Static flow of visco-plastic fluids

• Definition of a fluid?
  “A continuum that cannot resist a shear stress while at rest.”
  Newtonian fluids: hydrostatics…

• Visco-plastic fluids do not satisfy this definition.

• Static flow = any full solution that does not have non-zero velocity
  – Possible is to have a non-zero stress field, as long as the yield stress is not exceeded
  – How do we identify such situations?
Physical/dimensional setting

- Simplified setting:
  - Bingham fluid
- Flow only driven by body force
  - Velocity scale from balance with viscous stresses:
  - Balance with yield stress:
- For fixed geometry, expect zero velocity for large enough $B > B_{cr}$
Analytical procedure:

- Energy balance:
  \[ \langle \gamma^2(u) \rangle + B \langle \gamma(u) \rangle = \langle u \cdot f \rangle \]
  \[ \langle u \cdot f \rangle \leq ||u||_p ||f||_q \]
  \[ C_{TS} ||u||_p \leq \langle \gamma(u) \rangle \]

- Hölder inequality
  \[ \langle \gamma^2(u) \rangle = \langle |\nabla u|^2 \rangle \]
  \[ C_P \langle u^2 \rangle = C_P ||u||^2 \leq \langle |\nabla u|^2 \rangle = \langle \gamma^2(u) \rangle \]
  \[ C_P ||u||^2 + C_{TS} B ||u||_p \leq ||u||_p ||f||_q \]

- Temam & Strang 1980:
  \[ \text{– For } p \leq \frac{d}{d-1} \]

- Boundary conditions:
  \[ \text{Poincaré inequality:} \]
  \[ \text{Putting it all together, assuming } u \neq 0 \]
  \[ \text{– Contradiction if } B \text{ large} \]
  \[ \text{– Sufficient condition for } u=0 \]

\[ B \geq ||f||_q / C_{TS} \]
Not all forces promote motion

• Ex. 1. \( f=(0,-y) \)  
• Ex. 2. \( f=(0,-x) \)
More precise analysis of forces

• Helmholtz decomposition:
  – Conservative force vanishes:
  – Correct force scaling?

\[ f = -\nabla \phi + \nabla \times A \]

\[ \langle u \cdot f \rangle = \langle u \cdot (\nabla \times A) \rangle \]

\[ \hat{f}_0 = \|\nabla \times A\|_q. \]

\[ \langle \dot{\gamma}^2(u) \rangle = -\langle \dot{\gamma}(u) \rangle \left[ B - \frac{\langle u \cdot f \rangle}{\langle \dot{\gamma}(u) \rangle} \right] \leq -\langle \dot{\gamma}(u) \rangle \left[ B - \sup_{v \in \mathcal{V}, v \neq 0} \left\{ \frac{\langle v \cdot f \rangle}{\langle \dot{\gamma}(v) \rangle} \right\} \right] \]

• Return to energy balance, assuming \( u \neq 0 \)
  – Critical limit:

\[ B_{cr} = \sup_{v \in \mathcal{V}, v \neq 0} \left\{ \frac{\langle v \cdot f \rangle}{\langle \dot{\gamma}(v) \rangle} \right\} = \sup_{v \in \mathcal{V}, v \neq 0} \left\{ \frac{\langle v \cdot (\nabla \times A) \rangle}{\langle \dot{\gamma}(v) \rangle} \right\} \]

  – With above force scaling it follows that: \( B = 1/C_{TS} \)
Two-dimensional flows:

• **Scalar potential:** \( A = (0, 0, A(x, y)) \)

\[
\langle \mathbf{v} \cdot (\nabla \times \mathbf{A}) \rangle = \langle \nabla \psi_v \cdot \nabla A \rangle = -\langle A \nabla^2 \psi_v \rangle = -\langle \psi_v \nabla^2 A \rangle
\]

– Optimality aligns contours of stream-function and \( A(x, y) \)

• Constraints from boundary conditions, etc

– Vorticity interpretation
Examples: Poiseuille flow

• Channel

• Pipe
I. Exact methods for $B_{cr}$?

- Anti-plane shear flows: $\mathbf{u} = (0,0,w(x,y))$
  - Starting point, as before (mechanical energy balance)
    - $B' = B_{cr}$
    - $\Omega'$ is subdomain of $\Omega$
      - $B'$: largest ratio of area/perimeter??

$$B_{cr} = \sup_{\mathbf{u} \in H_0^1(\Omega); \mathbf{u} \neq 0} \frac{\int_\Omega u \, dx}{\int_\Omega |\nabla u| \, dx},$$

$$\int_\Omega B|\nabla u| - u \, dx \geq 0$$

$$B' \int_\Omega |\nabla u| \, dx \geq \int_\Omega u \, dx : \quad B' = \sup_{\Omega' \subseteq \Omega} \frac{\text{meas}(\Omega')}{\text{meas}(\partial \Omega')}$$
Optimal $\Omega'$?

- Optimal subdomain should have boundary that either coincides with that of $\Omega$, or is arc of circle, touching tangentially
- Many shapes are known: (Huilgol 2005)
  - Rectangle, square, triangle, L-shaped...
  - Pipe, annulus..
  - Kiwi, maple leaf...
- Example: square pipe $B_{cr} = 1/(\pi^{0.5}+2)$
• Settling of disc under gravity
• Plastic drag coefficient

\[ C_D^p = \frac{\hat{F}_s}{\hat{A}_{\perp} \hat{\tau}_Y} = \frac{\Delta \hat{\rho} \hat{g} \pi \hat{D}^2 / 4}{\hat{D} \hat{\tau}_Y} \]

– Randolph & Houlsby 1984
  • Perfect plasticity
– Tokpavi et al 2008

\[ C_{d,c}^p = 4\sqrt{2} + 2\pi \approx 11.94 \ldots \]

**Figure 5.** Limiting flow around a settling circular disc, from Tokpavi et al. (2008).
2. Exact methods for $B_{cr}$?

- 2D Planar flows forced at the boundary ($f=0$)
- Viscous dissipation plays no role in limit of zero flow
- Analogy with perfectly plastic materials (Hill 1950)?
  - Can we generate bounds for $B_{cr}$ from plastic flow?
  - Are these estimates good?
    - E.g. circular disc

$$
\tau \approx B \frac{\dot{\gamma}}{\|\dot{\gamma}\|}.
$$

$$
\nabla \cdot \tau - \nabla p = 0

\|\tau\| = B.
$$

Maybe

Sometimes
Example where plasticity gives exact answer

Flow around a diamond shaped particle (only ¼ shown)

Figure 6. Slipline network around the diamond: (a) \( \chi = 10 \); (b) \( \chi = 1 \); (c) \( \chi = 0.2 \). Due to symmetry only 1/4 of the flow is shown.

Figure 7. Calculated speed colourmap around diamond, close to the critical limit: (a) \( \chi = 10 \); (b) \( \chi = 1 \); (c) \( \chi = 0.2 \). The speed is normalized with the particle settling speed. Due to symmetry only 1/4 of the flow is shown.
Estimating $B_{cr}$?

1. Crude approximations

\[ f = (f_1, f_2) \]
\[ \dot{\gamma} \geq 2|u_x| = 2|v_y| \]

\[ |f_1(x, y)u(x, y)| = |f_1(x, y) \int_{x_{\partial \Omega}}^{x} u_x(s, y) \, ds| \]
\[ \leq \frac{\max_\Omega \{|f_1(x, y)|\}}{2} \int_{x_{\partial \Omega}}^{x} \dot{\gamma}(s, y) \, ds, \]

$\dot{x}_{\partial \Omega} = \text{coordinate of nearest boundary}$

\[ \int_{\Omega} f \cdot u \, dx \leq \frac{||f||_\infty (\ell_x + \ell_y)}{4} \int_{\Omega} \dot{\gamma}(u) \, dx \Rightarrow B_{cr} \leq \frac{||f||_\infty (\ell_x + \ell_y)}{4}. \]
• Usually approximations are poor
• Why use them?
• E.g.
Estimating $B_{cr}$?

- 2. Newtonian solution
- 3. Linear elastic solution
- 4. Regularized viscosity computed solution
So far.... finding $B_{cr}$

- In mechanical energy balance all terms vanish as $B \to B_{cr}$
- Viscous dissipation vanishes faster than other terms $\sim O([B_{cr} - B]^2)$
- Limit determined by balance between plastic dissipation and body forces
- Solution attains sup in the limit $B \to B_{cr}$
- Evaluating sup we can rescale: “shape” not magnitude
Computing $B_{cr}$?

1. **Direct method**
   - Set B, compute solution, compute $||u||$
   - Iterate on B
   - Issues: no serious ones
     - Time consuming, or maybe not at all
     - Use numerical approach that can compute $u=0$ exactly
     - One-sided limit
     - Error management/precision
Computing $B_{cr}$?

• 2. Re-scaling
  – Example: Poiseuille flow in channel
    • Method A: fixed pressure gradient
    • Method B: fixed mean velocity
More complex example:

Figure 10. Computed Oldroyd number as function of the Bingham number for the affine fracture of Fig. 9 and two other similar fractures. Blue squares: the two surface are symmetrical. Red circles: the two surfaces are uncorrelated. Black lozenges: the two surfaces are identical, shifted laterally (Fig. 9).

Figure 9. Computed examples of speed $|u|$ & streamlines for a fracture formed from two affine surfaces. Parameters are: $\tilde{B} = 100, 1000, 10000$, from top to bottom, with gray denoting unyielded plug regions.
Computing $B_{cr}$?

- 3. Informed guessing
  - Functional is scale invariant
  - Solution will approach minimizer as $B \to B_{cr}$

Ex: heated cavity

$$B_{cr} = \sup_{v \in \mathbf{V}, v \neq 0} \left\{ \frac{\langle v \cdot f \rangle}{\langle \dot{\gamma}(v) \rangle} \right\}$$

Figure 11. Contours of the computed speed as $B \to B_{cr}^- = 1/32$. Here computed at $B = 0.0311$. The white curves represent yield surfaces.

Karimfazli et al 2015