Module 5:

- Navier-Stokes Equations:
  - Apply mass and momentum conservation to a differential control volume
  - Derive the Navier-Stokes equations
  - Simple classical solutions of NS equations
  - Use dimensional analysis to simplify the NS equations
  - Introduce: inviscid flows, irotational flows and potential flows
Mass Conservation

\[ 0 = \int_{CV} \frac{\partial \rho}{\partial t} \, dv + \int_{CS} \rho \mathbf{V} \cdot \mathbf{n} \, dA \]

\[ V = (u, v, w) \]

Term 1:
\[ \int_{CV} \frac{\partial \rho}{\partial t} \, dv = \frac{\partial \rho}{\partial t} \, dx \, dy \, dz \]

Term 2: Sum of fluxes through all 6 faces. For Example, mass flux through left face:
\[ -\rho \mathbf{u} |_{x} \, dy \, dz \]

mass flux through right face:
\[ (\rho \mathbf{u}) |_{x+\Delta x} \, dy \, dz \]

\[ (\rho \mathbf{u}) |_{x+\Delta x} \approx (\rho \mathbf{u}) |_{x} + dx \left( \frac{\partial}{\partial x} (\rho \mathbf{u}) \right) |_{x} \]

Net mass flux in x direction is
\[ \left( \frac{\partial (\rho \mathbf{u})}{\partial x} \right) |_{x} \, dx \, dy \, dz \]
Mass Conservation, contd.

Similarly, next flux in y direction:

\[
\left. \frac{\partial (\rho v)}{\partial y} \right|_y \, dxdydz
\]

and z direction:

\[
\left. \frac{\partial (\rho w)}{\partial z} \right|_z \, dxdydz
\]

\[
0 = \int_{cv} \frac{\partial \rho}{\partial t} \, dv + \int_{cs} \rho \mathbf{V} \cdot \mathbf{n} \, dA
\]

\[
0 = \frac{\partial \rho}{\partial t} \, dxdydz + \left. \frac{\partial (\rho u)}{\partial x} \right|_x \, dxdydz + \left. \frac{\partial (\rho v)}{\partial y} \right|_y \, dxdydz + \left. \frac{\partial (\rho w)}{\partial z} \right|_z \, dxdydz
\]

Or in vector form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0
\]

Continuity equation
Chain rule ...

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = 0
\]

\[
\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{V} = 0
\]

Steady Flow ?

Incompressible flow?
An incompressible velocity field is given by

\[ u = a(x^2 - y^2) \quad v \text{ unknown} \quad w = b \]

where \( a \) and \( b \) are constants. What must the form of the velocity component \( v \) be?
Conservation of momentum

\[ \sum \mathbf{F} = \int_C \frac{\partial}{\partial t}(\rho \mathbf{V}) \, dv + \int_{CS} \rho \mathbf{V} (\mathbf{V} \cdot \mathbf{n}) \, dA \]

Consider first left-hand side:

\[ \sum \mathbf{F} = \sum \mathbf{F}_{body} + \sum \mathbf{F}_{pressure} + \sum \mathbf{F}_{Viscous} \]

Body force: acts upon entire mass within element (gravity, magnetism, etc)

\[ \sum \mathbf{F}_{body} = \int_{CV} \rho g \, dv = m_{CV} g = \rho \Delta x \Delta y \Delta z g \]

Surface force: acts only on the sides of control surface

\[ \sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \]

Pressure forces: 
Viscous forces
Conservation of momentum contd.

on left surface, the pressure force is

\[ p \left|_x \right. \, dydz \]

And on the right surface

\[ - p \left|_{x+\Delta x} \right. \, dydz = - \left( p \left|_x \right. +dx \frac{\partial p}{\partial x} \left|_x \right. \right) dydz \]

So the net force in \( x \) direction

\[- \frac{\partial p}{\partial x} \, dxdydz \]

Similarly in \( y \) direction

\[- \frac{\partial p}{\partial y} \, dxdydz \]

and in \( z \) direction

\[- \frac{\partial p}{\partial z} \, dxdydz \]
Conservation of momentum contd.

Viscous stresses: in $x$ direction:

$$
\left[ \tau_{xx} \mid_{x+\Delta x} - \tau_{xx} \mid_x \right] dydz = \left( \tau_{xx} \mid_x + \Delta x \frac{\partial \tau_{xx}}{\partial x} \mid_x - \tau_{xx} \mid_x \right) dydz = \frac{\partial \tau_{xx}}{\partial x} dxdydz
$$

$$
\left[ \tau_{xy} \mid_{y+\Delta y} - \tau_{xy} \mid_y \right] dxdz = \left( \tau_{xy} \mid_y + \Delta y \frac{\partial \tau_{xy}}{\partial y} \mid_y - \tau_{xy} \mid_y \right) dxdz = \frac{\partial \tau_{xy}}{\partial y} dxdydz
$$

$$
\left[ \tau_{xz} \mid_{z+\Delta z} - \tau_{xz} \mid_z \right] dxdy = \left( \tau_{xz} \mid_z + \Delta z \frac{\partial \tau_{xz}}{\partial z} \mid_z - \tau_{xz} \mid_z \right) dxdy = \frac{\partial \tau_{xz}}{\partial z} dxdydz
$$
Conservation of momentum contd.

Net forces in $x$ direction

$$\frac{\partial \tau_{xx}}{\partial x} dxdydz + \frac{\partial \tau_{xy}}{\partial y} dxdydz + \frac{\partial \tau_{xz}}{\partial z} dxdydz$$

Similarly in $y$ direction

$$\frac{\partial \tau_{xy}}{\partial x} dxdydz + \frac{\partial \tau_{yy}}{\partial y} dxdydz + \frac{\partial \tau_{yz}}{\partial z} dxdydz$$

and in $z$ direction

$$\frac{\partial \tau_{xz}}{\partial x} dxdydz + \frac{\partial \tau_{yz}}{\partial y} dxdydz + \frac{\partial \tau_{zz}}{\partial z} dxdydz$$

So far

$$\sum F = \sum F_{body} + \sum F_{pressure} + \sum F_{viscous}$$

$$\sum F = \rho g_x \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} dxdydz$$

$$\sum F = \rho g_y \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} dxdydz$$

$$\sum F = \rho g_z \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} dxdydz$$
Conservation of momentum

$$\sum \textbf{F} = \int_{CV} \frac{\partial}{\partial t} (\rho \textbf{V}) \, dv + \int_{CS} \rho \textbf{V} (\textbf{V} \cdot \textbf{n}) \, dA$$

Now lets look at right-hand side:
First term:
$$\int_{CV} \frac{\partial}{\partial t} (\rho \textbf{V}) \, dv = \frac{\partial}{\partial t} (\rho \textbf{V}) \, dxdydz$$

Second term of right-hand side:
Momentum in $x$ direction

$x$-faces: \[ \left[ (\rho uu)_{x+\Delta x} - (\rho uu)_x \right] \, dydz = \left( (\rho uu)_x + \Delta x \frac{\partial (\rho uu)}{\partial x} \right)_x - (\rho uu)_x \, dydz = \frac{\partial (\rho uu)}{\partial x} \, dxdydz \]

$y$-faces: \[ \left[ (\rho uv)_{y+\Delta y} - (\rho uv)_y \right] \, dxdz = \frac{\partial (\rho uv)}{\partial y} \, dxdydz \]

$z$-faces: \[ \left[ (\rho uw)_{z+\Delta z} - (\rho uw)_z \right] \, dxdy = \frac{\partial (\rho uw)}{\partial z} \, dxdydz \]
Conservation of momentum contd.

Putting it all together, for the $x$-component ...

$$\sum \mathbf{F} = \int_C \frac{\partial}{\partial t} (\rho \mathbf{V}) \, dv + \int_S \rho \mathbf{V} (\mathbf{V} \cdot \mathbf{n}) \, dA \bigg|_x$$

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} \bigg|_{\text{out}} + \frac{\partial \tau_{xy}}{\partial x} \bigg|_{\text{out}} + \frac{\partial \tau_{xz}}{\partial x} \bigg|_{\text{out}} \, dxdydz = \left[ \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho uw)}{\partial z} \right] dxdydz$$

And similarly in $y$ and $z$ direction:

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho vw)}{\partial z} = - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho g_y$$

$$\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho uw)}{\partial x} + \frac{\partial (\rho vw)}{\partial y} + \frac{\partial (\rho ww)}{\partial z} = - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z$$
Constitutive relationship for a Newtonian fluid

Not finished yet...
Need to express shear stress in terms of velocity ... How?

\[ \tau_{xx} = 2\mu \frac{\partial u}{\partial x} \]
\[ \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \]
\[ \tau_{yy} = 2\mu \frac{\partial w}{\partial z} \]
\[ \tau_{yx} = \tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ \tau_{zx} = \tau_{xz} = \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \]
\[ \tau_{yz} = \tau_{zy} = \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \]

- This relationship is attributed to Navier & Stokes.
  - Navier apparently made some mistakes and in fact Stokes only considered slow flows

- Possibly, Poisson & St-Venant also have as strong a claim... but...

- The basic momentum equation is due to Cauchy (many years earlier)
  - but without a closure for the stresses is not much use
  - Cauchy’s equation is also the basis for solid mechanics/dynamics, but we don’t use the material derivative
Incompressible flow in Cartesian Coordinates

Continuity
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

x-momentum
\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + \rho g_x
\]

y-momentum
\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] + \rho g_y
\]

z-momentum
\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] + \rho g_z
\]

See text for equations in cylindrical and spherical coordinates
Special cases

- Vector notation for incompressible Newtonian fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{V}] = 0$$

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + [\mathbf{V} \cdot \nabla] \mathbf{V} \right) = -\nabla P + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$$

- Steady Flow

- Incomressible
Boundary conditions

- Boundary conditions (BC’s) are critical to exact, approximate, and computational solutions.
- BC’s used in analytical solutions are discussed here
  - No-slip boundary condition
  - Interface boundary condition
- These are used in CFD as well, plus there are some BC’s which arise due to specific issues in CFD modeling.
  - Inflow and outflow boundary conditions
  - Symmetry and periodic boundary conditions
No-slip boundary condition

- For a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall.
Interface boundary conditions

- When two fluids meet at an interface, the velocity and shear stress balance:
  \[ \vec{V}_A = \vec{V}_B \quad \tau_{s,A} = \tau_{s,B} \]
- Normal stresses differ by an amount proportional to surface tension & inversely proportional to curvature.
- Negligible surface tension or surface nearly flat:
  \[ P_A = P_B \]
Interface boundary condition

- Degenerate case of the interface BC occurs at the free surface of a liquid.
- Same conditions hold

\[ u_{air} = u_{water} \]

\[ \tau_{s,water} = \mu_{water} \left( \frac{\partial u}{\partial y} \right)_{water} = \tau_{s,air} = \mu_{air} \left( \frac{\partial u}{\partial y} \right)_{air} \]

- Since \( \mu_{air} \ll \mu_{water} \),

\[ \left( \frac{\partial u}{\partial y} \right)_{water} \approx 0 \]

- As with general interfaces, if surface tension effects are negligible or the surface is nearly flat, \( P_{water} = P_{air} \)
What we covered:

- Derived Navier-Stokes & mass conservation equations:
  - Control volume approach, applied to differential volume of fluid
    \[
    \frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{V}] = 0
    \]
    \[
    \rho \left( \frac{\partial \mathbf{V}}{\partial t} + [\mathbf{V} \cdot \nabla] \mathbf{V} \right) = -\nabla P + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}
    \]
  - Special cases, e.g. steady flows, incompressible
- Discussed different boundary conditions and interfacial conditions
Clasical Solutions of NS equations

- There are about 80 known exact solutions to the NSE
- They can be classified as:
  - Linear solutions, where the convective term is zero:
    \[(\vec{V} \cdot \nabla) \vec{V}\]
    No acceleration = Stokes flows
  - Nonlinear solutions where convective term is not zero
    - Usually some other terms are zero, e.g. inviscid flows

- Solutions also classified by type or geometry, e.g.
  1. Couette flows
  2. Steady duct/pipe flows
  3. Unsteady duct/pipe flows
  4. Flows with moving boundaries
  5. Similarity solutions
  6. Asymptotic solutions
  7. Ekman flows…
Exact Solutions of the NS equations

Procedure for solving NS equations

1. Set up the problem and geometry, identifying all relevant dimensions and parameters
2. List all appropriate assumptions, approximations, simplifications, and boundary conditions
3. Simplify the differential equations as much as possible
4. Integrate the equations
5. Apply BC’s to solve for constants of integration
6. Verify results
Example 1: Fully Developed Couette Flow

For the given geometry and BC’s, calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate

- Step 1: Geometry, dimensions, and properties
Example: Fully Developed Couette Flow

Step 2: Assumptions and BC’s

- Assumptions
  1. Plates are infinite in x and z
  2. Flow is steady, $\frac{\partial}{\partial t} = 0$
  3. Parallel flow, $v=0$
  4. Incompressible, Newtonian, laminar, constant properties
  5. No pressure gradient
  6. 2D, $w=0$, $\frac{\partial}{\partial z} = 0$
  7. Gravity acts in the -z direction, $\vec{g} = -g \vec{k}, g_z = -g$

- Boundary conditions
  1. Bottom plate ($y=0$): $u=0$, $v=0$, $w=0$
  2. Top plate ($y=h$): $u=V$, $v=0$, $w=0$
Example: Fully Developed Couette Flow

- **Step 3: Simplify**

Continuity

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0
\]

Note: these numbers refer to the assumptions on the previous slide

This means the flow is “fully developed” or not changing in the direction of flow

X-momentum

\[
\rho \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right)
\]

\[
\frac{\partial^2 U}{\partial y^2} = 0
\]
Example: Fully Developed Couette Flow

Step 3: Simplify, cont.

Y-momentum

\[
\rho \left( \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right)
\]

\[
\frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad p = p(z)
\]

Z-momentum

\[
\rho \left( \frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right)
\]

\[
\frac{\partial p}{\partial z} = \rho g_z \quad \Rightarrow \quad \frac{dp}{dz} = -\rho g
\]
Example: Fully Developed Couette Flow

- **Step 4: Integrate**

**X-momentum**

\[
\frac{d^2 u}{dy^2} = 0 \quad \Rightarrow \quad \frac{du}{dy} = C_1 \quad \Rightarrow \quad u(y) = C_1 y + C_2
\]

**Z-momentum**

\[
\frac{dp}{dz} = -\rho g \quad \Rightarrow \quad p = -\rho g z + C_3
\]
Example: Fully Developed Couette Flow

Step 5: Apply BC’s

- @y = 0, u = 0 \implies C_2 = 0
- @ y = h, u = V \implies C_1 = V/h
- This gives
  \[ u(y) = V \frac{y}{h} \]
  \[ p(z) = p_0 - \rho g z \]

  - For pressure, no explicit BC, therefore \( C_3 \) can remain an arbitrary constant (recall only \( \nabla p \) appears in NSE).
    - Let \( p = p_0 \) at \( z = 0 \) (\( C_3 \) renamed \( p_0 \))

Step 6: Verify solution by back-substituting into differential equations

1. Hydrostatic pressure
2. Pressure acts independently of flow
Example: Fully Developed Couette Flow

Finally, calculate shear force on bottom plate

\[
\tau_{i,j} = \begin{pmatrix}
2\mu \frac{\partial U}{\partial x} & \mu \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) & \mu \left( \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right) \\
\mu \left( \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) & 2\mu \frac{\partial V}{\partial y} & \mu \left( \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right) \\
\mu \left( \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right) & \mu \left( \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} \right) & 2\mu \frac{\partial W}{\partial z}
\end{pmatrix} = \begin{pmatrix}
0 & \mu \frac{V}{h} & 0 \\
\mu \frac{V}{h} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Shear force per unit area acting on the wall

\[
\frac{\vec{F}}{A} = \tau_w = \mu \frac{V}{h} \hat{i}
\]

Note that \( \tau_w \) is equal and opposite to the shear stress acting on the fluid \( \tau_{yx} \) (Newton’s third law).
Example 2: Fully Developed Planar Poiseuille Flow

Find velocity profile of fully developed flow of a fluid moving between two fixed plate due to a constant pressure gradient $G$ in $x$ direction.

Assumptions:

1) steady, incompressible flow
2) Newtonian
3) infinitely large plates in $z$ direction (i.e. 2D flow)
4) fully developed flow
5) …
Example 2: Fully Developed Planar Poiseuille Flow

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{\partial v}{\partial y} = 0 \quad \Rightarrow v = 0$

$x$-mom: $\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

$-G + \mu \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow u = \frac{G}{2\mu} y^2 + C_1 y + C_2$

$y$-mom: $\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$

$-\frac{\partial p}{\partial y} = 0,$

BC: $@ y = \pm h, u = 0 \quad \Rightarrow C_1 = 0,$ and $C_2 = -\frac{G}{2\mu} h^2$

$u = -\frac{G}{2\mu} h^2 \left( 1 - \left( \frac{y}{h} \right)^2 \right) = u_{\text{max}} \left( 1 - \left( \frac{y}{h} \right)^2 \right)$
Example 3: Poiseuille Flow with a moving plate

Consider Poiseuille problem except that the top plate is moving with the speed $U$. Derive the velocity profile. Everything remains unchanged, except the boundary conditions:

$@y = 0, u = 0$ and $@ y = b, u = U$

$$u = -\frac{G}{2\mu} b^2 \left( \frac{y}{b} - \left( \frac{y}{b} \right)^2 \right)$$
Example 4: Rotational Couette Flow

The area between two very long cylinder is filled with a viscous fluid. The inner cylinder rotates at the angular speed of $\omega$, while the outer cylinder is stationary. Find velocity profile of flow in annulus.

Assumptions:
1) steady, incompressible flow
2) Newtonian
3) Parallel flow (in a polar sense)
4) Use polar coordinate
5) Long cylinders, so no axial flow
Continuity: \( \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0 \), no radial flow \( \Rightarrow \frac{\partial u_\theta}{\partial \theta} = 0 \) \( \Rightarrow u_\theta = u_\theta(r) \)

\( \theta \)-momentum:

\[
\rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r u_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right]
\]

\( \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} = 0 \) \( \Rightarrow \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r^2} = 0 \)

Euler equation: let \( u_\theta = r^k \) \( \Rightarrow r^{k-2} (k(k-1) + k - 1) = 0 \) \( \Rightarrow k = \pm 1 \)

\( u_\theta = C_1 r + \frac{C_2}{r} \)

BC: \( @r = r_i, \ u_\theta = r_i \omega \) and \( @r = r_o, u_\theta = 0 \)

\( u_\theta = \frac{\omega r_i^2 r_o^2 - r^2}{r r_o^2 - r_i^2} \)
Example 5: Stokes first problem

Consider a stationary flat plate beneath a semi-infinite layer of initially quiescent incompressible fluid. At \( t = 0 \), suddenly the plate is pulled with the speed \( U \).

Assumptions:
1) Unsteady, incompressible flow
2) Newtonian
3) Infinitely large in \( x \) and positive \( y \) direction
4) Parallel flow
5) No external pressure gradient

At \( t = 0 \), the plate is pulled with the speed \( U \).
Example 5: Stokes first problem

Continuity: \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) parallel flow \( v = 0 \)

x-mom: \( \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \)

\( \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \) or \( u_t = \nu u_{yy} \)

BC: @ \( y = 0, u = U \) and IC: @ \( t = 0, u = 0 \) everywhere

Solution is by change of variable. \( \eta = \frac{y}{\sqrt{4vt}} \) (How did we know this change of variable works?)

\( u_t = u_\eta u_\eta = -\frac{y}{2\sqrt{4vtt}} u_\eta \), and \( u_{xx} = \frac{1}{4vt} u_{\eta\eta} \)

\( u_{\eta\eta} = -2\eta u_\eta \)

\( u_\eta = C_1 \exp -\eta^2 \rightarrow u = C_1 \int_0^\eta \exp -x^2 \, dx + C_2 \)

\( \eta = 0, u = U \implies C_2 = U \)

\( \eta = \infty, u = 0 \implies C_1 \int_0^\infty \exp -x^2 \, dx + U = 0 \implies C_1 = -2U/\sqrt{\pi} \)

\( u = U \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-x^2} \, dx \right) \)
What we covered:

- Study classical examples of Navier-Stokes equation
  - Poiseuille flow
  - Couette flow
  - Mixture of Poiseuille and Couette flow
  - Stokes First problem
Dimensional analysis version 2.0

- **Scale the fundamental equations:**
  - To determine dimensionless variables and dependencies
  - In order to simplify the equations in a rational way

- **Example 1: Bernoulli’s equation**
  \[ p + \frac{1}{2} \rho V^2 + \rho g z = C \]
  - \( \{p\} = \{\text{force/area}\} = \{\text{mass x length/time x 1/length}^2\} = \{\text{m/(t}^2\text{L)}\} \)
  - \( \{1/2\rho V^2\} = \{\text{mass/length}^3 \times (\text{length/time})^2\} = \{\text{m/(t}^2\text{L)}\} \)
  - \( \{\rho g z\} = \{\text{mass/length}^3 \times \text{length/time}^2 \times \text{length}\} = \{\text{m/(t}^2\text{L)}\} \)

- **Non-dimensionalize all variables with scaling parameters**
  \[ V^* = \frac{V}{U_0} \quad z^* = \frac{z}{L} \quad \rho^* = \frac{\rho}{\rho_0} \quad p^* = \frac{p}{\rho_0 U_0^2} \quad g^* = \frac{g L}{U_0^2} \]
Non dimensional equations:

- Back-substitute $p$, $\rho$, $V$, $g$, $z$ into dimensional equation:

$$\rho_0 U_0^2 p^* + \frac{1}{2} \rho_0 \rho^* \left( U_0^2 V^*^2 \right) + \rho_0 \rho^* g^* U_0^2 z^* = C$$

- Divide by $\rho_0 U_0^2$ and set $\rho^* = 1$ (assume incompressible)

$$p^* + \frac{1}{2} V^*^2 + g^* z^* = \frac{C}{\rho_0 U_0^2} = C^*$$

- Since $g^* = 1/\text{Fr}^2$, where $\text{Fr} = \frac{U_0}{\sqrt{gL}}$ we can write this as:

$$p^* + \frac{1}{2} V^*^2 + \frac{1}{\text{Fr}^2} z^* = C^*$$

- Flow depends on 2 dimensionless parameters
Example 2:

Scale the incompressible Navier-Stokes equations appropriately for flow of water along a vertical square duct of side $D$, driven by a steady flow rate $Q$. How many dimensionless groups govern the velocity field?

Conservation of mass
\[ \nabla \cdot \mathbf{U} = 0 \]

Conservation of momentum
\[ \rho \left( \frac{\partial \mathbf{U}}{\partial t} + [\mathbf{U} \cdot \nabla] \mathbf{U} \right) = -\nabla P + \rho \mathbf{g} + \mu \nabla^2 \mathbf{U} \]
Example 3: Slider bearing

Use scaling to simplify the Navier-Stokes equations for steady flow in the slider bearing, as illustrated. Hence derive an expression for the force (per unit depth) exerted on the bearing:

\[
\nabla \cdot \mathbf{U} = 0
\]

\[
\rho [\mathbf{U} \cdot \nabla] \mathbf{U} = -\nabla P + \rho g + \mu \nabla^2 \mathbf{U}
\]

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Example 4: Boundary layer flow

Fluid flows in the x-direction past a semi-infinite flat plate positioned along the x-axis for \( x > 0 \). The far-field speed is \( U \).

Use scaling arguments to simplify the Navier-Stokes equations in the “boundary layer” separating the plate from the far-field.

\[
\begin{align*}
    u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
    u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \\
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{align*}
\]
Inviscid flows

- Consider the scaled system of Navier-Stokes equations
  - $\text{Re}$ represents balance of inertial to viscous stresses
  - $\text{Re} \to \infty$ represents inviscid limit
- We study only incompressible inviscid flows
  - Re-derive Bernoulli’s equation
  - Define vorticity and irrotational flows
  - Potential flows
Incompressible inviscid flows:

- Approximation to flow of e.g. air & water, far from solid surfaces:

- Euler equations:

\[
\begin{align*}
\rho g_x - \frac{\partial p}{\partial x} &= \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\rho g_y - \frac{\partial p}{\partial y} &= \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\
\rho g_z - \frac{\partial p}{\partial z} &= \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{align*}
\]

Vector form:

- General vector identity:

\[
(V \cdot \nabla)V = \frac{1}{2} \nabla(V \cdot V) - V \times (\nabla \times V)
\]

- Align z with gravity:

\[
-\rho g \nabla z - \nabla p = \frac{\rho}{2} \nabla(V \cdot V) - \rho V \times (\nabla \times V)
\]

- Euler’s eqns in vector form:

\[
\frac{\nabla p}{\rho} + \frac{1}{2} \nabla(V^2) + g \nabla z = V \times (\nabla \times V)
\]
Relation to Bernoulli’s equation?

- Dot product with ds, along streamline:
  \[
  \frac{\nabla p}{\rho} \cdot ds + \frac{1}{2} \nabla (V^2) \cdot ds + g \nabla z \cdot ds = [V \times (\nabla \times V)] \cdot ds
  \]

- Note that:
  \[ [V \times (\nabla \times V)] \cdot ds = 0 \]

- Differential along streamline:
  \[
  \frac{dp}{\rho} + \frac{1}{2} d(V^2) + g \ dz = 0
  \]

- Integrate to get:
  \[
  \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}
  \]

- Restrictions: incompressible, inviscid, steady, along streamline

- If \( \nabla \times V = 0 \), then we can write Bernoulli's equation between any two points (not necessarily on one streamline)

- So what is special about \( \nabla \times V = 0 \)?
Irrotational Flow

- We define $\omega = \nabla \times V$ and call it vorticity.
- Vorticity is a measure of local rotation of fluid particles.
- If $\omega = 0$, the flow is called irrotational.
- For incompressible irrotational flows, we can define $V$ as the gradient of a potential function $V = \nabla \Phi$ (This automatically satisfies $\omega = 0$. why?)
- Together with continuity, $\nabla^2 \Phi = 0$
  - Laplace’s equation, which has many analytical solutions.
- So $V = \nabla \Phi \Rightarrow u = \frac{\partial \Phi}{\partial x}$, and $v = \frac{\partial \Phi}{\partial y}$.
- Also remember , $u = \frac{\partial \Psi}{\partial y}$, and $v = -\frac{\partial \Psi}{\partial x}$.
- It can be easily shown that lines of iso-potential are normal to stream lines.
Some basic potential flows (2D)

1) Uniform Flow

\[ u = U, v = 0 \Rightarrow \frac{\partial \Phi}{\partial x} = U, \frac{\partial \Phi}{\partial y} = 0 \Rightarrow \Phi = Ux \]

\[ u = U, v = 0 \Rightarrow \frac{\partial \Psi}{\partial y} = U, -\frac{\partial \Psi}{\partial x} = 0 \Rightarrow \Psi = Uy \]

2) Source/Sink

\[ u_r = \frac{m}{2\pi r}, u_\theta = 0 \Rightarrow \frac{\partial \Phi}{\partial r} = \frac{m}{2\pi}, \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = 0 \Rightarrow \Phi = \frac{m}{2\pi} \ln r \]

\[ u_r = \frac{m}{2\pi r}, u_\theta = 0 \Rightarrow \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = \frac{m}{2\pi} \frac{\partial \Psi}{\partial r} = 0 \Rightarrow \Psi = \frac{m}{2\pi} \theta \]

\(m\) characterises the strength of the source and \(2\pi\) is only there conventionally.

3) Free Vortex

\[ u_\theta = \frac{\Gamma}{2\pi r}, u_r = 0 \Rightarrow \Phi = \frac{\Gamma}{2\pi} \theta, \text{ and } \Psi = -\frac{\Gamma}{2\pi} \ln r \]

\(\Gamma\) characterises the strength of vortex and \(2\pi\) is only there conventionally.

Since Laplace equation is linear, we can superimpose these solutions to create complex flows.
Example 1: Superimposing a source and a sink

Consider a source and sink of equal strength $m$ locating at a distance $2a$ as shown. Find stream function. Simplify your answer when $a$ is small.

\[ \Psi = \Psi_{source} + \Psi_{sink} = -\frac{m}{2\pi} (\theta_1 - \theta_2) \]

which can be written (test your trigonometry and prove this)

\[ \Psi = -\frac{m}{2\pi} \tan^{-1}\left(\frac{2ar \sin \theta}{r^2 - a^2}\right) \]

We know $\tan^{-1} \epsilon \approx \epsilon$ when $\epsilon \ll 1$, so when $a \ll 1$

\[ \Psi \approx -\frac{ma}{\pi} \frac{r \sin \theta}{r^2 - a^2} = \frac{ma}{\pi} \frac{r \sin \theta}{(r^2 - a^2)} \]
Example 2: Doublet

In the previous example, let \( a \to 0 \), while we increase \( m \), in such a way that \( K = \frac{ma}{\pi} \) is kept constant. This flow is called a doublet. Find stream and potential functions of a doublet.

\[
\Psi = -K \frac{r \sin \theta}{(r^2 - a^2)}
\]

Let \( a \to 0 \)

\[
\Psi = -\frac{K \sin \theta}{r}
\]

We can similarly derive

\[
\Phi = \frac{K \cos \theta}{r}
\]
Example 3: Flow around a fixed cylinder

Superimpose a uniform flow and doublet. Design this configuration such that you have a flow around a fixed cylinder of radius \(a\)

\[
\Psi = \Psi_{\text{uniform}} + \Psi_{\text{doublet}} = Uy - \frac{K \sin \theta}{r} = Ur \sin \theta - \frac{K \sin \theta}{r}
\]

\[
= r \sin \theta \left( U - \frac{K}{r^2} \right)
\]

If this is to be flow around a fixed cylinder, on the surface of cylinder, we must have zero radial velocity

\[
\text{@} r = a, u_r = 0 \Rightarrow \frac{\partial \Psi}{\partial \theta} = 0 \Rightarrow U - \frac{K}{a^2} = 0 \Rightarrow K = Ua^2
\]

So

\[
\Psi = Ur \sin \theta \left( 1 - \frac{a^2}{r^2} \right)
\]

Potential flow around a cylinder
Example 3: Flow around a fixed cylinder contd.

How does pressure changes in the previous example. Using the expression for pressure find the drag and lift forces:

Applying Bernoulli’s equation:

\[ \frac{p_s}{\rho} + \frac{1}{2} u_\theta^2 = \frac{p_\infty}{\rho} + \frac{1}{2} U_\infty^2 \]

\[ p_s - p_\infty = \frac{1}{2} \rho U_\infty^2 (1 - 4 \sin^2 \theta) \]

Lift = \( F_y = - \int_0^{2\pi} p_s \sin \theta \, a \, d\theta = 0 \) 
(because of symmetry)

Drag = \( F_x = - \int_0^{2\pi} p_s \cos \theta \, a \, d\theta = 0 \) 
(d’Alembert’s paradox)
Example 4: Flow around a rotating cylinder

Let's add a free vortex to the stream function for the flow around a cylinder.

\[ \Psi = \Psi_{\text{cylinder}} + \Psi_{\text{vortex}} = Ur \sin \theta \left(1 - \frac{a^2}{r^2}\right) - \frac{m}{2\pi} \ln r \]

Find tangential velocity at \( r = a \)

\[ u_{\theta s} = - \frac{\partial \Psi}{\partial r} = -2U \sin \theta + \frac{m}{2\pi a} \]

\[ u_{rs} = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = 0 \]

This resembles potential flow around a rotating cylinder. At \( \theta = \sin^{-1} \left(\frac{m}{4\pi Ua}\right) \), flow is stationary. We call this stagnation point.
Example 4: Flow around a rotating cylinder contd

Let’s find Pressure

\[ p_s - p_\infty = \frac{1}{2} \rho U^2 \left( 1 - 4 \sin^2 \theta + \frac{2m \sin \theta}{\pi a U} - \frac{m^2}{4\pi^2 a^2 U^2} \right) \]

Now if we calculate Drag and Lift

Drag = \( F_x = - \int_{0}^{2\pi} (p_s - p_\infty) \cos \theta \, d\theta = 0 \)

and

Lift = \( F_y = -(p_s - p_\infty) \sin \theta \, d\theta = -\rho U m \)

A counter clockwise rotation create a downward lift. The development of this lift is called *Magnus effect*. 