Visco-plastic Fluid Mechanics: L1
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From Cauchy to Navier-Stokes

- Continuum mechanics perspective, Cauchy:
  \[ \rho \mathbf{a} = \nabla \cdot \mathbf{\sigma} + \rho \mathbf{g} \]
  - \( \rho \) = density; \( \mathbf{a} \) = acceleration; \( \mathbf{\sigma} \) = stress tensor; \( \mathbf{g} \) = gravitational acceleration

- For convective accelerations & incompressible flows
  \( \nabla \cdot \mathbf{u} = 0 \)

  \[ \rho \frac{d\mathbf{u}}{dt} = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g} \]
  - \( p \) = pressure; \( \mathbf{u} \) = velocity; \( \mathbf{\tau} \) = deviatoric stress tensor
  - Material (convective) derivative and outer product

  \[ \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}), \quad \mathbf{u} \otimes \mathbf{v} = \mathbf{u} \mathbf{v}^T \]
Navier-Stokes

- Eulers equations: 1750’s (no deviatoric stress)
- Cauchy’s equation 1820’s (Cauchy stress tensor)

\[ \boldsymbol{\sigma} = -p \mathbf{I} + \boldsymbol{\tau}, \quad \sigma_{ij} = -p \delta_{ij} + \tau_{ij} \]

\[ p = -\frac{1}{3} \sigma_{ii}, \quad \tau_{ii} = 0 \]

- Navier/Stokes: specific form of \( \tau \) for viscous fluid
  - Galilean invariant: does not depend directly on velocity
  - Depends only on local variations in velocity, i.e. assumed linear dependence on velocity gradients
  - Fluid is isotropic, hence \( \tau \) is an isotropic tensor

- Stokes **constitutive law** for Newtonian fluid

\[ \boldsymbol{\tau} = \mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right], \quad \tau_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \]
Usual simplifications for Newtonian fluids

- Viscosity is constant
- Flow incompressible
- Navier-Stokes equations:
  \[ \nabla \cdot \tau = \nabla \cdot \left( \mu \left[ \nabla u + \nabla u^T \right] \right) = \mu \nabla^2 u \]
  \[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + g \]
  \[ \nabla \cdot u = 0 \]
  \[ \nu = \frac{\mu}{\rho} = \text{the kinematic viscosity (a diffusivity)} \]

- Non-Newtonian fluid? A fluid that does not satisfy:
  \[ \tau = \mu \left[ \nabla u + \nabla u^T \right] \]
  - Mathematically, fluid needs closure law (9 unknowns, 4 equations)
  - Physical and mathematical constraints on \( \tau \), e.g. frame invariance
  - Constitutive law for \( \tau \) must reflect actual mechanical behaviour
Non-Newtonian fluids approaches

- **Mechanical**
  - Traditionally a continuum level description of fluids in terms of PDE’s
  - Constitutive models
  - Different physically and mathematically admissible departures from linear Newtonian law

- **Physico-chemical**
  - Build micro-scale models based on molecular theories
    - Not necessarily PDE based
  - Upscaling/homogenization, or work directly with micro-scale

- **Phenomenological**
  - What new types of flow do we observe?
  - Are they explainable with different types of model
  - How do we classify different types of fluid

- **Rheometrical**
  - Design specific flows that allow us to measure either parameters or generic features of fluid
    - Device = rheometer
  - Use rheometer in different ways to determine underlying fluid behaviour
Polymers

- Polymer solutions, e.g. HDPE, LDPE
  - Concentrated or dilute
  - Linear or branched, end groups, cross-linked, networked
  - Synthetic or biological
    - Molecular weights: water \( \approx 18 \text{ g/mol} \); synthetic polymers \( 10^4-10^6 \text{ g/mol} \); biological polymers up to \( 10^8 \text{ g/mol} \)
  - Often modeled as polymer phase + solvent phase. Mechanical properties of polymer that lead to non-Newtonian behaviours, e.g. memory, normal stresses, nonlinearity

https://science-in-chains.natfak2.uni-halle.de/microscopic-insight/
http://www.globalspec.com/reference/34787/203279/chapter-5-cyclic-polysiloxanes
Suspensions

- **Classifications:**
  - Concentrated or dilute
  - Monodisperse or polydisperse
    - Particle shape & size, e.g. fibers to spheres
    - Particle mechanics: flexible, elastic, hard
  - Brownian, Stokesian, inertial
  - Colloidal or non-colloidal
  - Active and passive, smart

- **Different origins, e.g.**
  - Mined suspensions
  - Geophysical (mud slides, avalanches...)
  - Polymer solutions, food, drink, cosmetics
  - Pulp fiber suspensions, oilfield fluids
  - ER/MR fluids, biological, effluent
  - Model laboratory suspensions


Other

- **Emulsions**
  - Dispersed & continuous phase; liquid in liquid

- **Liquid foams**
  - Gas bubbles bordered by metastable lamellar liquid films
  - Typically high void fractions, e.g. 85-95%

- **Bubbly liquids**
  - Gas in liquid, separated by bulk liquid

- **Granular flows**
  - Solids in gas at high volume fraction
  - E.g. dense inertial granular media modelled with $\mu(I)$ “rheology”
    - Mathematical analogy
Non-Newtonian phenomena 1

- Viscosity varies with shear rate
- Shear-thinning is most common
  - In transporting fluids, pressure drop increases less than linearly with flow rate
  - Newtonian: linear variation
- Experiment: 2 fluids of same density

A: small particle settling
B: draining of tube under gravity
Non-Newtonian phenomena 2

- **Shear generates normal stress differences**
  - The normal stress differences generate observable flow effects
    - Viscoelastic effects
    - Note: isotropic normal stresses (pressure) only generate motion via gradients

- **Examples:**
  - Rod climbing (Weisssenberg effect)
  - Die swell
  - Tanner’s tilted trough
  - Hole pressure effect

![Diagram of flow](image)

*Bird, Armstrong, Hassager, "Dynamics of Polymeric Liquids, Vol. 1"*
Non-Newtonian phenomena 3

- Extensional and memory effects
  - Elastic recoil
  - Tubeless siphon
  - Relaxation timescales in rheometry

- Examples
  http://web.mit.edu/nnf/
Non-Newtonian phenomena 4

- Negative wakes & velocity jumps/hysteresis, e.g.
  - Fraggedakis et al 2016
Non-Newtonian phenomena 5

- Viscoelastic turbulence
- Yielding and plug zones
- Secondary flows in rotating spheres & lid-driven cavities
- Filament formation and instability – beads on a string
- Bubbles shapes
- Drag reduction in turbulent flows
Types of fluids

- **Generalised Newtonian fluids**
  - Shear-thinning/pseudo-plastic
  - Shear-thickening/dilatant
  - Visco-plastic

- **Viscoelastic fluids**
  - The stress tensor evolves dynamically in time, typically satisfying a differential or integral equation
  - Elastic effects: memory, relaxation, swelling...

- **Thixotropic fluids**
  - Model parameters are functions of a “structural” variable
  - Structural variable varies due to destruction and reforming/growth of structure

- **Most real fluids combine a number of the above**

The viscosity depends nonlinearly on the invariants of the rate of strain tensor.
Announcement of the grand opening of

RHEOLOGY DRUGSTORE

Our motto: “Fit The Data”
Proprietor: Daniel D. Joseph

“To make your experiment agree with your theory you should have the right fluids.”

We carry many different fluids, corresponding to the thirty or forty models currently considered most realistic.

Standard brandname Fluids (well advertised):

- Maxwell
- Jeffreys
- BKZ
- KBKZ
- Doi-Edwards
- Curtiss-Bird
- White-Metzner
- Phan Thien-Tanner
- Newtonian
- Reiner-Rivlin
- Johnson-Segalman
- Lodge’s
- Green-Tobolsky
- Oldroyd
- Giesekus

Graded Fluids:

- Single integral
- Multiple order integral
- 1st, 2nd, 3rd order, etc.
- Fluids of complexity 1, 2, 3, etc.

Composite Fluids:

- With Springs and Dumbbells
- With Beads and Chains
- With Reptating Snakes

Retarded fluids with a strong backbone and fading memory

Mathematician’s Delight:

- Models with 1, 2, or 3 Fréchet derivatives
- Less good fluids with only 1, 2, or 3 Gateaux derivatives

Less expensive fluids:

- Liquid gold
- Milky Way dust
- Water with c=1 cm/sec
Rod climbing – mechanism?

- Flow is observed to be steady & axisymmetric with velocity $v(r)$ in azimuthal direction
  - Consider a deep container, to neglect $z$-gradients

$$-\frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \tau_{rr}\right] + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{r\theta} + \frac{\partial}{\partial z} \tau_{rz} - \frac{\tau_{\theta\theta}}{r}$$

$$0 = -\frac{1}{r^2} \frac{\partial}{\partial \theta} \left[r^2 \tau_{\theta r}\right] + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \tau_{\theta r}\right]$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$
Newtonian fluids?

- Consider distribution of $p$ along a horizontal plane immersed in fluid
  - Solve $\theta$ equation with boundary conditions to give azimuthal $v(r)$
  - Integrate $r$ equations:
    
    
    \[ -\frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{\tau_{\theta\theta}}{r} \]

    \[ 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \tau_{\theta r} \right] \]

    \[ 0 = -\frac{\partial p}{\partial z} - \rho g \]

- Pressure increases with $r$
  - Fluid rises near wall to compensate via static pressure
  - Constant pressure at surface, might be used to find shape

\[ p(r, z) = \int_{r_i}^{r} \frac{\rho v(\tilde{r})^2}{\tilde{r}} \tilde{r} d\tilde{r} + k(z) \]
Viscoelastic fluid

- Consider distribution of $p-\tau_{zz}$ along a horizontal plane immersed in fluid
  - Use radial momentum equation:
    \[
    \frac{\partial}{\partial r} [p - \tau_{zz}] = -\frac{\partial \tau_{zz}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} [r \tau_{rr}] - \frac{\tau_{\theta \theta}}{r} + \frac{\rho v^2}{r}
    \]
    \[
    = -\left[\frac{\tau_{\theta \theta} - \tau_{rr}}{r}\right] + \frac{\partial}{\partial r} \left[\tau_{rr} - \tau_{zz}\right] + \frac{\rho v^2}{r}
    \]
  - Distribution of $p-\tau_{zz}$ is due to varying normal stress differences with $r$
  - Change of variables (exercise)
    \[
    \frac{\partial}{\partial \ln r} [p - \tau_{zz}] = \rho v^2 - \left[\tau_{\theta \theta} - \tau_{rr}\right] - 2\tau_{\theta r} \frac{\partial}{\partial \tau_{\theta r}} \left[\tau_{rr} - \tau_{zz}\right]
    \]

\[
- \frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} [r \tau_{rr}] - \frac{\tau_{\theta \theta}}{r}
\]
\[
0 = \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \tau_{\theta r}]
\]
\[
0 = -\frac{\partial p}{\partial z} - \rho g
\]
\[
\frac{\partial}{\partial \ln r} [p - \tau_{zz}] = \rho v^2 - [\tau_{\theta\theta} - \tau_{rr}] - 2\tau_{\theta r} \frac{\partial}{\partial \tau_{\theta r}} [\tau_{rr} - \tau_{zz}]
\]

- \([\tau_{\theta\theta} - \tau_{rr}]\) is called 1\(^{\text{st}}\) normal stress difference \(N_1\)
  - 1\(^{\text{st}}\) implies \(\tau_{\theta\theta}\) in direction of flow minus \(\tau_{rr}\) in direction of shear

- \([\tau_{rr} - \tau_{zz}]\) is called 2\(^{\text{nd}}\) normal stress difference \(N_2\)
  - 2\(^{\text{nd}}\) implies \(\tau_{rr}\) in direction of shear minus \(\tau_{zz}\) in direction of orthogonal to shear plane

- If \(N_1 > 0\) it can counteract effects of inertia so that \(p - \tau_{zz}\) decreases with \(r\)
  - Fluid is pushed up near the centre
  - Effects of \(N_2\) are more complex
  - Suggestion that \(N_2\) depending on shear stress

(Images of Newtonian liquid and Polymeric fluid)
2nd Problem: free surface deformation

- Inclined open channel
  - Flow in x-direction
  - Surface bulges upwards

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \cos \beta \\
0 &= -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\
0 &= -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \sin \beta
\end{align*}
\]
**2\textsuperscript{nd} Problem: free surface deformation**

- **Assumptions and analysis**
  - $\tau_{ij}$ depend only on $y$, if channel is deep
  - $|\tau_{zy}| << |\tau_{xy}|$
  - $p$ is independent of $x$

  \[ \tau_{xy} = -y\rho g \cos \beta \]

  \[
  \begin{align*}
  \frac{\partial}{\partial y} [p - \tau_{zz}] &= \frac{\partial}{\partial y} [\tau_{yy} - \tau_{zz}] \\
  \frac{\partial}{\partial z} [p - \tau_{zz}] &\approx \frac{\partial p}{\partial z} = -\rho g \sin \beta
  \end{align*}
  \]

  - At free surface $z = z_i(y)$, $p - \tau_{zz}$ balances ambient pressure:

    \[
    p_a = [\tau_{yy} - \tau_{zz}] (y) - \rho g z_i(y) \sin \beta + C
    \]

    \[
    z_i(y) = \left( \frac{[\tau_{yy} - \tau_{zz}] (y) + C - p_a}{\rho g \sin \beta} \right)
    \]
2\textsuperscript{nd} Problem: free surface deformation

- Again see that normal stress difference is responsible for free surface variation
  - $\tau_{yy} - \tau_{zz} = N_2(y) = 2\textsuperscript{nd}$ normal stress difference
  - At centre, expect $\tau_{yy} - \tau_{zz} = 0$, as velocity gradients vanish
  - At wall, $z_i = 0$

- Positive bulge implies negative $N_{2,\text{wall}}$
  - Note that with no shear, there is no bulging
  - Amount of shear at the wall: $\tau_{xy} = -y \rho g \cos \beta$
  - Measuring $z_i(0)$ provides a way of estimating $N_2(\tau_{xy})$

\[ p_a = \left[ \tau_{yy} - \tau_{zz} \right](y) - \rho g z_i(y) \sin \beta + C \]
\[ z_i(y) = \frac{N_2(y) + C - p_a}{\rho g \sin \beta} \]
\[ N_{2,\text{wall}} = p_a - C \]
\[ z_i(0) = -\frac{N_{2,\text{wall}}}{\rho g \sin \beta} \]
Viscometric functions

- First viscometric function is the effective viscosity $\eta$
- Two other viscometric functions are used to characterize normal stress differences
  - Normal stress differences generated by shear:
    \[
    \tau_{11} - \tau_{22} = N_1(\dot{\gamma}) = \psi_1(\dot{\gamma})\dot{\gamma}^2
    \]
    \[
    \tau_{22} - \tau_{33} = N_2(\dot{\gamma}) = \psi_2(\dot{\gamma})\dot{\gamma}^2
    \]
  - $\psi_1$ = primary normal stress coefficient
  - $\psi_2$ = secondary normal stress coefficient
  - Even functions of the shear rate
  - For polymeric liquids (and more generally) we find $\psi_1 > 0$
  - For polymeric liquids usually $\psi_2 < 0$, and is significantly smaller than $\psi_1$

Source: Thermopedia
$N_1 =$ part of standard suite of rheometric tests and characterizations for general complex fluids

**Example:** CTAB, wormlike micellar solution

**FIG. 1.** Linear viscoelastic moduli for 16.7 wt % CTAB in D$_2$O at 32 °C. Lines give fits to Eq. (6). Inset shows Cole–Cole representation of the data and fit.

**FIG. 2.** Steady state shear viscosity (top) and stress (bottom) for 16.7 wt % CTAB in D$_2$O at 32 °C. Both strain controlled (squares) and stress controlled (circles) measurements are shown. Lines give corresponding predictions from the G-D model under viscometric (dashed) and inhomogeneous (solid) flow.

Hegelson et al., J. Rheol. 2009
Deborah and Weissenberg

• Deborah number

\[ De = \frac{\lambda}{t_{\text{flow}}} = \frac{\text{Timescale of fluid}}{\text{Timescale of flow}} \]

• \( \lambda \) depends on fluid
  – Largest timescale describing slowest molecular motions
  – Characteristic of rheometry test (e.g. relaxation time)
  – Term in a constitutive law

• Flow timescale more ambiguous
  – Inverse of a representative strain rate
  – Experimental duration
  – Convective or viscous timescale

• Weissenberg number

\[ Wi = \lambda \dot{\gamma}_{\text{flow}} \left( = \frac{\tau_{11} - \tau_{22}}{\tau_{12}} = \frac{\text{Elastic stresses}}{\text{Viscous stresses}} \right) \]

  • \( \lambda \) depends on fluid, as with De
  • For some flows De and Wi can be same
  • For simple shear and some models the elastic:viscous balance is evident

• De is not specific to non-Newtonian fluids, but relates fluid timescale to timescales of interest
  • If \( t_{\text{flow}} \) very large then \( De = 0 \)