Variational Methods for Stokes Flow
Incompressible flow setting

- Domain $\Omega$, length & velocity scales:
  \[ x = \frac{x}{\hat{L}}, \quad t = \frac{t\hat{U}_0}{\hat{L}}, \quad u = \frac{\hat{u}}{\hat{U}_0}, \quad p = \frac{\hat{p}\hat{L}}{\hat{\mu}_0\hat{U}_0}, \quad \tau_{ij} = \frac{\hat{\tau}_{ij}\hat{L}}{\hat{\mu}_0\hat{U}_0}, \quad f_i = \frac{\hat{f}_i\hat{L}^2}{\hat{\mu}_0\hat{U}_0}, \]  

- Scaled Navier-Stokes
  \[ \text{Re} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i, \]
  \[ 0 = \frac{\partial u_i}{\partial x_i}. \]  

- Constitutive law:
  \[ \tau_{ij}(u) = \left[ \eta_v(\dot{\gamma}(u)) + \frac{B}{\dot{\gamma}(u)} \right] \dot{\gamma}_{ij}(u) \iff \tau(u) > B, \]
  \[ \dot{\gamma}(u) = 0 \iff \tau(u) \leq B, \]
Ideal visco-plastic fluids

\[ \tau_{ij}(\mathbf{u}) = \left[ \eta_v(\dot{\gamma}(\mathbf{u})) + \frac{B}{\dot{\gamma}(\mathbf{u})} \right] \dot{\gamma}_{ij}(\mathbf{u}) \quad \iff \quad \tau(\mathbf{u}) > B, \]
\[ \dot{\gamma}(\mathbf{u}) = 0 \quad \iff \quad \tau(\mathbf{u}) \leq B, \]

- **Effective viscosity:** \( \eta(\dot{\gamma}) = \eta_v(\dot{\gamma}) + B/\dot{\gamma} \)
- **Bingham number:** \( B = \frac{\dot{\tau}_Y \hat{L}}{\hat{U}_0 \hat{\mu}_0} \)
- **Viscous part of effective viscosity**
  - Bingham
  - Casson
  - Herschel-Bulkley

\[ \dot{\gamma}(\mathbf{u}) = \left[ \frac{1}{2} \sum_{i,j=1}^{3} [\dot{\gamma}_{ij}(\mathbf{u})]^2 \right]^{1/2} \]
\[ \tau(\mathbf{u}) = \left[ \frac{1}{2} \sum_{i,j=1}^{3} [\tau_{ij}(\mathbf{u})]^2 \right]^{1/2} \]
Essential parts of these models:

- Yield stress (dimensionless version): \( B \)
  - Von-Mises yield criterion
- Increasing flow curve
- Nonlinear effective viscosity for \( \tau > B \)
  - Not very different from e.g. power law, Carreau, etc fluid models
- Effective viscosity approaches \( \infty \) as \( \tau \to B \)
- Strain rate approaches 0 as \( \tau \to B \)
- Plug regions for \( \tau < B \)
  - Kinematically: rigid body motions
- Can we directly model the 2 regions?
Classical formulation?

Momentum conservation, linear & angular

\[ 0 = \int_{\Omega_p(t)} \left( \dot{u}_{c,i} + \epsilon_{ijk}(x_j - x_{c,j})\dot{\omega}_{c,k} \right) \, dx - \int_{\Gamma(t)} \sigma_{ij} n_j \, ds \]
\[- \int_{\Omega_p(t)} f_i \, dx + \int_{\Gamma(t)} \left( u_{c,i} + \epsilon_{ijk}(x_j - x_{c,j})\omega_{c,k} \right) \times \]
\[ \left[ (u_{c,l} + \epsilon_{lmn}(x_m - x_{c,m})\omega_{c,n}) n_l - u_p \right] \, ds \]

\[ 0 = \int_{\Omega_p(t)} \epsilon_{ijk}[x_j - x_{c,j}][\dot{u}_{c,k} + \epsilon_{klm}(x_l - x_{c,l})\dot{\omega}_{c,m}] \, dx - \]
\[ \int_{\Omega_p(t)} \epsilon_{ijk}[x_j - x_{c,j}] f_k \, dx + \]
\[ \int_{\Gamma(t)} \epsilon_{ijk}[x_j - x_{c,j}][u_{c,k} + \epsilon_{klm}(x_l - x_{c,l})\omega_{c,m}] \times \]
\[ \left[ (u_{c,l} + \epsilon_{lmn}(x_m - x_{c,m})\omega_{c,n}) n_l - u_p \right] \, ds. \]
\[- \int_{\Gamma(t)} \epsilon_{ijk}(x_j - x_{c,j})\sigma_{kl} n_l \, ds \]

At yield surface:
- Continuity of velocity
- Continuity of traction

What is \( u_p \)?
- Assumed \( u_p = u_p n \)

No “evolution” equation for yield surface
Classical vs weak formulation

- For general flows (2D, 3D transient) classical formulation is impossible to use
  - Useable computationally with viscosity regularization
  - What is wrong with regularization?
    - Rigid regions are a key features of the yield stress
    - Velocity is known to converge as $\epsilon \to 0$; stress convergence is unknown
    - Regularization smooths the flow, but yield surfaces often show geometric features such as corners

- Classical formulation still used for e.g.
  - Analytical solutions (1D, axisymmetric, etc)
  - Aspect ratio/lubrication/thin film problems
  - Some mapping methods for special 2D flows
  - Hydrodynamic stability: linear, weakly nonlinear, sometimes

- More general descriptions of NS equations are weak (variational) formulations

Frigaard & Nouar, JNNFM 2005
Stokes flows:

- More prevalent (high viscosity)
- Sufficient to understand rheological differences with generalized Newtonian fluids

\[
\sigma_{ij} n_j = g_i \]

\[
\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega.
\]

\[
\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{f}, \text{ in } \Omega,
\]

\[
\sigma = -p\delta + \mathbf{\tau}
\]

\[
\tau(\mathbf{u}) = \left[ \eta_v(\dot{\gamma}(\mathbf{u})) + \frac{B}{\dot{\gamma}(\mathbf{u})} \right] \dot{\gamma}(\mathbf{u}) \iff \tau(\mathbf{u})
\]

\[
\dot{\gamma}(\mathbf{u}) = 0 \iff \tau(\mathbf{u})
\]

\[
a : b = \frac{1}{2} a_{ij} b_{ij}
\]

\[
\dot{\gamma} = \sqrt{\dot{\gamma} : \dot{\gamma}}, \quad \mathbf{\tau} = \sqrt{\mathbf{\tau} : \mathbf{\tau}}.
\]
• Admissible velocities and stresses

\[ \mathcal{V} = \{ \mathbf{v} : \mathbf{v} = \mathbf{u}_s \text{ on } \partial \Omega_v; \quad \nabla \cdot \mathbf{v} = 0, \text{ in } \Omega \} . \]

\[ \mathcal{S} = \{ \tilde{\sigma} = -\tilde{\rho} \delta + \tilde{\tau} : \tilde{\sigma} \cdot \mathbf{n} = s_s \text{ on } \partial \Omega_t; \quad -\nabla \tilde{p} + \nabla \cdot \tilde{\tau} + f \text{ in } \Omega \} \]

• Principle of virtual power:

\[ \mathbf{v} \in \mathcal{V} \text{ and } \tilde{\sigma} \in \mathcal{S}, \]

\[ 0 = \int_{\partial \Omega_v} u_{s,i} \tilde{\sigma}_{ij} n_j \, ds + \int_{\partial \Omega_t} v_i g_i \, ds + \int_{\Omega} f_i v_i \, dx - \int_{\Omega} \tilde{\tau} : \dot{\gamma}(\mathbf{v}) \, dx \]

• 2 alternate forms also useful

(i) If \( \sigma(\mathbf{u}) \) is the stress associated with the solution \( \mathbf{u} \) and if \( \mathbf{v} \in \mathcal{V} \),

\[ \int_{\Omega} \tau(\mathbf{u}) : [\dot{\gamma}(\mathbf{v}) - \dot{\gamma}(\mathbf{u})] \, dx = \int_{\partial \Omega_t} (\mathbf{v} - \mathbf{u}) \cdot \mathbf{g} \, ds + \int_{\Omega} f \cdot (\mathbf{v} - \mathbf{u}) \, dx \]

(ii) If \( \tilde{\sigma} \in \mathcal{S} \):

\[ \int_{\Omega} [\tau(\mathbf{u}) - \tilde{\tau}] : \dot{\gamma}(\mathbf{v}) \, dx = \int_{\partial \Omega_v} u_{s,i} (\sigma_{ij} - \tilde{\sigma}_{ij}) n_j \, ds. \]
Mechanical energy balance:

• Simply the principle of virtual power, evaluated at the solution \((u, \sigma)\):

\[
\int_{\Omega} \tau(u) : \dot{\gamma}(u) \, dx = \int_{\partial \Omega_v} u_{s,i} \sigma_{ij} n_j \, ds + \int_{\partial \Omega_t} g \cdot u \, ds + \int_{\Omega} f \cdot u \, dx
\]

- Rate of visco-plastic dissipation of energy
- Work done at the boundary
- Work done by body forces

• In dimensional form, each term has dimension of power
1. Velocity minimization

\[ \phi(\dot{\gamma}) = \int_{0+}^{\dot{\gamma}} \tau(s) \, ds = \int_{0+}^{\dot{\gamma}} \eta_v(s) \, ds + B \dot{\gamma} = \phi_v(\dot{\gamma}) + B \dot{\gamma}. \]

- **Examples:**
  - **Newtonian:** \[ \phi(\dot{\gamma}) = \frac{\dot{\gamma}^2}{2}, \]
  - **Bingham:** \[ \phi(\dot{\gamma}) = \frac{\dot{\gamma}^2}{2} + B \dot{\gamma}, \]
  - **Herschel-Bulkley:** \[ \phi(\dot{\gamma}) = \frac{\dot{\gamma}^{n+1}}{n + 1} + B \dot{\gamma}, \]
  - **Casson:** \[ \phi(\dot{\gamma}) = \frac{\dot{\gamma}^2}{2} + \frac{4B^{1/2}\dot{\gamma}^{3/2}}{3} + B \dot{\gamma}. \]

- **Velocity potential:**
  \[ \Phi(v) = \int_{\Omega} \phi(\dot{\gamma}(v)) \, dx - \int_{\Omega} f \cdot v \, dx - \int_{\partial\Omega_t} v \cdot g \, ds. \]
Theorem 2.1. The solution $u$ minimizes $\Phi(v)$ over $\mathcal{V}$.

Proof. We need to show that $\Phi(v) \geq \Phi(u)$, $\forall v \in \mathcal{V}$.

$$
\Phi(v) - \Phi(u) = \int_\Omega \phi(\dot{\gamma}(v)) - \phi(\dot{\gamma}(u)) \, dx - \int_\Omega f \cdot (v - u) \, dx \\
- \int_{\partial \Omega} (v - u) \cdot g \, ds \\
= \int_\Omega \phi(\dot{\gamma}(v)) - \phi(\dot{\gamma}(u)) \, dx - \int_\Omega \tau(u) : [\dot{\gamma}(v) - \dot{\gamma}(u)] \, dx \\
\geq \int_\Omega \phi(\dot{\gamma}(v)) - \phi(\dot{\gamma}(u)) - \tau(u)\dot{\gamma}(v) + \tau(u)\dot{\gamma}(u) \, dx \\
= \int_\Omega \left( \int_{\dot{\gamma}(u)}^{\dot{\gamma}(v)} [\tau(s) - \tau(u)] \, ds \right) \, dx \geq 0.
$$
2. Stress maximization:

\[ \dot{\gamma} = \dot{\gamma} = B + \eta \dot{\gamma} \ddot{\gamma} \]

\[ \lambda(\dot{\gamma}) \ddot{u} + \ddot{q} = \ddot{d} \]

\[ \lambda(\dot{\gamma}) \ddot{u} + \ddot{q} = \ddot{d} \]

\[ \dot{\gamma}(\tau) = \frac{1}{2} \psi(\tau) \]

\[ \psi(\tau) = \int_{B}^{\tau} \dot{\gamma}(s) \, ds \]
2. Stress maximization:

\[ \psi(\tau) = \int_B^{\tau} \dot{\gamma}(s) \, ds, \]

Newtonian: \[ \psi(\tau) = \frac{\tau^2}{2}, \]

Bingham: \[ \psi(\tau) = \frac{(\tau - B)^2}{2}, \]

Herschel-Bulkley: \[ \psi(\tau) = \frac{(\tau - B)^{1/n+1}}{1/n+1}, \]

Casson: \[ \psi(\tau) = \left( \frac{\tau^2}{2} - \frac{4B^{1/2}\tau^{3/2}}{3} + B\tau - \frac{B^2}{6} \right) \]

Stress potential: \[ \Psi(\tilde{\sigma}) = - \int_{\Omega} \psi(\tilde{\tau}) \, dx + \int_{\partial \Omega_v} u_{s,i} \tilde{\sigma}_{ij} n_j \, ds. \]
Theorem 2.2. The solution stress tensor, $\sigma$ maximizes $\Psi(\tilde{\sigma})$ over $S$.

Proof. Let the velocity solution be $u$ with associated stress $\sigma = -p\delta + \tau$. We need to show that $\Psi(\sigma) \geq \Psi(\tilde{\sigma})$, $\forall \tilde{\sigma} \in S$.

$$
\Psi(\sigma) - \Psi(\tilde{\sigma}) = \int_{\Omega} \psi(\tilde{\tau}) - \psi(\tau) \, dx + \int_{\partial\Omega_v} u_{s,i} (\sigma_{ij} - \tilde{\sigma}_{ij}) n_j \, ds,
$$

$$
= \int_{\Omega} \psi(\tilde{\tau}) - \psi(\tau) + (\tau(u) - \tilde{\tau}) : \dot{\gamma}(u) \, dx,
$$

$$
\geq \int_{\Omega} \psi(\tilde{\tau}) - \psi(\tau) + (\tau(u) - \tilde{\tau})\dot{\gamma}(u) \, dx
$$

$$
= \int_{\Omega} \psi(\tilde{\tau}) - \psi(\tau) + (\tau - \tilde{\tau})\dot{\gamma}(\tau) \, dx
$$

$$
= \int_{\Omega} \left( \int_{\tau}^{\tilde{\tau}} [\dot{\gamma}(s) - \dot{\gamma}(\tau)] \, ds \right) \, dx \geq 0.
$$
Theorem 2.3. Let the velocity solution be \( \mathbf{u} \) with associated stress \( \sigma = -p\delta + \tau \). Then for all \( \bar{\sigma} \in \mathcal{S} \) and all \( \mathbf{v} \in \mathcal{V} \) we have:

\[
\Psi(\bar{\sigma}) \leq \Psi(\sigma) = \Phi(\mathbf{u}) \leq \Phi(\mathbf{v}).
\]

Proof. This result combines the velocity minimization and stress maximization, so all that remains to be shown is that \( \Psi(\sigma) = \Phi(\mathbf{u}) \).

\[
\begin{align*}
\Phi(\mathbf{u}) - \Psi(\sigma) &= \int_{\Omega} \phi(\dot{\gamma}(\mathbf{u})) \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x} - \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{g} \, ds \\
&\quad + \int_{\Omega} \psi(\tau) \, d\mathbf{x} - \int_{\partial\Omega_{v}} u_{s,ij} \sigma_{ij} n_{j} \, ds \\
&= \int_{\Omega} \phi(\dot{\gamma}(\mathbf{u})) + \psi(\tau) \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x} - \int_{\partial\Omega} u_{i} \sigma_{ij} n_{j} \, ds \\
&= \int_{\Omega} [\dot{\gamma}\tau](\mathbf{u}) \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} + \frac{\partial}{\partial x_{j}} [u_{i} \sigma_{ij}] \, d\mathbf{x} \\
&= \int_{\Omega} [\dot{\gamma}\tau](\mathbf{u}) \, d\mathbf{x} - \int_{\Omega} \tau(\mathbf{u}) : \dot{\gamma}(\mathbf{u}) \, d\mathbf{x} = 0.
\end{align*}
\]
Uses of the variational expressions

• Basis of results concerning existence & uniqueness of (weak) solutions
• Underlie computational methods
  – Sometimes directly, e.g. FEM
• Allows qualitative analysis of solutions for a whole class of viscoplastic fluids
• Sometimes used to guess solutions and estimate features of flow
• Velocity minimization equivalent to elliptic variational inequality

\[ \langle \Phi'_v(u), v - u \rangle + B \int_\Omega \dot{\gamma}(v) - \dot{\gamma}(u) \, dx \geq L(v - u), \]

• \( \Phi_v(u) \) is viscous part of \( \Phi(u) \)

\[ \langle \Phi'_v(u), v - u \rangle = \int_\Omega \eta_v(\dot{\gamma}(u)) \dot{\gamma}(u) : [\dot{\gamma}(v) - \dot{\gamma}(u)] \, dx \]

\[ L(v) = \int_{\partial \Omega_t} v \cdot g \, ds + \int_\Omega f \cdot v \, dx \]

• E.g. Prop 2.2., chap 5 of Glowinski (1984)

• General theorems for existence & uniqueness of solutions of V.I.

• E.g. Thm 2.1, chap 5 of Glowinski (1984)
Comments:

- Function space of solution is determined by behaviour of $\Phi_v(u)$ at large $\|u\|$
- Method of analysis is well established, methods for weak solutions are standard
  - Extend methods to steady inertial flows
    - Use ellipticity/convexity for small enough Re
  - Unsteady Navier-Stokes:
    - Duvaut & Lions 1976 (also other results)
- Other authors have dealt with other fluids, have studied regularity etc.
  - e.g. Cioranescu 1976; Fuchs & Seregin 2002; Malek et al 2005
Qualitative analysis:

- Simplify problem
  - Bingham fluid
  - Boundary conditions
- Velocity potential
  \[ \Phi(u) = \int_{\Omega} \frac{1}{2} \dot{\gamma}(u) : \dot{\gamma}(u) + B\dot{\gamma}(u) \, dx = \frac{1}{2} \langle u, u \rangle + Bj(u) \]
  - 1\textsuperscript{st} term in V.I. is elliptic bilinear form

What is needed?
- Poincare, Cauchy Schwarz & Korn inequalities
- Results from Temam & Strang 1980
Monotonicity & bulk flow

• Starting point V.I.: \( f_1 \rightarrow u_1 \) & \( f_2 \rightarrow u_2 \)

\[
\langle \Phi'_\nu(u), v - u \rangle + B \int_\Omega \gamma(v) - \gamma(u) \, dx \geq L(v - u).
\]

– \( u_2 \) is a test function for \( u_1 \) & vice versa
– sum 2 variation inequalities, cancel \( j \)

\[
0 \leq \langle u_1 - u_1, u_2 - u_1 \rangle \leq \int_\Omega (u_2 - u_1) \cdot (f_2 - f_1) \, dx.
\]

• This says the integral of \( u \) increases in the direction of \( f \):
  – So what?
Examples

• Flow in a pipe or duct:
  – Single component of velocity, single component of $f$ = applied pressure gradient
  – Flow rate along the pipe increases monotonically with $f$
  – Can show that increase with $f$ is strict, unless the flow rate equals zero

• Extends to flows of 2 fluids along a pipe:
  – Plug-cementing stability
  – Frigaard & Scherzer 1998; Frigaard & Crawshaw 1999; Frigaard & Scherzer 2000; Moyers-Gonalez & Frigaard 2004
Plug cementing stability:

- Fluids must remain static for hours to allow cement to set
- Any negative density difference promotes instability
- Rheology must be designed to keep fluids in place

**Good**

- 1400 kg/m$^3$ Mud
- Spacer

**Bad**

- 1900 kg/m$^3$ Cement
- Pill
- 1400 kg/m$^3$ Mud
From Frigaard & Crawshaw 1999

- Flows of interest are 1D with 2 fluids in a pipe
- Different rheologies and different densities
- How to find these flows?
- How large must yield stresses be in order to stop plug falling?

Figure 2. A slumping exchange flow observed in an experimental rig at Schlumberger Cambridge Research (UK) at different times after placement; density difference 1:1.1 SG, $D = 0.2 \text{ m}$, $\beta = 45$ degrees.
Pipe, horizontal interface, different rheology, no density difference

Pipe, horizontal interface, two yield stress fluids & density difference
Increase pressure gradient until flow rates match

Pipe, horizontal interface, different rheology, no density difference
Other:

- Flows around particles/droplets, etc
  - Yoshioka, Adachi & co-workers, 1970’s
    - Estimates of particle drag
    - Generalisation of Faxen-laws
  - Putz & Frigaard 2010
    - Drag force opposes settling motion
    - Monotonicity implies interchange between *mobility* and *resistance* problems
Exercise: Show that the solutions are continuous with respect to $f$.

Exercise: For Bingham numbers $B_1$ and $B_2$, with corresponding solutions $u_1$ & $u_2$, show that:

\[
\begin{align*}
\langle u_1 - u_1, u_2 - u_1 \rangle & \leq C_1 |B_1 - B_2|^2, \\
\langle \dot{\gamma}(u) \rangle & \leq C_2 |B_1 - B_2|, \\
||u_1 - u_1||_{[H^1(\Omega)]^d} & \leq C_3 |B_1 - B_2|,
\end{align*}
\]
• Show that the plastic dissipation $j(u)$ decreases with $B$, and strictly unless $u=0$

• Show that $-\Psi(\tau(u))$ decreases with $B$, and strictly unless $u=0$