# Math 605D Exercises 1: due October 9th

**Problem 1:**

Consider the steady plane Poiseuille flow of a Bingham fluid (rheological parameters: $\hat{\eta}_\infty, \hat{\tau}_Y$), between 2 parallel plates separated a distance $2\hat{d}$. The mean imposed velocity is $\hat{U}_0$.

a) Simplify the full 2D equations of motion to give a 1D momentum balance in the x-direction and a constitutive equation involving only the shear stress and shear strain rate.

b) Scale the equations using $\hat{\eta}_\infty \hat{U}_0/\hat{d}$ as your stress scale, $\hat{d}$ for lengths and $\hat{U}_0$ for velocity. Your equations should have only the Bingham number, $B = (\hat{\tau}_Y \hat{d})/(\hat{\eta}_\infty \hat{U}_0)$.

c) Integrate the momentum balance to find the shear stress in terms of the dimensionless wall shear stress (say $\tau_w$). Show that the velocity is given by:

$$U(y) = \begin{cases} \frac{B}{2y_Y}(1-y_Y)^2, & |y| \le y_Y \\ \frac{B}{2y_Y}[(1-y_Y)^2 - (|y|-y_Y)^2], & y_Y < |y| \le 1 \end{cases}$$

where $y_Y = B/\tau_w$ is the (unknown) position of the yield surface.

d) As you have scaled with the mean flow, $\int_0^1 U(y)dy = 1$. Use this flow rate constraint to derive the following cubic for $y_Y$, which has a single root in [0,1].

$$0 = y_Y^3 - 3y_Y\left(1 + \frac{2}{B}\right) + 2$$

e) As $B \to 0$, show that $y_Y \sim a_1 B + a_2 B^2 + ..$ . Similarly show that as $B \to \infty$, show that $y_Y \sim c_0 + c_1 B^{-1/2} + c_2 B^{-1} + ..$ (i.e. find the coefficients in each case).

**Problem 2:**

Consider the **axial** flow of a Bingham fluid between two concentric cylinders of radii $\hat{R}_1$ and $\hat{R}_2$ driven by a constant pressure gradient $\hat{p}_z = -\hat{G}$, in the z-direction.

a) Write down the Navier Stokes equations for a Bingham fluid, specifying the components of the deviatoric stress and strain rate tensors in cylindrical coordinates

b) Scale the equations using $\hat{R}_2$ as a length-scale, $\hat{G}\hat{R}_2$ as the stress scale, $\hat{G}\hat{R}_2^2/\hat{\mu}$ as the velocity scale. Show that 3 dimensional groups result: a Reynolds number, a Bingham number ($B = \hat{\tau}_Y/\hat{G}\hat{R}_2$) and a radius ratio.

c) Simplify the equations for the described case of axial annular flow, by assuming that $\boldsymbol{u}=(0,0,W(r))$ and that $\tau_{zr}$ is the only non-zero component of the deviatoric stress. These equations should depend only on the Bingham number and radius ratio.

d) Solve for the shear stress $\tau_{zr}$ and hence show that the flow either has 2 yield surfaces within the annulus or is static, according to the value of *B*. Derive an algebraic expression for the positions of the yield surfaces and for the critical value of *B* above which there is no flow.

e) Solve the above algebraic expression numerically for radius ratio of 0.5 and plot the solution $W(r)$ for $B=0.75B_c$, $0.5B_c$, $0.25B_c$.

**Problem 3:**

Consider the **azimuthal** flow of a Bingham fluid between two cylinders of radii $\hat{R}_1$ and $\hat{R}_2$, that rotate about a common axis with angular speeds $\hat{\Omega}_1>0$ and $\hat{\Omega}_2$, respectively. Use the gap width $\hat{d}=\hat{R}_2-\hat{R}_1$ and the inner cylinder angular velocity to define suitable scales that make the Navier-Stokes equations dimensionless, in the following way:

$$u_t + Re_1(u\cdot\nabla)u = -\nabla p + \nabla\cdot\tau, \qquad Re_1 = \frac{\hat{\rho}\hat{R}_1\hat{\Omega}_1\hat{d}}{\hat{\mu}_p}.$$

$$\nabla\cdot u = 0,$$

$$\tau_{ij} = \left(1+\frac{B}{\dot{\gamma}}\right)\dot{\gamma}_{ij} \iff \tau > B, \qquad B = \frac{\hat{\tau}_y\hat{d}}{\hat{\mu}_p\hat{R}_1\hat{\Omega}_1}$$

$$\dot{\gamma} = 0 \iff \tau \leqslant B,$$

$$Re_2 = \frac{\hat{\rho}\hat{R}_2\hat{\Omega}_2\hat{d}}{\hat{\mu}_p}, \qquad \eta = \frac{\hat{R}_1}{\hat{R}_2}.$$

These equations have a steady 1D azimuthal velocity solution in which the fluid flows azimuthally with velocity V(r).

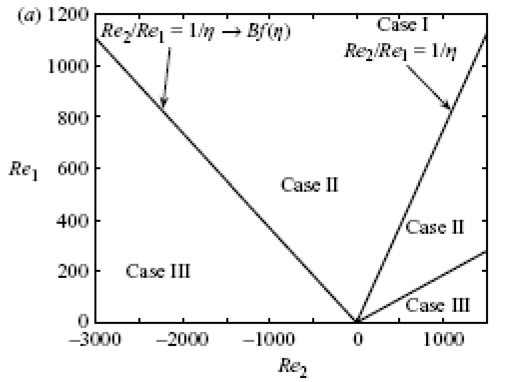
a) Write down the 1D azimuthal momentum equation, constitutive laws and boundary conditions satisfied by V(r)

b) Suppose that the (dimensionless) inner fluid wall shear stress is specified: $\tau_{r\theta}=\tau_i$. Integrate the momentum equation to the outer wall and show that the following 3 types of flow may be found:

| | | |
|---|---|---|
| I. | Solid body rotation | $\frac{\lvert\tau_i\rvert}{B} \leqslant 1,$ |
| II. | Partial plug | $1 < \frac{\lvert\tau_i\rvert}{B} \leqslant \left(\frac{1}{\eta}\right)^2,$ |
| III. | No plug | $\left(\frac{1}{\eta}\right)^2 < \frac{\lvert\tau_i\rvert}{B}.$ |

c) Find V(r) in these 3 cases and plot a solution in each case

d) Show that in the plane of ($Re_1$,$Re_2$) the different solution types occur in regimes that look like:




where the boundary between case II and case III is given by:

$$\frac{Re_2}{Re_1} = \frac{1}{\eta} \pm Bf(\eta). \qquad\qquad f(\eta) = \frac{1+\eta}{2\eta^2} - \frac{\ln(1/\eta)}{1-\eta},$$