Flow Stability

Lecture 21?

Initial nonlinear analysis due to Bristeau 1975

- Applied to 1D plane channel flows
- Extended to many 1D flows, e.g. Chatzmina et al 2005
  - Easy to find B_{cr}

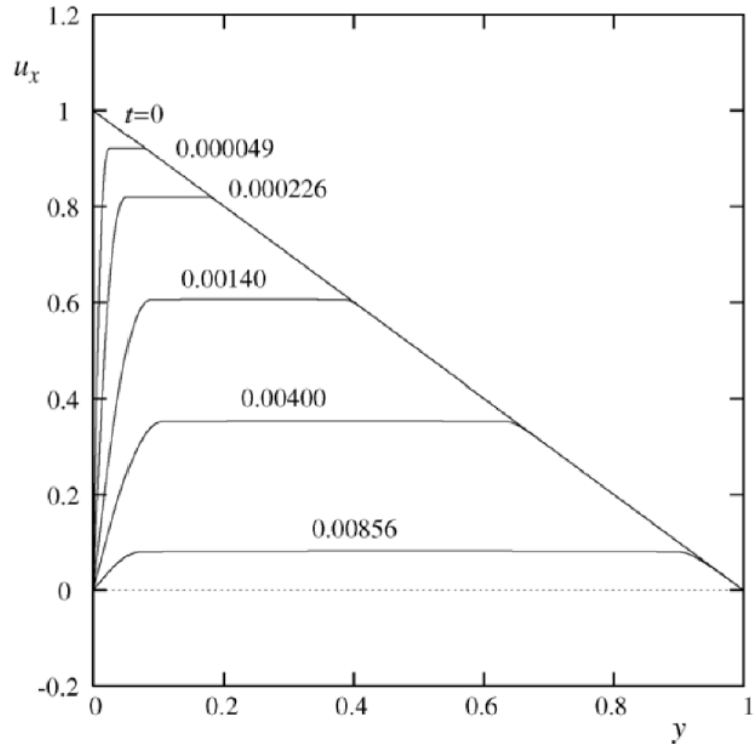
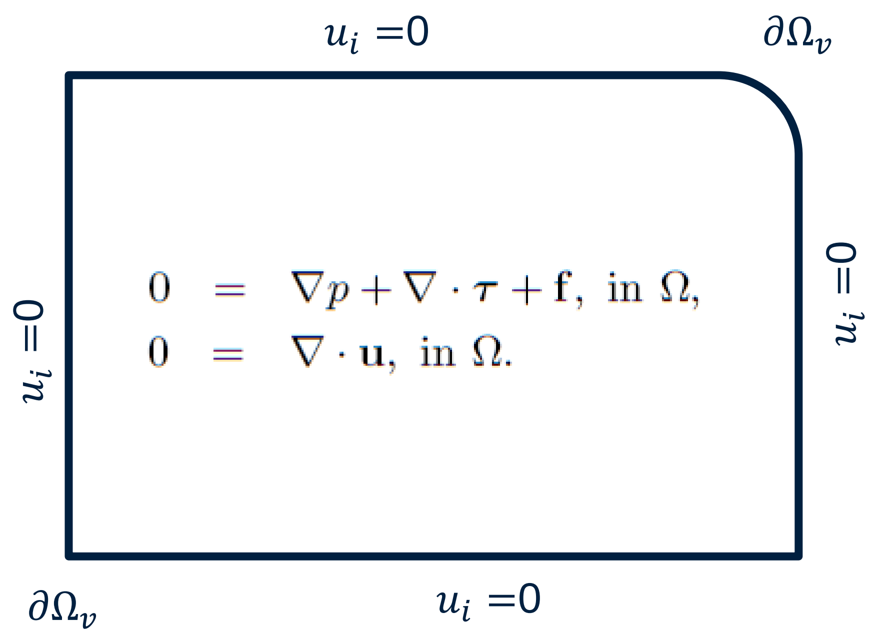


Figure 12. Cessation of 1D Couette flow for $B = 20$, from Chatzimina et al. (2005). At $t = 0$ the initial condition is the linear profile between parallel walls. For $t > 0$ the moving wall become stationary and the velocity decays.

More general flows: Recap…. finding B_{cr}

$$B_{cr} = \sup_{\mathbf{v}\in\mathcal{V},\mathbf{v}\neq 0} \left\{ \frac{\langle \mathbf{v}\cdot\mathbf{f}\rangle}{\langle\dot{\gamma}(\mathbf{v})\rangle} \right\}$$

- In mechanical energy balance all terms vanish as $B \rightarrow B_{cr}$
- Viscous dissipation vanishes faster than other terms $\sim O([B_{cr}-B]^2)$
- Limit determined by balance between plastic dissipation and body forces
- Solution attains sup in the limit $B \rightarrow B_{cr}$
- Different ways to find B_{cr}

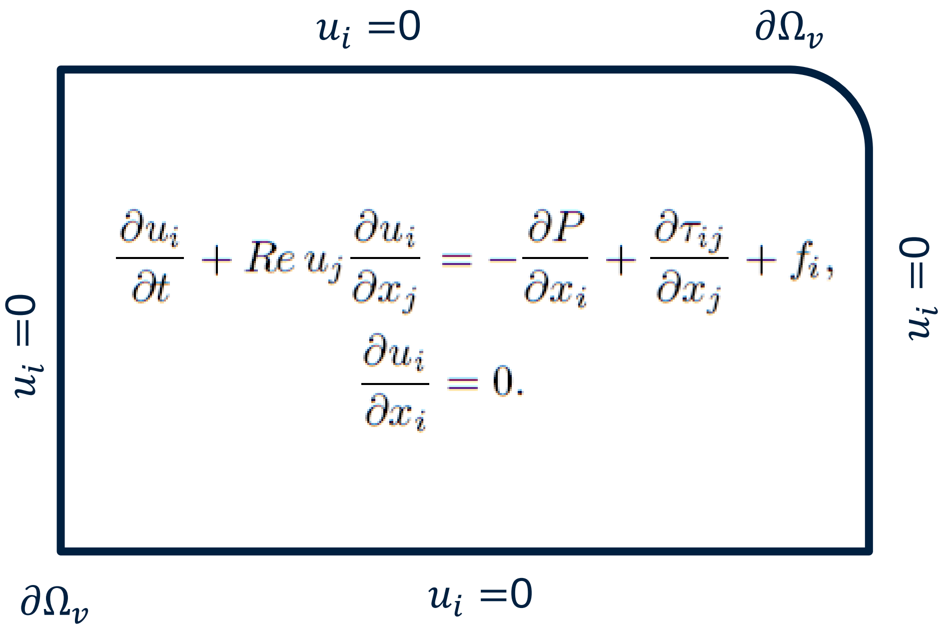


Basic setup:

- Navier-Stokes problem in closed domain

$$Re = \frac{\hat{\rho}\hat{U}_0\hat{L}}{\hat{\mu}_0} \left(= \frac{\hat{\rho}\hat{f}_0\hat{L}^3}{\hat{\mu}_0^2} \right)$$

- Can extend to usual temporal stability situations, e.g. with periodic domains…



$$\frac{\partial u_i}{\partial t} + Re\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i,$$

$$\frac{\partial u_i}{\partial x_i} = 0.$$

Steady inertial flows

- Derive steady energy balance:

$$Re\langle u_j \frac{\partial u_i}{\partial x_j} u_i \rangle + \langle \dot{\gamma}^2(\mathbf{u}) \rangle + B\langle \dot{\gamma}(\mathbf{u}) \rangle = \langle \mathbf{u} \cdot \mathbf{f} \rangle$$

  – First term integrates to zero

$$\langle \dot{\gamma}^2(\mathbf{u}) \rangle + B\langle \dot{\gamma}(\mathbf{u}) \rangle = \langle \mathbf{u} \cdot \mathbf{f} \rangle$$

  – Identical balance to Stokes flow!
    - Non-zero steady solutions ⇔ $B<B_{cr}$
    - Same critical limit $B_{cr} = \sup_{\mathbf{v} \in \mathcal{V}, \mathbf{v} \neq 0} \left\{ \frac{\langle \mathbf{v} \cdot \mathbf{f} \rangle}{\langle \dot{\gamma}(\mathbf{v}) \rangle} \right\}$
    - Re affects non-zero solution(s), but not their occurrence

Stability for $B > B_{cr}$

- Transient energy equation

$$\frac{d}{dt}\frac{\|\mathbf{u}\|^2}{2} + \langle \dot{\gamma}^2(\mathbf{u})\rangle + B\langle \dot{\gamma}(\mathbf{u})\rangle = \langle \mathbf{u}\cdot\mathbf{f}\rangle$$

- Analyse as for the steady problem
  - Inertial terms absent
  - No “base flow” to transfer energy from

$$\begin{aligned}\frac{d}{dt}\frac{\|\mathbf{u}\|^2}{2} &= -\langle \dot{\gamma}^2(\mathbf{u})\rangle - \langle \dot{\gamma}(\mathbf{u})\rangle \left[B - \frac{\langle \mathbf{u}\cdot\mathbf{f}\rangle}{\langle \dot{\gamma}(\mathbf{u})\rangle}\right] \\ &\le -\langle |\nabla \mathbf{u}|^2\rangle - \langle \dot{\gamma}(\mathbf{u})\rangle \left[B - B_{cr}\right] \\ &\le -C_P\|\mathbf{u}\|^2 - C_{TS}\|\mathbf{u}\|_p \left[B - B_{cr}\right].\end{aligned}$$

Finite time decay

Exercise: show that if $d=2$, we have finite time decay of $\mathbf{u}$, being zero after:

$$t_{stop} = \frac{1}{C_P} \ln\left(1 + \frac{C_P \|\mathbf{u}\|(0)}{C_{TS}(B - B_{cr})}\right).$$

Physically – why?

Setup:

- Differentially heated square cavity
- Navier-Stokes + Heat equation
- Boussinesq approximation

- Analysis needs generalizing B_{cr} = 1/32

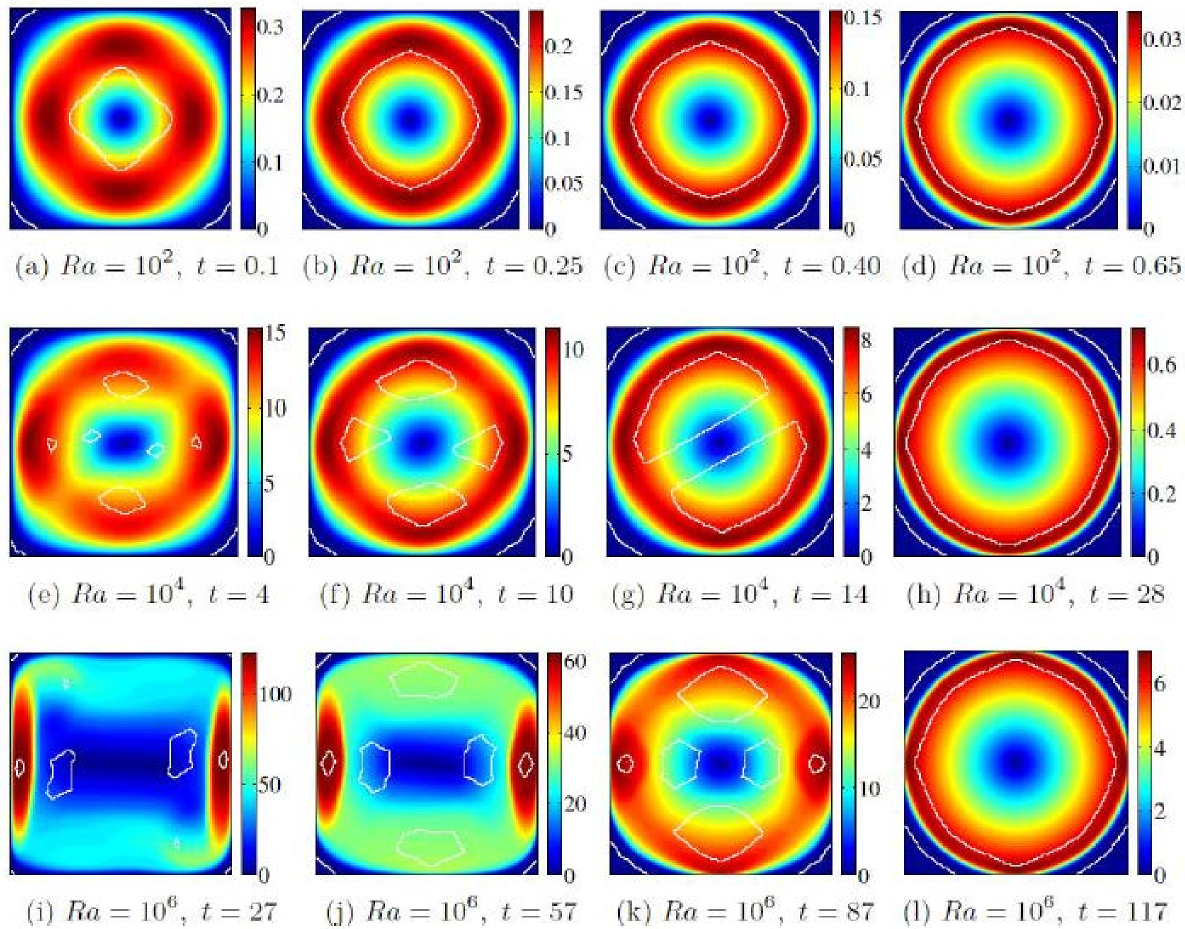


Figure 13. Colour maps of the speed at intermediate stages in the transition from the steady Newtonian flow to a completely static state, due to the introduction of a large yield stress $B = 0.075$ for $t > 0$. White curves represent the yield surfaces; $Pr = 1$. See Karimfazli et al. (2015) for further details.

Finite time decay

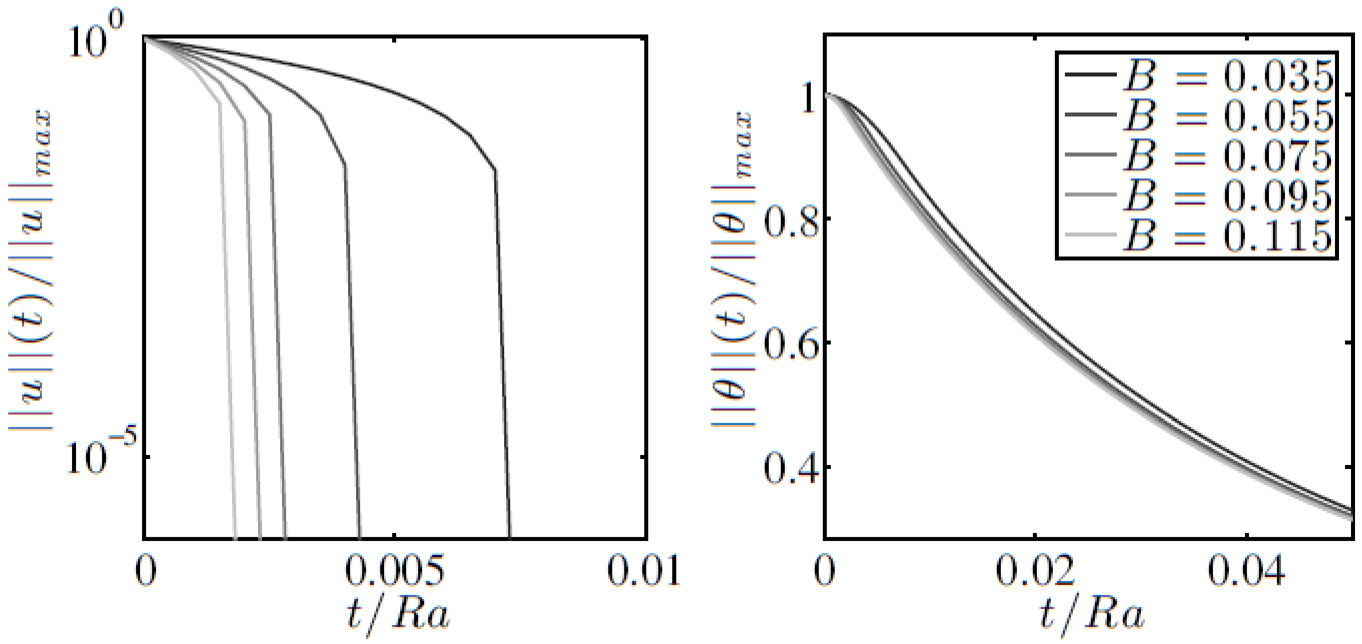


Figure 14. Development of $\|\mathbf{u}\|$ and $\|\theta\|$ after the introduction of $B > B_{cr}$ for $t > 0$, all for $Ra = 10^4$ (decay trends are quite similar at lower and higher Ra). See Karimfazli et al. (2015) for further details.

Disc settling in Bingham fluid

- Initially fluid Newtonian, steady settling

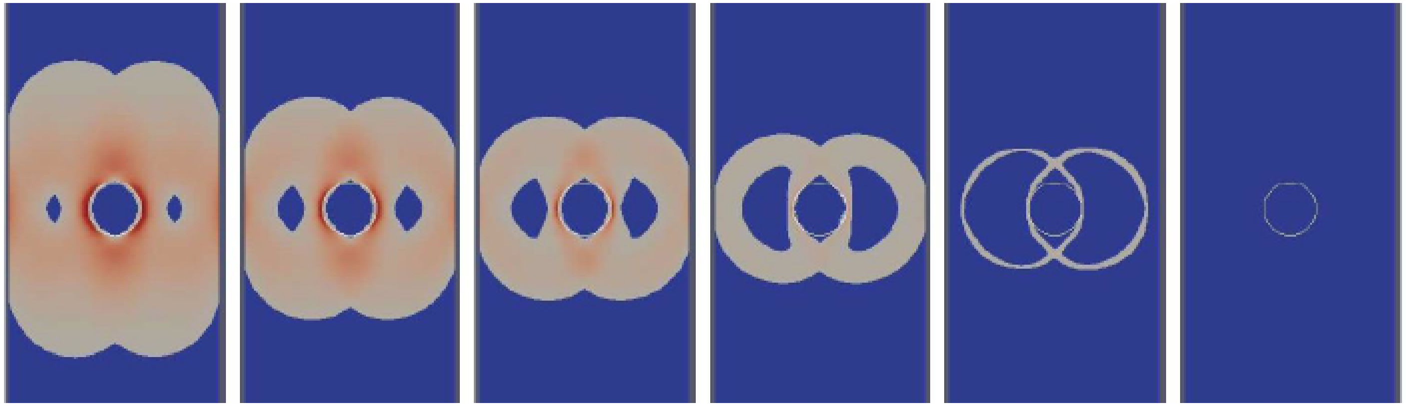


Figure 15. From Wachs and Frigaard (2016): Colourmap of the strain rate around a settling circular disc at successive times. The disc has initially the same steady settling velocity as in a Newtonian fluid. Here $B > B_{cr}$ for $t > 0$ and the motion comes to a complete stop in a finite time. The frame of reference is translated upwards as the particle falls.

Stability when $B<B_{cr}$

- Steady solution: $\mathbf{U}(\mathbf{x})\neq 0$
- Perturbed solution: $\mathbf{U}(\mathbf{x})+\mathbf{u}(\mathbf{x},t)$
  - Substitute in NS, subtract, multiply by $\mathbf{u}(\mathbf{x},t)$ and integrate over Ω

$$\frac{d}{dt}\frac{\|\mathbf{u}\|^2}{2} + Re\langle \mathbf{u}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{U}]\rangle + \langle(\tau(\mathbf{U}+\mathbf{u})-\tau(\mathbf{U})):\dot{\gamma}(\mathbf{u})\rangle = 0.$$

  - Reynolds-Orr equation
  - Evolution of kinetic energy of the perturbation
- Stress perturbation is nonlinear & hard to deal with, but is positive definite
  - Can bound below:

$$\frac{d}{dt}\frac{\|\mathbf{u}\|^2}{2} \leq -Re\langle \mathbf{u}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{U}]\rangle - \langle\dot{\gamma}^2(\mathbf{u})\rangle$$

Analysis of the inertial terms:

- From steady flow: $\langle \dot{\gamma}^2(\mathbf{U}) \rangle \le [B_{cr} - B] \langle \dot{\gamma}(\mathbf{U}) \rangle$

$$\le [B_{cr} - B] |\Omega|^{1/2} \langle \dot{\gamma}^2(\mathbf{U}) \rangle^{1/2}.$$

  – Cauchy-Schwarz: $\langle |\nabla \mathbf{U}|^2 \rangle = \langle \dot{\gamma}^2(\mathbf{U}) \rangle \le [B_{cr} - B]^2 |\Omega|$

- Embedding results: $-\langle \mathbf{u} \cdot [(\mathbf{u} \cdot \nabla)\mathbf{U}] \rangle \le C_b \|\mathbf{u}\|_{H^1}^2 \|\mathbf{U}\|_{H^1}$
- Finally: $-\langle \mathbf{u} \cdot [(\mathbf{u} \cdot \nabla)\mathbf{U}] \rangle \le C_b \left(1 + \frac{1}{C_P}\right) (B_{cr} - B) |\Omega|^{1/2} \|\mathbf{u}\|_{H^1}^2$
- Reynolds-Orr inequality becomes:

$$\frac{d}{dt} \frac{\|\mathbf{u}\|^2}{2} \le \left[Re C_b \left(1 + \frac{1}{C_P}\right) (B_{cr} - B) |\Omega|^{1/2} - 1 \right] \langle |\nabla \mathbf{u}|^2 \rangle$$

- RHS negative if $Re < Re_E = \dfrac{1}{C_b \left(1 + \frac{1}{C_P}\right) (B_{cr} - B) |\Omega|^{1/2}}$

  – Perturbation decays: $\|\mathbf{u}\|(t) \le \|\mathbf{u}\|(0) \mathrm{e}^{-C_P (Re_E - Re) t}.$

Notes:

- Body force and Re contribute to $\mathbf{U}(\mathbf{x})$, but not explicit in bound
- As $B \to B_{cr}$ from below see that $Re_E \to \infty$

$$Re_E = \frac{1}{C_b\left(1+\frac{1}{C_P}\right)(B_{cr}-B)\,|\Omega|^{1/2}}$$

  – decay timescale shortens

$$\|\mathbf{u}\|(t) \leq \|\mathbf{u}\|(0)\mathrm{e}^{-C_P(Re_E - Re)t}.$$

  – Recover same results as for $B > B_{cr}$
- Specific flows will have stronger results, i.e. larger Re_E is possible
    - E.g. Nouar & Frigaard 2001