Module 4: Dimensional analysis

- Understand dimensions, units and dimensional homogeneity
- Understand the Buckingham Pi theorem
  - Use method of repeating variables to find dimensionless $\Pi$'s
  - Use these to solve example problems
- Different types of similarity
- Look at model & lab experiments, plus extrapolation to prototype development.
Dimensions and Units

- **Dimension**: Measure of a physical quantity, e.g., length, time, mass

- **Units**: Assignment of a number to a dimension, e.g., (m), (sec), (kg)

- **7 Primary Dimensions**:
  1. Mass \( m \) (kg)
  2. Length \( L \) (m)
  3. Time \( t \) (sec)
  4. Temperature \( T \) (K)
  5. Current \( I \) (A)
  6. Amount of Light \( C \) (cd)
  7. Amount of matter \( N \) (mol)

- **Non-primary dimensions can all be formed by a combination of the 7 primary dimensions**

- **Examples**
  - \{Velocity\} = \{Length/Time\} = \{L/t\}
  - \{Force\} = \{Mass Length/Time^2\} = \{mL/t^2\}
  - \{Viscosity\} =
  - \{Density\} =
  - \{Pressure\} =
Dimensional Homogeneity

- Law of dimensional homogeneity (DH): every additive term in an equation must have the same dimensions.

- Example: Bernoulli equation

\[ p + \frac{1}{2} \rho V^2 + \rho g z = C \]

- \( \{p\} = \{\text{force/area}\} = \{\text{mass} \times \text{length/time} \times 1/\text{length}^2\} = \{\text{m/(t}^2\text{L)}\} \)
- \( \{1/2 \rho V^2\} = \{\text{mass/length}^3 \times (\text{length/time})^2\} = \{\text{m/} (t^2 \text{L)}\} \)
- \( \{\rho gz\} = \{\text{mass/length}^3 \times \text{length/time}^2 \times \text{length}\} = \{\text{m/(} t^2 \text{L)}\} \)
Dimensional analysis: why?

- In many real-world flows, the equations are either unknown or too difficult to solve.
  - Experimentation is the main method of obtaining reliable information
  - Geometrically-scaled models are used (saves time and money)
  - Experimental conditions/results must be properly scaled so that results are meaningful for the full-scale prototype
- The above is also true in designing complex computations
  - These are not “free”, e.g. actual power costs of supercomputing clusters & researcher time to analyze and post-process results
- Primary purposes of dimensional analysis
  - To generate non-dimensional parameters that help in the design of experiments (physical and/or numerical) and in reporting of results
  - To obtain scaling laws so that prototype performance can be predicted from model performance
  - To predict trends in the relationship between parameters
  - To simplify physical model equations in a rational way so that you can perform analysis and solve equations to give physically meaningful answers
    - E.g. when are flows inviscid?
Buckingham Pi Theorem

Theorem:

A process that satisfies physical dimensional homogeneity and involves \( n \) variables can be reduced to a relation between \( k \) dimensionless variables: \( \Pi_1, \Pi_2, \Pi_3, ... \Pi_k \).

- The reduction \( j=n-k \) is always less than or equal to the number of dimensions describing the variables.
- A dimension should occur at least once in the variables, or not at all.

This lecture:

- How is this useful? What does it mean? How to apply it?
Advantages of non-dimensionalization

- Non-dimensionalization is the method of reduction referred to in the Buckingham Pi Theorem
  - Decreases number of parameters in the process/problem
    - Easier communication
    - Fewer experiments
    - Fewer simulations
  - Can allow increased insight about key parameters
  - Can allow determination of relationships from a small amount of data, or at least educated guesses to be made
  - When applied properly it means you will extrapolate results to untested conditions in a sensible way
Application of Buckingham Pi Theorem

- Non-dimensional parameters $\Pi_j$ can be generated by several methods.
- We will use the Method of Repeating Variables
- Six steps
  1. List the parameters in the problem and count their total number $n$.
  2. List the primary dimensions of each of the $n$ parameters
  3. Set the reduction $j$ as the number of primary dimensions. Calculate $k$, the expected number of $\Pi$'s, $k = n - j$.
  4. Choose $j$ repeating parameters.
  5. Construct the $k$ $\Pi$'s, and manipulate as necessary.
  6. Write the final functional relationship and check algebra.
Guidelines for non-dim variables

1. Never pick the dependent variable. Otherwise, it may appear in all the $\Pi$'s.

2. Chosen repeating parameters must not be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the $\Pi$'s.

3. Chosen repeating parameters must represent all the primary dimensions.

4. Never pick parameters that are already dimensionless.

5. Never pick two parameters with the same dimensions or with dimensions that differ by only an exponent.

6. Choose dimensional constants over dimensional variables so that only one $\Pi$ contains the dimensional variable.

7. Pick common parameters since they may appear in each of the $\Pi$'s.

8. Pick simple parameters over complex parameters.
Example 1:

Apply the Buckingham Pi theorem to a ball falling in a vacuum

- Step 1: List relevant parameters.
  \[ z = f(t, w_0, z_0, g) \Rightarrow n = 5 \]
- Step 2: Primary dimensions of each parameter
  \[
  \begin{align*}
  z &= \{L^1\} \\
  t &= \{t^1\} \\
  w_0 &= \{L^1t^{-1}\} \\
  z_0 &= \{L^1\} \\
  g &= \{L^1t^{-2}\}
  \end{align*}
  \]
- Step 3: As first guess, reduction \( j \) is set to 2 which is the number of primary dimensions (\( L \) and \( t \)).
- Number of expected \( \Pi \)'s is:
  \[ k = n - j = 5 - 2 = 3 \]
- Step 4: Choose repeating variables \( w_0 \) and \( z_0 \)
- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the \( \Pi \)'s.
Example 1, continued

- \( \Pi_1 = zw_0^{a_1}z_0^{b_1} \)
  \[ \{ \Pi_1 \} = \{ L^0 t^0 \} = \{ zw_0^{a_1}z_0^{b_1} \} = \{ L^1 (L^1 t^{-1})^{a_1} L^{b_1} \} \]
  - \( a_1 \) and \( b_1 \) are constant exponents which must be determined.
  - Use primary dimensions identified in Step 2 and solve for \( a_1 \) and \( b_1 \).
  - Time equation:
    \[ \{ t^0 \} = \{ t^{-a_1} \} \rightarrow 0 = -a_1 \rightarrow a_1 = 0 \]
  - Length equation:
    \[ \{ L^0 \} = \{ L^1 L^{a_1} L^{b_1} \} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1 \]
  - This results in:
    \[ \Pi_1 = zw_0^0 z_0^{-1} = \frac{z}{z_0} \]

- \( \Pi_2 = tw_0^{a_2}z_0^{b_2} \)
  \[ \{ \Pi_2 \} = \{ L^0 t^0 \} = \{ tw_0^{a_2} z_0^{b_2} \} = \{ t^1 (L^1 t^{-1})^{a_2} L^{b_2} \} \]
  - Time equation:
    \[ \{ t^0 \} = \{ t^1 t^{-a_2} \} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1 \]
  - Length equation:
    \[ \{ L^0 \} = \{ L^{a_2} L^{b_2} \} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1 \]
  - This results in:
    \[ \Pi_2 = tw_0^1 z_0^{-1} = \frac{w_0 t}{z_0} \]
Example 1 continued

- \[ \Pi_3 = gw_0^{a_3}z_0^{b_3} \]
  \[ \{\Pi_3\} = \{L^0 t^0\} = \{gw_0^{a_3}z_0^{b_3}\} = \{L^1 t^{-2}(L^1 t^{-1})^{a_3}L^{b_3}\} \]

  - Time equation: \[ \{t^0\} = \{t^{-2}t^{-a_3}\} \rightarrow 0 = -2 - a_3 \rightarrow a_3 = -2 \]

  - Length equation:
    \[ \{L^0\} = \{L^1 L^{a_3}L^{b_3}\} \rightarrow 0 = 1 + a_3 + b_3 \rightarrow b_3 = -1 - a_3 \rightarrow b_3 = 1 \]

  - This results in:
    \[ \Pi_3 = gw_0^{-2}z_0^{1} = \frac{gz_0}{w_0^2} \]
    \[ \Pi_{3,modified} = \left(\frac{gz_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{gz_0}} = Fr \]

- Step 6:
  - Double check that the \( \Pi \)'s are dimensionless.
  - Write the functional relationship between \( \Pi \)'s
    \[ \Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \frac{z}{z_0} = f \left( \frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{gz_0}} \right) \]

  - Or, in terms of non-dimensional variables
    \[ z^* = f(t^*, Fr) \]
Other comments?

- Success of the method depends on initially describing the variables that influence “the problem”
- Experience is needed to know what is relevant, but a sensible rubric is to group in terms of:
  - Geometric parameters
  - Material properties
  - External effects, e.g. imposed flow rates, stresses, accelerations
- After some practice it is common to find the $\Pi$ groups by inspection, using the Buckingham theorem only to verify the correct number of groups
Example 2: Drag on a ball

Use the Buckingham Pi theorem to predict a relationship for the drag on a spherical ball in a uniform flow
Drag on falling ball in fluid

\[ \text{Cd} \]

\[ \text{Re} \]
Example 3: Thermo Nuclear Explosion

- What is the radius (R) of a fireball as a function of time (t), and energy, (E).
  - Note: We know nothing about the physics of the problem other than time and energy (what else?) are important…

An early stage in the "Trinity" fireball, photographed by Berlyn Brixner
Example 4: Hosepipe experiment

A jet of liquid directed against a block can tip over the block. Assume the velocity, $V$, needed to tip over the block is a function of the fluid density, $\rho$, the diameter of the jet, $D$, the weight of the block, $W$, the width of the block, $b$, and the distance, $d$, between the jet and the bottom of the block.

(a) Determine a set of dimensionless groups for this problem.
(b) Use the momentum equation to determine an equation for $V$ in terms of the other variables.
(c) Compare the results.
Similarity of flows

- **So far:**
  - Buckingham Π theorem
    - Useful even when you don’t know the detailed physics of the problem
    - Simplify experimental results, reduce number of independent unknowns.
    - Concept of Reynolds (viscosity) independence (drag coefficient)
    - Examples of using the Π theorem

- **Today:**
  - Meaning of dimensionless groups?
  - Look at model & lab experiments, plus extrapolation to prototype development.
**Dimensionless numbers:**

**Buckingham Pi theorem:** The phenomena present in a flow should depend only on the values of a minimal set of dimensionless \( \Pi \) groups, relevant to the flow. Many of these \( \Pi \) groups have special names. These \( \Pi \) groups represent balances between physical effects. Fluid mechanicists use these to qualitatively understand the types of flows.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Qualitative ratio of effects</th>
<th>Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reynolds number</td>
<td>( R_E = \frac{\rho U L}{\mu} )</td>
<td>Inertia, Viscosity</td>
<td>Always</td>
</tr>
<tr>
<td>Mach number</td>
<td>( M_A = \frac{U}{A} )</td>
<td>Flow speed, Sound speed</td>
<td>Compressible flow</td>
</tr>
<tr>
<td>Froude number</td>
<td>( F_r = \frac{U^2}{gL} )</td>
<td>Inertia, Gravity</td>
<td>Free-surface flow</td>
</tr>
<tr>
<td>Weber number</td>
<td>( W_e = \frac{\rho U^3 L}{\gamma} )</td>
<td>Inertia, Surface tension</td>
<td>Free-surface flow</td>
</tr>
<tr>
<td>Cavitation number</td>
<td>( C_a = \frac{p - p_v}{\rho U^2} )</td>
<td>Pressure, Inertia</td>
<td>Cavitation</td>
</tr>
<tr>
<td>Prandtl number</td>
<td>( P_r = \frac{C_p \mu}{k} )</td>
<td>Dissipation, Conduction</td>
<td>Heat convection</td>
</tr>
<tr>
<td>Eckert number</td>
<td>( E_c = \frac{U^2}{c_p T_o} )</td>
<td>Enthalpy, Kinetic energy</td>
<td>Dissipation</td>
</tr>
<tr>
<td>Specific-heat ratio</td>
<td>( \gamma = \frac{c_p}{c_v} )</td>
<td>Internal energy</td>
<td>Compressible flow</td>
</tr>
<tr>
<td>Strouhal number</td>
<td>( S_t = \frac{\omega L}{U} )</td>
<td>Oscillation, Mean speed</td>
<td>Oscillating flow</td>
</tr>
<tr>
<td>Roughness ratio</td>
<td>( \frac{\varepsilon}{L} )</td>
<td>Wall roughness</td>
<td>Turbulent, rough walls</td>
</tr>
<tr>
<td>Grashof number</td>
<td>( G_r = \frac{\beta \Delta T g L^3 \rho^2}{\mu} )</td>
<td>Wall temperature, Buoyancy, Viscosity</td>
<td>Natural convection</td>
</tr>
<tr>
<td>Temperature ratio</td>
<td>( \frac{T_s}{T_o} )</td>
<td>Stream temperature, Static pressure</td>
<td>Heat transfer</td>
</tr>
<tr>
<td>Pressure coefficient</td>
<td>( C_p = \frac{p - p_v}{1/2 \rho U^2} )</td>
<td>Dynamic pressure</td>
<td>Aerodynamics, hydraulics</td>
</tr>
<tr>
<td>Lift coefficient</td>
<td>( C_L = \frac{L}{1/2 \rho U^2 A} )</td>
<td>Lift force, Dynamic force</td>
<td>Aerodynamics, hydraulics</td>
</tr>
<tr>
<td>Drag coefficient</td>
<td>( C_D = \frac{D}{1/2 \rho U^2 A} )</td>
<td>Lift force, Dynamic force</td>
<td>Aerodynamics, hydraulics</td>
</tr>
</tbody>
</table>
Physical effects associated with $\Pi$ groups

- Reynolds number: ratio of inertia forces to viscous forces in a moving fluid
  - Extremely important in characterising flow instability & turbulence

- Froude number: ratio of inertia forces to gravitational forces
  - Applications in free surface flows, e.g. waves & open channel flows

- Strouhal number: ratio of local inertia forces to convective inertia forces.
  - Local inertia forces are frequently due to some form of imposed oscillation
**Fluids in the news:**

**Slip at the micro scale (in main text)** A goal in chemical and biological analyses is to miniaturize the experiment, which has many advantages including reduction in sample size. In recent years, there has been significant work on integrating these tests on a single microchip to form the “lab-on-a-chip” system. These devices are on the millimeter scale with complex passages for fluid flow on the micron scale (or smaller). While there are advantages to miniaturization, care must be taken in moving to smaller and smaller flow regimes, as you will eventually bump into the continuum assumption. To characterize this situation, a dimensionless number termed the Knudsen number, $\text{Kn} = \lambda / \ell$, is commonly employed. Here $\lambda$ is the mean free path and $\ell$ is the characteristic length of the system. If $\text{Kn}$ is smaller than 0.01, then the flow can be described by the Navier–Stokes equations with no-slip at the walls. For $0.01 < \text{Kn} < 0.3$, the same equations can be used, but there can be slip between the fluid and the wall so the boundary conditions need to be adjusted. For $\text{Kn} > 10$, the continuum assumption breaks down and the Navier–Stokes equations are no longer valid.

**Modeling parachutes in a water tunnel (in main text)** The first use of a parachute with a free-fall jump from an aircraft occurred in 1914, although parachute jumps from hot-air balloons had occurred since the late 1700s. In more modern times parachutes are commonly used by the military and for safety and sport. It is not surprising that there remains interest in the design and characteristics of parachutes, and researchers at the Worcester Polytechnic Institute have been studying various aspects of the aerodynamics associated with parachutes. An unusual part of their study is that they are using small-scale parachutes tested in a water tunnel. The model parachutes are reduced in size by a factor of 30 to 60 times. Various types of tests can be performed, ranging from the study of the velocity fields in the wake of the canopy with a steady free-stream velocity to the study of conditions during rapid deployment of the canopy. According to the researchers, the advantage of using water as the working fluid, rather than air, is that the velocities and deployment dynamics are slower than in the atmosphere, thus providing more time to collect detailed experimental data. (See Problem 7.57.)
Dimensional analysis - similarity

- **Similarity basic idea:**
  - If all the $\Pi$ groups for the model and prototype are identical than the flows are identical (just scaled)
  - We can perform the experiment (test) on the model and rescale to get correct answers for the prototype
  - In practice we achieve similarity in stages that are intuitive to implement, starting with...

- **Geometric Similarity** - the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.
**Kinematic Similarity** - velocity at any point in the model must be proportional to that in prototype.

**Kinematically Similar Low Speed Flows**

**Kinematically Similar Free Surface Flows**
Dynamic Similarity - *all forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow.

Dynamic Similarity for Flow through a Sluice Gate
How do we achieve these scaling

- The result of achieving similarity by the above means is that relevant non-dimensional dependent variables, e.g. $C_D$, $C_p$, $C_f$, or $Nu$, etc., are then equal for both the model and prototype.
- This result would then indicate how the relevant dependent results, e.g. drag force, pressure forces, viscous forces, are to be scaled for the model to properly represent the prototype.
- Equality of the relevant non-dimensional independent variables, $Re$, $Ma$, $x/L$, etc., indicates how the various independent variables of importance should be scaled.
Example 1: Drag on a sphere

- Relevant Pi groups were $Cd = f (Re)$

- We have dynamic similarity when ...

\[
\left( \frac{\rho VD}{\mu} \right)_{\text{model}} = \left( \frac{\rho VD}{\mu} \right)_{\text{prototype}}
\]

- Then ...

\[
\left( \frac{F}{\rho V^2 D^2} \right)_{\text{model}} = \left( \frac{F}{\rho V^2 D^2} \right)_{\text{prototype}}
\]
Example 2:

A one-twelfth-scale model of an airplane is to be tested at 20°C in a pressurised wind tunnel. The prototype is to fly at 240 m/s at 10 km standard altitude. What should the tunnel pressure be in order to scale both the Mach number and Reynolds number?
Example 3: rapid prototyping

Model tests are to be performed to study the flow through a large check valve with 2-ft-diameter inlet carrying water at a flowrate of 30 cfs, as shown. The model inlet diameter is 3 in. The fluid in the model is water at the same temperature as that in the prototype. Complete geometric similarity exists.

**Q.** Determine the flowrate required in the model to achieve complete similarity.
For ship hydrodynamics, $Fr$ similarity is maintained while $Re$ is allowed to be different.

**Why?** Look at complete similarity:

$$Re_p = \frac{V_p L_p}{\nu_p} = Re_m = \frac{V_m L_m}{\nu_m} \Rightarrow \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p} \frac{V_p}{V_m}$$

$$Fr_p = \frac{V_p}{\sqrt{g L_p}} = Fr_m = \frac{V_m}{\sqrt{g L_m}} \Rightarrow \frac{L_m}{L_p} = \left(\frac{V_p}{V_m}\right)^2$$

To match both $Re$ and $Fr$, the viscosity in the model test is a function of the scale ratio!

**This is not feasible.**

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$
Experimental Testing & Incomplete Similarity

- Flows with free surfaces present unique challenges in achieving complete dynamic similarity.
- For hydraulics applications, depth is very small in comparison to horizontal dimensions. If geometric similarity is used, the model depth would be so small that other issues would arise
  - Surface tension effects (Weber number) would become important.
  - Data collection becomes difficult.
- Distorted models are therefore employed, which requires empirical corrections & correlations to extrapolate model data to full scale.
**Fluids in the news:**

*“Galloping Gertie”* (in main text) One of the most dramatic bridge collapses occurred in 1940 when the Tacoma Narrows Bridge, located near Tacoma, Washington, failed due to aerodynamic instability. The bridge had been nicknamed “Galloping Gertie” due to its tendency to sway and move in high winds. On the fateful day of the collapse the wind speed was 65 km/hr. This particular combination of a high wind and the aeroelastic properties of the bridge created large oscillations leading to its failure. The bridge was replaced in 1950, and a second bridge parallel to the existing structure was opened in 2007. To determine possible wind interference effects due to two bridges in close proximity, wind tunnel tests were run in a 9 m × 9 m wind tunnel operated by the National Research Council of Canada. *Models* of the two side-by-side bridges, each having a length scale of 1:211, were tested under various wind conditions. Since the failure of the original Tacoma Narrows Bridge, it is now common practice to use wind tunnel model studies during the design process to evaluate any bridge that is to be subjected to wind-induced vibrations. (See Problem 7.84.)

**Jurassic Tank** Geologists use *models* involving water flowing across sand- or soil-filled tanks to investigate how river beds and river valleys are formed. The only variables in these studies are the model flowrate, the bottom slope, and the type of sand or soil material used. Now researchers at the University of Minnesota have developed the “Jurassic Tank,” the first apparatus to use a “sinking floor.” The bottom of the 40-ft-long, 20-ft-wide, 5-ft-deep tank contains 432 honeycomb funnels on top of which rests a rubber membrane floor. The floor can be programmed to sink in any uneven fashion via computer control by removing the supporting gravel within the honeycombs. By running sediment-loaded water into the tank and studying the patterns of sediment deposition as the basin floor is lowered, it is possible to determine how sinking of the earth’s crust interacts with sediment buildup to produce the sediment layers that fill ocean sedimentary basins. The name Jurassic Tank comes from its ability to model conditions during the Jurassic era at the beginning of the formation of the Atlantic Ocean (about 160 million years ago).

**Old Man River in (large) miniature** (in main text) One of the world’s largest scale models, a Mississippi River model, resides near Jackson, Mississippi. It is a detailed, complex model that covers many acres and replicates the 1,250,000-acre Mississippi River basin. Built by the Army Corps of Engineers and used from 1943 to 1973, today it has mostly gone to ruin. As with many hydraulic models, this is a *distorted model*, with a horizontal scale of 1:2000 and a vertical scale of 1:100. One step along the model river corresponds to one mile along the river. All essential river basin elements such as geological features, levees, and railroad embankments were sculpted by hand to match the actual contours. The main purpose of the model was to predict floods. This was done by supplying specific amounts of water at prescribed locations along the model and then measuring the water depths up and down the model river. Because of the length scale, there is a difference in the time taken by the corresponding model and prototype events. Although it takes days for the actual floodwaters to travel from Sioux City, Iowa, to Omaha, Nebraska, it would take only minutes for the simulated flow in the model.
What we covered:

- How to make non-dimensional parameters using Buckingham Pi theorem
- Demonstrated how non-dimensional parameters reduce number of independent variables, because of dimensional homogeneity.
- Demonstrated how non-dimensional parameters provide insight and allow prediction with limited experiments
- Examined types of similarity:
  - Geometric, Kinematic, Dynamic
  - Scale up forces, etc from model to prototype
  - Dynamic similarity requires
    - Geometric similarity
    - Matching relevant Pi Groups
- Incomplete similarity for models