Particles in Yield Stress Fluids

Lecture 22 & 23
Industrial motivations to both use & understand YSF

- Drilling muds used ~100 years to remove cuttings from wells
- Plasticity recognised as an important component of drilling mud from late 1920’s
  - Bingham’s work on yield stress fluids 1916, popularised in 1920s
  - 1930s/1940s: elementary flows solved
- Efforts at quantifying transport effectiveness from ~1950
  - Simple rule: flow velocity > slip velocity of particles
- Similar vintage of technical literature in other industries?
  - Slurry transport
  - Concrete industry

Ability of Drilling Mud to Lift Bit Cuttings:
Hall et al., SPE Journal 1950

“The force causing slip in a true fluid is: ....

In a plastic fluid such as mud, however, this force is reduced due to the yield strength and the resulting force, causing slip is:.....”

Spherical Particles:

\[
V_s = \frac{gD^2 (\rho_s - \rho_m) - 6gD \Delta \rho}{18 (x)(n)}
\]  \hspace{1cm} (18)

Flat Disks:

\[
V_s = \frac{3\pi tDg (\rho_s - \rho_m) - 6\pi gt (D+2t)}{64 (x)(n)}
\]  \hspace{1cm} (19)
Misconceptions & Confusion:

- Early days: how could a particle move through a large expanse of unyielded fluid?
  - Particles yield the fluid locally only
  - What does the yielded envelope look like?
- When flowing these fluids typically have a plug
  - Poiseuille flows
- Cuttings not generally “held suspended” in the plug
  - (Viscous) drag acts on the particles
    - Drag force is increased by the yield stress
    - Shear stresses vary linearly across duct act to “reduce” yield stress available to suspend particles
- Role of yield stress in such systems is to hold particles static when there is no flow
What have we learned about the static single particle problem?

• Heavy particle settling within a region of yield stress fluid
  – From 1960s understood that fluid yields locally around the particle as it settles
  – Andres (1960): “sphere of influence”
  – Whitmore & co-authors – some of first experimental studies
• As $Y$ increases yielded region contracts inwards
  – Eventually stops
• Evolution of ideas about flow field about particle

\[
Y = \frac{\hat{\tau}_y}{\left[\hat{\rho}_p - \hat{\rho}_l\right]g\hat{L}}
\]

$\hat{\tau}_y$ = Fluid yield stress
$\hat{\rho}_p$ = Particle density
$\hat{\rho}_l$ = Fluid density
$\hat{g}$ = gravity
$\hat{L}$ = Length scale

• Various studies on particle motion and drag for different shapes:
Single particle problem

- **3D Mechanics:**
  - Solve NSE in the fluid domain \((u, p)\) [4]
  - Solve linear momentum equations for particle velocity at centroid \(u_c(t)\) [3]
  - Solve angular momentum equations for particle rotation about centroid \(\omega_c(t)\) [3]

- At particle we have
  - Continuity of velocity
  - Continuity of traction

- Other boundary conditions in far-field

- Two key problems are studied
  - **Mobility problem**
    - Force/torque on particle is given
    - Motion to be found
    - E.g. steady settling
  - **Resistance problem**
    - Particle motion is specified
    - Force/torque to be calculated
    - E.g. tracking motion

- In a general computational setting, may need to work with either
Revisit the classical formulation?

At yield surface:
- Continuity of velocity
- Continuity of traction

$\mathbf{u}_p$ was speed of yield surface
- Assumed $\mathbf{u}_p = u_p \mathbf{n}$

Momentum conservation, linear & angular

\[
0 = \int_{\Omega_p(t)} \left( \dot{\mathbf{u}}_{c,i} + \epsilon_{ijk}(x_j - x_{c,j})\dot{\omega}_{c,k} \right) \mathbf{d}x - \int_{\Gamma(t)} \sigma_{ij} n_j \mathbf{d}s \\
- \int_{\Omega_p(t)} f_i \mathbf{d}x + \int_{\Gamma(t)} \left( u_{c,i} + \epsilon_{ijk}(x_j - x_{c,j})\omega_{c,k}\right) \times \\
\left[ (u_{c,l} + \epsilon_{lmn}(x_m - x_{c,m})\omega_{c,n})n_l - u_p \right] \mathbf{d}s
\]  

Delete for particle motion

\[
0 = \int_{\Omega_p(t)} \epsilon_{ijk}[(x_j - x_{c,j})(\dot{u}_{c,k} + \epsilon_{klm}(x_l - x_{c,l})\dot{\omega}_{c,m})] \mathbf{d}x - \int_{\Omega_p(t)} \epsilon_{ijk}(x_j - x_{c,j})f_k \mathbf{d}x + \\
\int_{\Gamma(t)} \epsilon_{ijk}(x_j - x_{c,j})\mathbf{u}_c + \epsilon_{klm}(x_l - x_{c,l})\omega_{c,m} \times \\
\left[ (u_{c,l} + \epsilon_{lmn}(x_m - x_{c,m})\omega_{c,n})n_l - u_p \right] \mathbf{d}s, \\
- \int_{\Gamma(t)} \epsilon_{ijk}(x_j - x_{c,j})\sigma_{kn} \mathbf{d}s
\]  

Delete for particle motion

No “evolution” equation for yield surface
Creeping motion of a sphere through a Bingham plastic

By A. N. BERIS, J. A. TSAMOPOULOS, R. C. ARMSTRONG AND R. A. BROWN

Department of Chemical Engineering, Massachusetts Institute of Technology,
Cambridge, MA 02139

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A solid sphere falling through a Bingham plastic moves in a small envelope of fluid with shape that depends on the yield stress. A finite-element/Newton method is presented for solving the free-boundary problem composed of the velocity and pressure fields and the yield surfaces for creeping flow. Besides the outer surface, solid occurs as caps at the front and back of the sphere because of the stagnation points in the flow. The accuracy of solutions is ascertained by mesh refinement and by calculation of the integrals corresponding to the maximum and minimum variational principles for the problem. Large differences from the Newtonian values in the flow pattern around the sphere and in the drag coefficient are predicted, depending on the dimensionless value of the critical yield stress $Y_g$ below which the material acts as a solid. The computed flow fields differ appreciably from Stokes' solution. The sphere will fall only when $Y_g$ is below 0.143. For yield stresses near this value, a plastic boundary layer forms next to the sphere. Boundary-layer scalings give the correct forms of the dependence of the drag coefficient and mass-transfer coefficient on yield stress for values near the critical one. The Stokes limit of zero yield stress is singular in the sense that for any small value of $Y_g$ there is a region of the flow away from the sphere where the plastic portion of the viscosity is at least as important as the Newtonian part. Calculations for the approach of the flow field to the Stokes result are in good agreement with the scalings derived from the matched asymptotic expansion valid in this limit.

- Landmark computation
  - Resolved plug regions adaptively
  - Good error control using variational formulation: “duality gap”
  - Resolved 20-year old debate about the form of the solution
    - Polar cap plugs found
    - Critical yield number found
    - Results for Stokes drag
- 30 years later results are still believed!
**Figure 1.** Plastic and solid regions for the flow surrounding a solid sphere falling in a Bingham plastic material.

**Figure 3.** The finite-element mesh B2 in the original coordinate system for the calculation with $Y_r = 0.1$.

**Figure 5.** Comparison of values of variational integrals $H$ and $K$ computed using the finite-element method (——) and the approximate methods of Yoshioka et al. (1971). The values of the integrals $H$ and $K$ computed using the finite-element results are indistinguishable. The error bars for the approximate calculations indicate the difference between $H$ (upper bound) and $K$ (lower bound) respectively.
Stokes drag coefficient \( C_s = \frac{F}{6\pi\mu_\infty V_0 R_0} \)

Force for settling sphere: \( F = \frac{4}{3} \pi R_0^3 (\rho_s - \rho) g \)

Gravity yield number: \( Y_g = \frac{2\tau_y \pi R_0^2}{F} = \frac{3\tau_y}{2(\rho_s - \rho) g R_0} \)

Bingham number: \( N_B = \frac{2R_0 \tau_y}{\mu_\infty R_0} = 6C_s Y_g \)

**Figure 6.** Dependence of the Stokes drag coefficient on the yield-stress parameter \( Y_g \).

**Figure 7.** Dependence of Stokes drag coefficient on the Bingham number. Finite-element results are represented by the continuous curve and the experimental data of Ansley & Smith (1967) by (○). The upper and lower bounds calculated by Yoshioka et al. (1971) are shown as error bars.
Other computations

- Various papers by Mitsoulis & co-workers
  - Different shapes and configurations
  - Mostly slow flows
- Tokpavi, Jossic, Jay, Magnin: detailed computations of flow around a cylinder
- Wachs & Yu
- Liu, Denn & Muller
- Recently: many repetitive studies using LBM & regularization
- The above mostly use regularized models
- There are 2 distinct problems:
  - Drag on objects, often far from stopping
  - Yield limit, where $C_S \to \infty$
Viscous drag

- Valentik & Whitmore (1967)
  - Different clay suspensions and different balls
  - Force divided into yield and viscous component
    \[ \mathcal{F} = \frac{1}{2} \rho \pi v^2 D^2 + \frac{1}{8} C_D \rho \pi v^2 D^2 = \frac{1}{8} \pi (2\nu + C_D \rho v^2) D^2 \]
  - Drag coefficient defined
FIG. 1. The steady-state shear stress $\tau$ plotted as a function of shear rate $\dot{\gamma}$ for the two concentrations of Carbopol studied (squares: $\phi=0.5$, circles: $\phi=2.0$). Data indicated by solid symbols were measured with the...

FIG. 3. The depth as a function of time for the variable-density ping-pong ball falling through a 0.5% Carbopol suspension. The different symbols correspond to different densities as indicated.

FIG. 4. The higher-density data from Fig. 3 plotted with linear axes. The terminal velocity is the slope of the linear portion of the curve at long times.
FIG. 6. The drag coefficient as a function of the dynamic parameter $Q$ for the data of Fig. 5. The symbols have the same meaning as in that figure. The solid line is the prediction of Eq. (6) using the numerically calculated values of the coefficients.

\[ C_D = \frac{4 \, \text{Bi}}{Y \, \text{Re}} = 24X(n) \left( \frac{1 + k \, \text{Bi}}{\text{Re}} \right) = \frac{24X(n)}{Q} \]

\[ Q = \frac{\text{Re}}{1 + k \, \text{Bi}} \]

Re = \frac{\rho_0 V^2}{K(V/d)^n},

Bi = \frac{\tau_c}{K(V/d)^n},

Y = \frac{3 \tau_c}{gd\Delta \rho}.

FIG. 7. The data of Fig. 6 plotted as $Y^{-1}$, the reciprocal of the yield number, as a function of Bi$^{-1}$, the reciprocal of the Bingham number. The inset shows the same data plotted with linear axes. In both cases the solid line corresponds to Eq. (7); the dotted line shows the deviation of data from the theory.
Jossic & Magnin (2001)

- 4 Carbopol solutions
- Measure force and control velocity

\[ F_b = \frac{\pi}{6} g \Delta \rho d^3 \]

\[ Re = \frac{\rho l U^{2-n}}{K} \]

\[ C_d^* = \frac{F_d}{A \cdot \tau_0} \]

\[ Bi = \frac{\tau_0}{K(U/l)^n} \]

Figure 1. Experimental setup for measuring drag force at prescribed velocity.

Table 1. Shape of Objects Used

<table>
<thead>
<tr>
<th>Surface</th>
<th>Dimensions (mm)</th>
</tr>
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<tbody>
<tr>
<td>Sphere</td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td>( d = 20 )</td>
</tr>
<tr>
<td>Rough</td>
<td>( d = 22 )</td>
</tr>
<tr>
<td>Disc</td>
<td></td>
</tr>
<tr>
<td>( h/d = 0.02 )</td>
<td>( d = 35.4; h = 1.2 )</td>
</tr>
<tr>
<td>( h/d = 0.14 )</td>
<td>( d = 35.7; h = 2.7 )</td>
</tr>
<tr>
<td>Cylinder</td>
<td></td>
</tr>
<tr>
<td>( h/d = 1 )</td>
<td>( d = 35; h = 5 )</td>
</tr>
<tr>
<td>( h/d = 5 )</td>
<td>( d = 36.5; h = 6.5 )</td>
</tr>
<tr>
<td>Cylinder</td>
<td></td>
</tr>
<tr>
<td>( h/d = 1 )</td>
<td>( d = 21.1; h = 21.4 )</td>
</tr>
<tr>
<td>Cylinder</td>
<td></td>
</tr>
<tr>
<td>( h/d = 5 )</td>
<td>( d = 11.5; h = 51.5 )</td>
</tr>
<tr>
<td>Cube</td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td>( a = 20 )</td>
</tr>
<tr>
<td>Rough</td>
<td>( a = 22 )</td>
</tr>
<tr>
<td>Cone</td>
<td></td>
</tr>
<tr>
<td>( \alpha = 90^\circ )</td>
<td>( d = 35.2; h = 17.6 )</td>
</tr>
<tr>
<td>Smooth</td>
<td>( d = 35.4; h = 17.7 )</td>
</tr>
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</table>

\*\( d \): diameter, \( h \): height, \( a \): edge, \( \alpha \): top angle.
Figure 4. Change in drag coefficient $C_d^*$ of a horizontal cylinder as a function of slenderness.

Figure 5. Change in drag coefficient $C_d^*$ of a vertical cylinder as a function of slenderness.

Figure 6. Change in drag coefficient $C_d^*$ as a function of ratio $S_{lat}/A$.

Figure 7. Stability criterion of “rough” objects as a function of their shape and orientation.

Table 3. Experimental Results*

<table>
<thead>
<tr>
<th>Position</th>
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<th>$C_d^*$ = $F_d/A \cdot \tau_0$</th>
<th>$Y_{max}$</th>
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<td>9.8</td>
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<tr>
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<td>9.9</td>
<td>0.027</td>
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*See text for definition of orientation.
UBC: theoretical & computational contributions from Andreas Putz, PhD 2010; Emad Chaparian, PhD 2018

- Two distinct problems:
  - Resistance problem [R]
    - Motion imposed, compute drag force
    - Bingham number ($B$)
  - Mobility problem [M]
    - Force is imposed, compute motion
    - Yield number ($Y$)

- One-to-one mapping between the 2 problems
  - Provided that the particle moves $Y < Y_c$
  - Can compute flow using either formulation

- Critical ratio $Y_c$ defined:

$$Y_c = \sup_{v \in \mathcal{V}, v \neq 0} \frac{L(v)}{j(v)}$$

$$Y_c = \sup \frac{\text{Flux in fluid in direction of gravity}}{\text{Plastic dissipation rate in fluid}}$$

$$Y = \frac{\hat{\tau}_y}{[\hat{\rho}_p - \hat{\rho}_l]g\hat{L}}$$

\[ L(v) = \pi V = \pi V \cdot e_y = - \int_{\Omega \setminus \tilde{X}} v \cdot e_y \, dA. \]

\[ a(v, w) = \int_{\Omega \setminus \tilde{X}} \dot{\gamma}(v) : \dot{\gamma}(w) \, dA, \]

\[ j(v) = \int_{\Omega \setminus \tilde{X}} \| \dot{\gamma}(v) \| \, dA, \]
• Resistance:

\[ a(u, v) + B[j(u + v) - j(u)] \geq L_R(v), \quad u \in V_R, \forall v \in V_{R,0}, \]

• Mobility:

\[ a(u, v - u) + B[j(v) - j(u)] \geq L_M(v - u), \quad u \in V_M, \forall v \in V_M, \]

• Relationship:

\[
\begin{align*}
& a(u, v - u) + B[j(v) - j(u)] \\
& \geq L_R(v - u) + F^{[u]} \cdot [U^{[v]} - U^{[u]}] \\
& \quad + M^{[u]} \cdot [\omega^{[v]} - \omega^{[u]}], \quad u \in V_M, \forall v \in V_M,
\end{align*}
\]

0 \leq a(u_1 - u_2, u_1 - u_2) \leq [F^{[u_1]} - F^{[u_2]}] \cdot [U^{[u_1]} - U^{[u_2]}] \\
\quad + [M^{[u_1]} - M^{[u_2]}] \cdot [\omega^{[u_1]} - \omega^{[u_2]}].

\[ a(u, v) = \int_{\Omega \setminus \bar{P}} \dot{\gamma}(u) : \dot{\gamma}(v) \, dx, \]

\[ j(v) = \int_{\Omega \setminus \bar{P}} \dot{\gamma}(v) \, dx, \]

\[ L_R(v) = \frac{\rho_r}{1 - \rho_r} \int_{\Omega \setminus \bar{P}} e_g \cdot v \, dx, \]

\[ L_M(u) = L_R(u) + \frac{V_p}{1 - \rho_r} U \cdot e_g + \omega \cdot M_b. \]
Yielding problem

• Related $Y \to Y_c$ with $B \to \infty$, via plastic drag coefficient
• As $Y \to Y_c$, have shown that viscous dissipation is not important

$$a(u, u) = O([Y_c - Y]^2), \quad j(u) = O(Y_c - Y),$$
$$\frac{a(u, u)}{j(u)} = O(Y_c - Y).$$

• As particles stop, plastic dissipation balances work done by settling

$$Y j(u) \sim L(u) = \pi U,$$

• Strong temptation to ignore viscosity if we want to understand static flows
  – Hill’s theory of perfect plasticity

$$\tau \approx B \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad \nabla \cdot \tau - \nabla p = 0$$
$$\|\tau\| = B.$$
2D particles & tools

- Augmented Lagrangian computation for [M] or [R]
  - FreeFEM & mesh adaptivity (Hecht 2003)
  - Rheolef from P. Saramito also good
- Developed method of characteristics for perfect plasticity (slipline method)
  - Lower bounds from stresses computed
  - Upper bounds from velocity formulation
- Asymptotics & guesswork within variational context

- Comparisons made via plastic drag coefficient

\[
C_{D}^{p} = \begin{cases}
\left[ \frac{\hat{F}}{\hat{\ell}_{\perp} \hat{\tau}_{Y}} \right]^{[R]} = \left[ \frac{F^{*}}{\ell_{\perp} B} \right]^{[R]} & \text{for problem [R]}, \\
\left[ \frac{\hat{\rho} \hat{g} \hat{A}_{p}}{\hat{\ell}_{\perp} \hat{\tau}_{Y}} \right]^{[M]} = \left[ \frac{\pi}{\ell_{\perp} Y} \right]^{[M]} & \text{for problem [M]}
\end{cases}
\]

\[
C_{D,c} = \frac{\pi}{\ell_{\perp} Y_{c}} = \left[ C_{D}^{p} \right]_{Y \rightarrow Y_{c}}^{[M]} = \left[ C_{D}^{p} \right]_{B \rightarrow \infty}^{[R]}
\]
Perfect plasticity/sliplines?

- Lower bound solutions from stress often give very good estimate of $C_{D,c}^p$
  - Sometimes even exact!
  - But flows are not the same!
- Circular disc: $C_{D,c}^p = 4\sqrt{2} + 2\pi \approx 11.94$
  - Plasticity: Randolph & Houlsby (1984)

Chaparian & Frigaard, JFM (2017)
Symmetric particles, aspect ratio $\chi$

\begin{align*}
\chi &= 1.6 \\
2\hat{a}_e &= \hat{e}_\perp \\
2\hat{a}_r &= \hat{e}_\perp \\
2\hat{b}_e &= \hat{e}_\parallel \\
2\hat{b}_r &= \hat{e}_\parallel \\
2\hat{a}_d &= \hat{e}_\perp \\
2\hat{b}_d &= \hat{e}_\perp
\end{align*}

\begin{align*}
\chi &= 0.625 \\
2\hat{a}_e &= \hat{e}_\perp \\
2\hat{a}_r &= \hat{e}_\perp \\
2\hat{b}_e &= \hat{e}_\parallel \\
2\hat{b}_r &= \hat{e}_\parallel \\
2\hat{a}_d &= \hat{e}_\perp \\
2\hat{b}_d &= \hat{e}_\perp
\end{align*}

**Figure 5.** Dimensional geometries considered.

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<th>Ellipse</th>
<th>Rectangle</th>
<th>Diamond</th>
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<td>$2\hat{a}_e$</td>
<td>$2\hat{a}_r$</td>
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<tr>
<td>$\hat{e}_\parallel$</td>
<td>$2\hat{b}_e$</td>
<td>$2\hat{b}_r$</td>
<td>$2\hat{b}_d$</td>
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<tr>
<td>$\hat{L}$</td>
<td>$\sqrt{\hat{a}_e\hat{b}_e}$</td>
<td>$2\sqrt{\hat{a}_r\hat{b}_r}/\pi$</td>
<td>$\sqrt{2\hat{a}_d\hat{b}_d}/\pi$</td>
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<tr>
<td>$\hat{e}<em>\parallel/\hat{e}</em>\perp$</td>
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<tr>
<td>$\hat{e}_\perp$</td>
<td>$2\chi^{-1/2} = 2a_e$</td>
<td>$\sqrt{\pi\chi^{-1/2}} = 2a_r$</td>
<td>$\sqrt{2\pi\chi^{-1/2}} = 2a_d$</td>
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<tr>
<td>$\hat{e}_\parallel$</td>
<td>$2\chi^{1/2} = 2b_e$</td>
<td>$\sqrt{\pi\chi^{1/2}} = 2b_r$</td>
<td>$\sqrt{2\pi\chi^{1/2}} = 2b_d$</td>
</tr>
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</table>

**Table 1.** Dimensional and dimensionless parameters for the three geometries.
Symmetric particles, aspect ratio $\chi$

**Figure 6.** Characteristic network around an ellipse: (a) $\chi = 10$; (b) zoom of frontal plug region for $\chi = 10$; (c) $\chi = 0.2$.

**Figure 10.** (Colour online) Velocity magnitude colour map, computed from problem [R] at $B = 10^4$ (the white lines are the yield surfaces): (a) $\chi = 10$, (b) $\chi = 1$, (c) $\chi = 0.2$. 

Chaparian & Frigaard, JFM (2017)
Symmetric particles, aspect ratio $\chi$

Figure 13. (Colour online) Velocity magnitude colour map, computed from problem [R] at $B=10^4$ (the white lines are the yield surfaces): (a) $\chi = 10$; (b) $\chi = 1$; (c) $\chi = 0.2$. 

Rectangle

Ellipse
FIGURE 14. Characteristic network around the diamond: (a) $\chi = 10$; (b) $\chi = 1$; (c) $\chi = 0.2$.

FIGURE 16. (Colour online) Velocity magnitude contours around the diamond: (a) $\chi = 10$; (b) $\chi = 1$; (c) $\chi = 0.2$. 

Chaparian & Frigaard, JFM (2017)
Cloaking: What determines $Y_c$ or $C_{D,c}^p$

- Different particles can have same $C_{D}^p$ and $C_{D,c}^p$
- Unyielded envelope around particle determines $C_{D,c}^p$
- For problem [M] can establish equivalence by using mean density of material within unyielded envelope
- Have developed a rubric for calculating the unyielded envelope around symmetric particles

Chaparian & Frigaard, JNNFM (2017)
Extensions?

• Various calculations of axisymmetric 3D particles
  – Analytical tools from perfect plasticity do not carry over
  – Concepts such as cloaking/unyelded envelope very relevant

• Symmetric particles with drift/orientation

• Multiple particles
Necklaces of particles:

- In-line sequences of particles (disks)
- Scale problem as for single disk, using radius of 1st disk
  - $\chi_i = \text{radius ratio of i-th disk to 1st disk}$
  - Consider only $\chi_i \leq 1$
- For single disk

$$Y_c = \left( \frac{\hat{\tau}_y}{[\hat{\rho}_p - \hat{\rho}_l] \hat{g}\hat{L}} \right)_c = 0.1316...$$

$$Y_k = Y_1 / \chi_k$$
Poor man’s necklace: 2 disks: $\chi \neq 1$

- 4 generic behaviours:
  - Static
  - Only small disk static
  - Moving in same envelope
  - Moving at same speed
- Small disk motion onset not $Y_2 < Y_c$
  - Stress is non-local
  - Flow is influenced by $Y$, $\chi$ and separation $\ell$ in an intuitive way
  - $\ell = $ centre-to-centre distance
• Vary separation distance for each radius ratio
  – Identify $Y_c(\chi, \ell)$
  – Transitions between flow regimes, e.g. at fixed $Y=0.11$

• Some analytical estimates for possible $Y_c$ if particles are connected
Two disks: $\chi=1$

- Disks move at same speed for $\chi=1$
  - Symmetry argument
- Same 4 regimes
- Calculate settling velocity $V_k(Y_1, \ell_1)$
Three disks: $\chi=1$

- Two separation distances
- 7 regimes, assuming $Y > Y_c$
- Symmetry about $\ell_1 = \ell_2$
- Central particle moves fastest
- *Difficult* to have all 3 connected

$Y=0.13$

- Isolated static disks
- 1 static
- 2 connected
- 3 connected

Centreline velocity $y > 0$

- Middle disk
- Outer disk

$Y=0.15$

- Isolated static disks
- 1 static
- 2 connected
- 3 connected

Owl?
How about 4 disks: $\chi=1$?

- Central 2 particles connect
- Outer particles less clear
- Appears that as $Y$ is increased: $V_k \to 0$ with $V_4/V_3 < 1$?
• Central 3 particles connect
• Outer particles do not
• Again as $Y$ is increased: $V_k \to 0$ with $V_5/V_3 < 1$?
• Can we get >3 disks together for any non-zero separation?
Can we move through these regimes?

3-disk necklaces in $\mathbb{R}^2$

$\ell_1(t)$

$\ell_2(t)$

$V_1$

$V_2$

$V_3$

Isolated static disks

1 static

2 connected

3 connected

All mobile & not connected

All mobile 2 connected

$Y = 0.15$
3-disk necklaces in $\mathbb{R}^2$

Relative motion governed by 2-variable autonomous system of DEs

$$\frac{d}{dt} l_1 = V_1(l_1, l_2) - V_2(l_1, l_2) = f_1(l_1, l_2)$$

$$\frac{d}{dt} l_2 = V_2(l_1, l_2) - V_3(l_1, l_2) = f_2(l_1, l_2)$$

$$\dot{x} = f(x), \quad x = (l_1, l_2)^T$$
3-disk necklaces in $\mathbb{R}^2$

\[
\begin{align*}
\frac{dl_1}{dt} &= V_1(l_1, l_2) - V_2(l_1, l_2) = f_1(l_1, l_2) \\
\frac{dl_2}{dt} &= V_2(l_1, l_2) - V_3(l_1, l_2) = f_2(l_1, l_2)
\end{align*}
\]

\[
\dot{x} = f(x), \quad x = (l_1, l_2)^T
\]
Uniform initial $\epsilon(0)$ with static particles

N-particle chain?
FIG. 4. Color maps of the fluid velocity magnitude with particles overlaid in gray and yield surface indicated by the solid black lines for $Y = 0.087$ with $\phi = 0.01$ (left) and $\phi = 0.05$ (right) for five different configurations.
FIG. 5. Color maps of $\log_{10}(\dot{\gamma}/\dot{G})$ for $\phi = 0.01$ at $Y = (0, 0.022, 0.065, 0.087, 0.13)$ (left panel) and $\phi = 0.05$ at $Y = (0, 0.043, 0.087, 0.13, 0.17)$ (right panel).
Koblitz et al. Phys. Rev Fluids 2018

• Impressive computations
  – Many other interesting images in paper
  – Moving towards understanding a static suspension and yielding

• But…
  – These are instances of Stokes flow for specific configurations
  – No time-dependency & not DNS
  – Statistical aspects needs better defining, e.g. variability in $Y_c$ for different configurations
Other theoretical avenues: Resurrecting Mosolov & Miasnikov

Main contribution from J. Iglesias & G. Mercier

Mosolov & Miasnikov:

- Studied anti-plane shear flows, i.e. flows along infinite ducts with a 2D cross-section: pipes, annuli, etc..
- Identified $Y_c$ functionally

\[ \tau_0 \int_\omega |\nabla h| d\omega - c \int_\omega h d\omega \geq 0 \]  \hspace{1cm} (2.1)

for all $h$ defined in the domain $\omega$ and satisfying the boundary condition (1.6).

- Awakened us to its geometric meaning

\[ K \int_\omega |\nabla h| d\omega \geq \int_\omega h d\omega, \quad K = \sup_{\omega \subseteq \omega} \frac{\text{mes } \omega'}{\text{mes } \Gamma} \]  \hspace{1cm} (2.2)
Mosolov & Miasnikov

- Algorithmic method for computing $Y_c$
  - Exploited e.g. by Huilgol JNNFM 2006
- Other results on size of plug regions etc..

- Frigaard et al 2017:
  - Anti-plane shear flow with “particles”
  - Moved from M&M’s geometric understanding of $Y_c$ to a set-theoretic formulation
  - Allowed us to solve many problems directly in an analytic way

Notes
- Gravity acts out of plane
- Net flow = 0: fluids are incompressible & infinite duct closed at end

Notes
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Fig. 4. Determination of the yield surface for a pipe of rectangular cross-section.
Problem involves finding 3 sets, $\Omega_c$ & $\Omega_{1c}$

1. Solve the Cheeger problem for $\Omega \setminus \Omega_s$. Let $\Omega_c$ be the maximal Cheeger set and $\lambda_c := \frac{\text{Per}(\Omega_c)}{|\Omega_c|}$ its Cheeger constant.
2. Obtain the minimal (with respect to $\subset$) minimizer $\Omega_{1c}$ of

$$\text{Per}(E) + \lambda_c |E| \text{ over } \{E : E \cap \Omega_c = \emptyset \text{ and } \Omega_s \subset E\}.$$

- Find $\Omega_c$. This is the (maximal) subset of the blue grey area that has the smallest ratio of perimeter to area.
- Minimize the function: $[\text{Perimeter}(E) + \lambda_c \text{Area}(E)]$ over all sets $E$ that enclose $\Omega_s$ but do not intersect $\Omega_c$. This defines $\Omega_{1c}$. 

Problem involves finding 3 sets, $\Omega_c$ & $\Omega_{1c}$

In other words: one set surrounds the particle and moves at speed 1. The other set moves negatively at a speed to balance the flux. The remaining set is stuck to the wall.
Critical yield number: $Y_c$?

• Use $\Omega_c$ and $\Omega_{1c}$:

\[ S(E_1, E_-) = \text{Per}(E_1) + \frac{|E_1|}{|E_-|} \text{Per}(E_-) \]

**Theorem 4.10.** The pair $(\Omega_{1c}, \Omega_c)$ minimizes $S$

• Finally find critical yield number:

\[ Y_c = 1/S(\Omega_{1c}, \Omega_c) \]

• 3-valued TV-minimizer approximates $w(x,y)$

\[ u_c(E_1, E_-) = 1_{E_1} - \frac{|E_1|}{|E_-|} 1_{E_-} \]
Aspect ratio?

\[ r_1(\beta) = \frac{L}{2} \left( 1 + \frac{\beta + 1/\beta}{2L} \right) \left( 1 - \sqrt{1 - (1 - \pi/4) \frac{1 - 1/L^2}{(\beta + 1/\beta)^2}} \right) \]

\[ r_2(\beta) = \frac{3L - \beta}{8(1 - \pi/4)} \left( 1 - \sqrt{1 - 8(1 - \pi/4) \frac{L(L - \beta)}{(3L - \beta)^2}} \right) \]

\[ r_1(\beta) = 1/\lambda_{c,1}(\beta) \]

\[ r_2(\beta) = 1/\lambda_{c,2}(\beta) \]
Fig. 5. Different mechanisms for the rectangle as $\beta$ is varied for $L = 3$: a) $\lambda_c(\beta)$; b) $Y_c(\beta)$. The optimal values are in solid black and sub-optimal are in broken red.

\[ Y_c(\beta) = \frac{1}{\text{Per}(\Omega_{1c}) + \min\{\lambda_{c,k}(\beta)\}|\Omega_{1c}|} = \frac{1}{2(\beta + 1/\beta) + \min\{\lambda_{c,k}(\beta)\}}. \]
Sets and solutions are non-unique
Proximity to boundary

\[ L = 3.33, l = 1, d = 1.1, Y_c = 0.176 \]

\[ \Omega_- = \text{Open}_r (\Omega) \setminus \Omega_s \]
\[ r = 0.60 \]

\[ L = 3.33, l = 1, d = 0.18, Y_c = 0.188 \]

\[ \Omega_- = \text{Open}_r (\Omega \setminus \Omega_s) \]
\[ r = 0.78 \]

**Fig. 6.** In this case, area and perimeter of \( \Omega, \Omega_s \) are constant. We change the distance between \( \partial \Omega \) and \( \Omega_s \). The critical yield number is larger if the inner set \( \Omega_s \) is close to \( \partial \Omega \).
Orientation:

\[ R = 2.36, \, l = 0.71, \, d = 1.40, \, Y_c = 0.145 \]

\[ R = 2.36, \, l = 0.71, \, d = 1.20, \, Y_c = 0.145 \]

Note: \( \Omega_{lc} \subset \Omega_s \)
Fig. 9. Upper row, left: Setup for the periodic case. Upper row, right: Dependence of the critical yield number on $\delta$, for $L = 12$, $N = 12$ and $a = 0.4$. The corner in the graph corresponds to the transition from trivial to bridged optimal sets. Lower row: Optimal sets for $\delta = 0.04$ and $\delta = 0.2$, when $L = 12$, $N = 12$ and $a = 0.4$. 