \tau_{22} - \tau_{33} = N_2(\dot{\gamma}) = \psi_2(\dot{\gamma})\dot{\gamma}^2 \]
  - $\psi_1$ = primary normal stress coefficient
  - $\psi_2$ = secondary normal stress coefficient
  - Even functions of the shear rate
  - For polymeric liquids (and more generally) we find $\psi_1 > 0$
  - For polymeric liquids usually $\psi_2 < 0$, and is significantly smaller than $\psi_1$

Source: Thermopedia
$N_1 = \text{part of standard suite of rheometric tests and characterizations for general complex fluids}$

**Example:** CTAB, wormlike micellar solution

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**FIG. 1.** Linear viscoelastic moduli for 16.7 wt % CTAB in D$_2$O at 32 °C. Lines give fits to Eq. (6). Inset shows Cole–Cole representation of the data and fit.

**FIG. 2.** Steady state shear viscosity (top) and stress (bottom) for 16.7 wt % CTAB in D$_2$O at 32 °C. Both strain controlled (squares) and stress controlled (circles) measurements are shown. Lines give corresponding predictions from the G-D model under viscometric (dashed) and inhomogeneous (solid) flow.

**FIG. 3.** First normal stress difference under steady shear for 16.7 wt % CTAB in D$_2$O at 32 °C. Closed and open symbols represent measurements made on a cone and plate rheometer and predictions from birefringence using Eq. (22), respectively. Lines give corresponding predictions from the G-D model under viscometric (dashed) and inhomogeneous (solid) flow.

Hegelson et al., J. Rheol. 2009
Deborah and Weissenberg

- Deborah number
  \[ De = \frac{\lambda}{t_{flow}} = \frac{\text{Timescale of fluid}}{\text{Timescale of flow}} \]
- \( \lambda \) depends on fluid
  - Largest timescale describing slowest molecular motions
  - Characteristic of rheometry test (e.g. relaxation time)
  - Term in a constitutive law
- Flow timescale more ambiguous
  - Inverse of a representative strain rate
  - Experimental duration
  - Convective or viscous timescale

- Weissenberg number
  \[ Wi = \lambda \dot{\gamma}_{flow} \left( = \frac{\tau_{11} - \tau_{22}}{\tau_{12}} = \frac{\text{Elastic stresses}}{\text{Viscous stresses}} \right) \]
  - \( \lambda \) depends on fluid, as with \( De \)
  - For some flows \( De \) and \( Wi \) can be same
  - For simple shear and some models the elastic:viscous balance is evident
- \( De \) is not specific to non-Newtonian fluids, but relates fluid timescale to timescales of interest
  - If \( t_{flow} \) very large then \( De = 0 \)
Flow timescale examples

- Plane channel flow of a GNF under an oscillating pressure gradient
- Squeeze flow of a power law fluid