Viscoplastic Lubrication II

Chapter 2: sections 2.1 – 2.4

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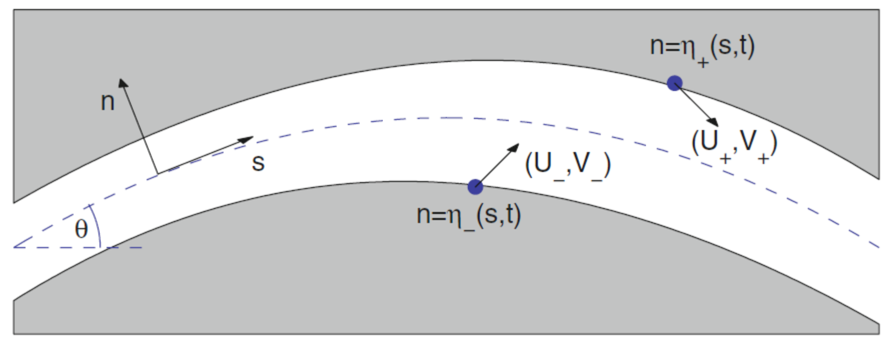
After seeing how Walton & Bittleston resolved the lubrication paradox, goal is to study this more generally

- 2D Stokes flow in a long-thin curvilinear geometry
  - Could also be inertial
  - s = arc length, n = normal coordinate
  - Walls at n=$\eta_\pm$, moving?
  - Velocity **u**=(u,v) in (s,n) directions
  - Curvature κ
- Equations of motion:
- Constitutive law & strain rates



$$\frac{\partial u}{\partial s} + (1-\kappa n)\frac{\partial v}{\partial n} - \kappa v = 0,$$

$$\frac{\partial \tau_{\rm ss}}{\partial s} + (1-\kappa n)\frac{\partial \tau_{\rm sn}}{\partial n} - 2\kappa\tau_{\rm sn} = \frac{\partial p}{\partial s},$$

$$\frac{\partial \tau_{\rm sn}}{\partial s} + (1-\kappa n)\frac{\partial \tau_{\rm nn}}{\partial n} + \kappa(\tau_{\rm ss} - \tau_{\rm nn}) = \frac{\partial p}{\partial n},$$

$$v = V_\pm = \frac{\partial \eta_\pm}{\partial t} + \frac{U_\pm}{1-\kappa\eta_\pm}\frac{\partial \eta_\pm}{\partial s} \qquad \text{with } u = U_\pm$$

$$\tau_{ij} = \mu(\dot\gamma)\dot\gamma_{ij} + \tau_Y\frac{\dot\gamma_{ij}}{\dot\gamma}, \quad \text{if } \sqrt{\tau_{\rm ss}^2 + \tau_{\rm sn}^2} > \tau_Y$$

$$\dot\gamma \equiv \sqrt{\dot\gamma_{\rm ss}^2 + \dot\gamma_{\rm sn}^2}$$

$$\dot\gamma_{\rm ss} = \frac{2}{1-\kappa n}\left(\frac{\partial u}{\partial s} - \kappa v\right), \quad \dot\gamma_{\rm nn} = 2\frac{\partial v}{\partial n}, \quad \dot\gamma_{\rm sn} = \frac{1}{1-\kappa n}\left(\frac{\partial v}{\partial s} + \kappa u\right) + \frac{\partial u}{\partial n},$$

- Scale problem:
  - $\mathcal{L}, \mathcal{H}, \mathcal{P}, \mathcal{U}$ scalesfor length, width, pressure and velocity
  - $\varepsilon = \mathcal{H}/\mathcal{L} << 1$
  - Curvature: $\varepsilon/\mathcal{L}$

$$\hat{s} = \frac{s}{\mathcal{L}}, \quad (\hat{n}, \hat{\eta}_\pm) = \frac{1}{\mathcal{H}}(n, \eta_\pm), \quad \hat{u} = \frac{u}{\mathcal{U}}, \quad \hat{v} = \frac{v\mathcal{L}}{\mathcal{U}\mathcal{H}}$$

$$\hat{t} = \frac{\mathcal{U}t}{\mathcal{L}}, \quad \hat{p} = \frac{p}{\mathcal{P}}, \quad (\hat{\tau}, \hat{\sigma}) = \frac{\mathcal{L}}{\mathcal{H}\mathcal{P}}(\tau_{\mathrm{sn}}, \tau_{\mathrm{ss}}),$$

- Reduced problem:
    - Drop ^ notation
    - $\tau = \tau_{sn}$ = shear stress

$$\frac{\partial \hat{u}}{\partial \hat{s}} + \frac{\partial \hat{v}}{\partial \hat{n}} = 0 \qquad \frac{\partial \hat{p}}{\partial \hat{s}} = \frac{\partial \hat{\tau}}{\partial \hat{n}}, \quad \frac{\partial \hat{p}}{\partial \hat{n}} = 0.$$

  - Integrate:
    - τ_- = shear stress on lower wall

$$p = p(s,t) \qquad \& \qquad \tau = \tau_-(s,t) + (n - \eta_-)p_s$$

$$\tau_+ - \tau_- = hp_s \qquad h = \eta_+ - \eta_-$$

  - Constitutive laws simplified:
    - $B = \tau_Y/(\varepsilon\mathcal{P})$

$$[\dot{\gamma}_{\mathrm{ss}}, \dot{\gamma}_{\mathrm{sn}}] = [2\epsilon u_s, u_n + \epsilon\kappa u] + O(\epsilon^2)$$

$$\dot{\gamma} \sim |\dot{\gamma}_{\mathrm{sn}}| \sim |u_n| \qquad \& \qquad \tau \sim \mu(\dot{\gamma})u_n + B\,\mathrm{sgn}(u_n)$$

  - Integrate w.r.t. n for velocity at leading order:

Will always find lubrication paradox at leading order, wherever there is a “plug”

General resolution of paradox: whenever $u_n = O(\varepsilon)$

- Second pseudo-plug solution:

$$u \sim u_p(s,t) + \epsilon u_1(s,n,t) + \ldots \qquad \& \qquad (\dot{\gamma}_{ss}, \dot{\gamma}_{sn}) = \epsilon(2u_{ps}, u_{1n} + \kappa u_p) + O(\epsilon^2). \tag{15}$$

The viscous part of the shear stress is then small, but the yield stress contribution demands the stress state is dictated by

$$(\tau, \sigma) = \frac{(\dot{\gamma}_{ns}, \dot{\gamma}_{ss})}{\dot{\gamma}} \qquad \& \qquad \tau^2 + \sigma^2 = B^2. \tag{16}$$

- Tasks:
  - A: find where $u_n = O(\varepsilon)$, e.g. using regular shear flow solution
  - B: matching between the 2 solutions. In terms of asymptotics, both are “outer” solutions
  - C: Possibly refine analysis to find true plugs & yield surfaces

Problem 1: Squeeze flow

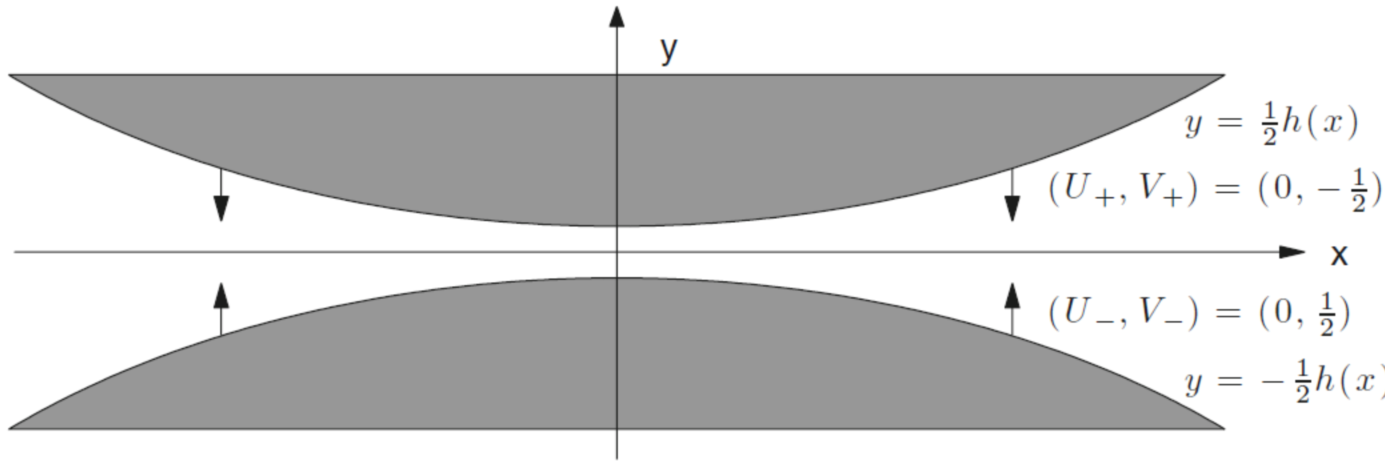


Fig. 2 Sketch of a Bingham squeeze flow

- Cartesian coordinates (s,n) = (x,y); $\eta_{\pm} = \pm 0.5h(x)$
- Vertical motion of the 2 symmetric plates
- Solution for sheared layers:
  - Strain rate →0 as |y|→Y

$$u_y = (|y| - Y)p_x \operatorname{sgn}(y), \qquad Y = \frac{B}{|p_x|},$$

$$u = -\frac{1}{2}p_x\left(\frac{1}{2}h - |y|\right)\left(\frac{1}{2}h - 2Y + |y|\right)$$

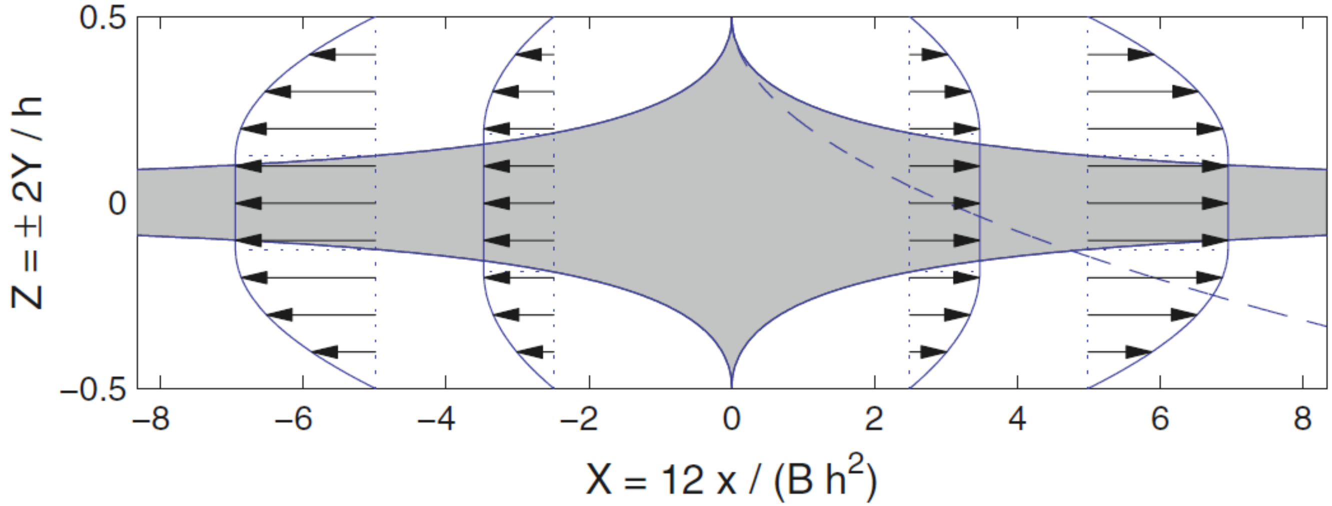


Fig. 3 The velocity profile and pseudoplug (shaded) in Bingham squeeze flow. The dashed line shows the approximation, $Y \sim \frac{1}{2}h - \sqrt{x/B}$, for $B \gg 1$

Problem 2: Slider bearing

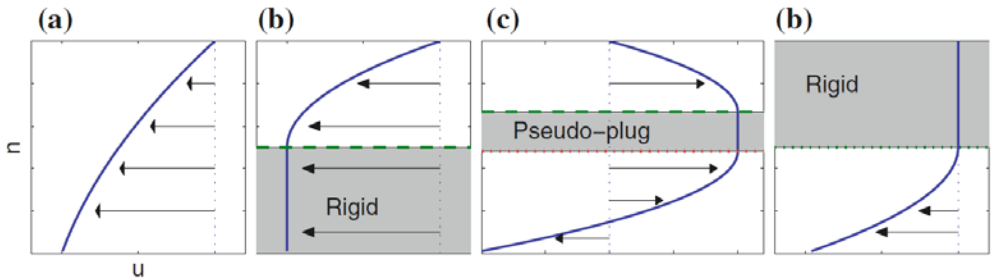


Fig. 5 Examples of the four possible flow configurations

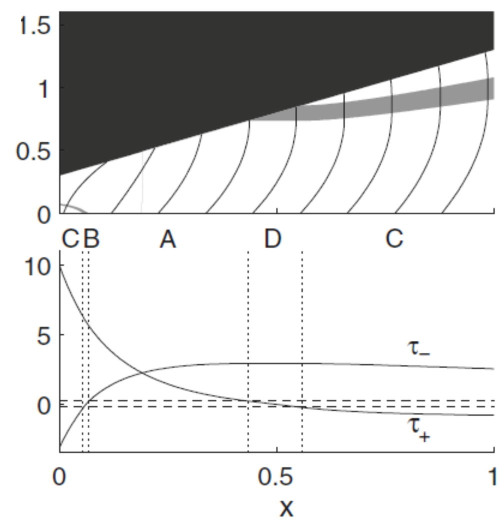


Fig. 7 A viscoplastic slider bearing solution, showing on top the geometry, true and pseudo plugs (shaded) and sample horizontal velocity profiles (solid lines), and on the bottom the surface shear stresses $\tau_{\pm}(x)$. The inclined slider (of length 1) moves to the right with speed 1. The dotted lines show the borders of the regions with different flow configurations (as indicated), and the dashed lines show $\pm B = 0.2$ (with power-law index $n = 1$). There are small C and B regions underneath the slider near the narrowest part of the gap

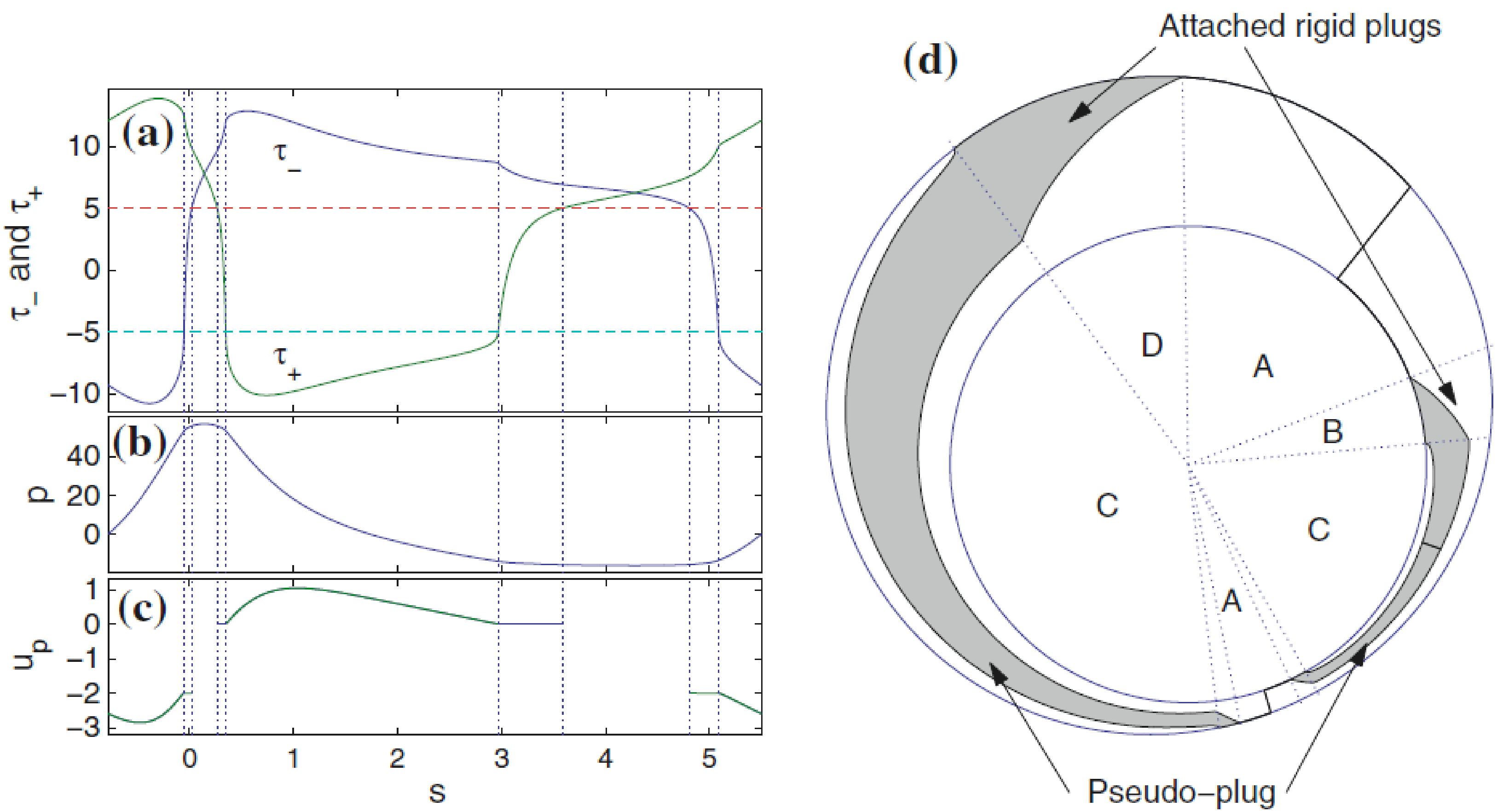


Fig. 6 A viscoplastic journal bearing solution, showing **a** $\tau_\pm(s)$, **b** $p(s)$, **c** $u_p(s)$ and **d** the fully yielded regions, true plugs and pseudo-plugs. The dotted lines show the borders of the regions with different flow configurations (as indicated in (**d**)). In (**a**), the dashed lines show $\pm B$. The origin of s (which corresponds to angle) is the location of the minimum gap. The outer cylinder (of radius 1.1) is fixed in place whilst the inner cylinder (of radius 0.8) rotates with angular speed 2 and its centre moves in the direction of the line of centres so as to close the minimum gap at speed 1. The fluid has a Bingham number of $B = 5$ and a power-law index of $n = \frac{1}{2}$