MATH104-106, Polynomial approximations

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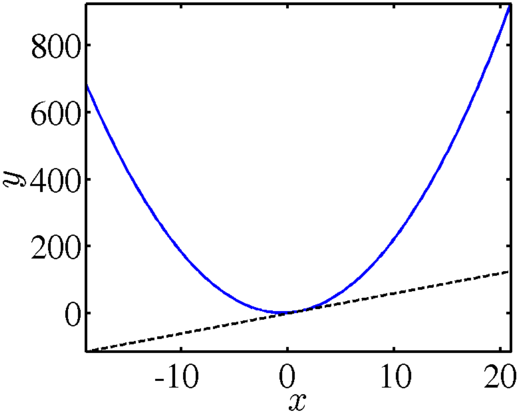
Linear approximation

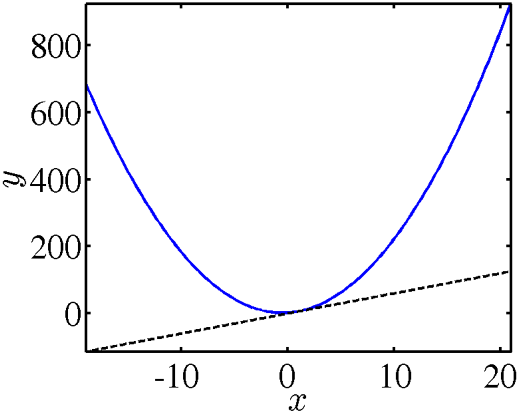


$y = 2x^2 + 2x + 1,$

$-20 < x - 1 < 20$

dashed line is the tangent line at $x = 1$

Linear approximation

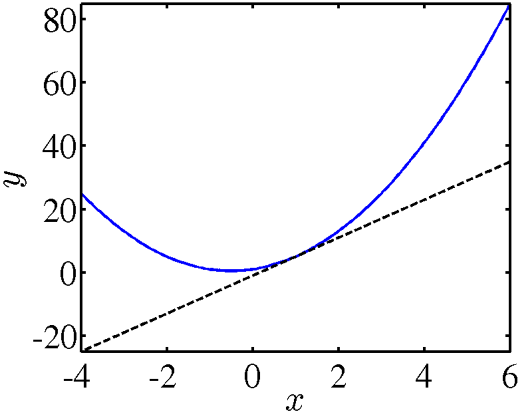


$y = 2x^2 + 2x + 1,$
$-20 < x - 1 < 20$

dashed line is the tangent line at $x = 1$

Let's zoom in around $x = 1$

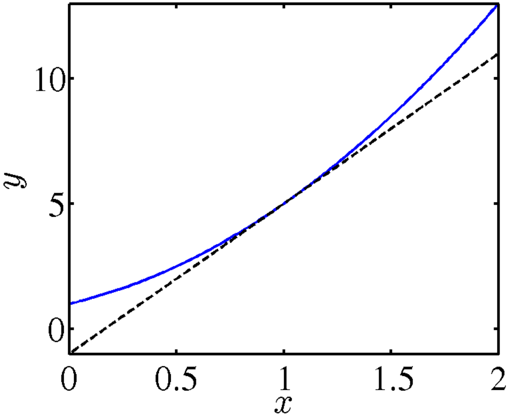
Linear approximation



$$y = 2x^2 + 2x + 1,$$
$$-5 < x - 1 < 5$$

dashed line is the tangent line at $x = 1$

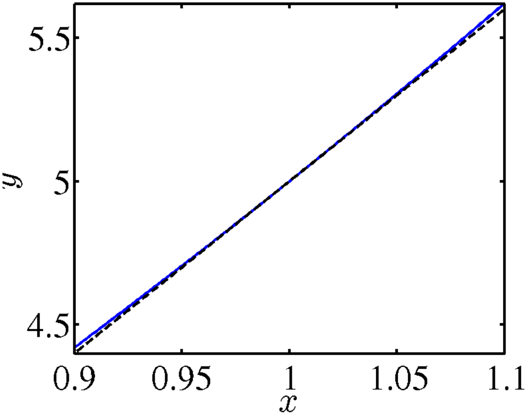
Linear approximation



$y = 2x^2 + 2x + 1,$
$-1 < x - 1 < 1$

dashed line is the tangent line at $x = 1$

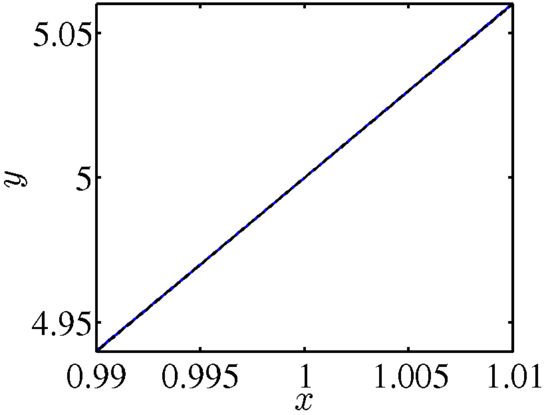
Linear approximation



$y = 2x^2 + 2x + 1,$
$-0.1 < x - 1 < 0.1$

dashed line is the tangent line at $x = 1$

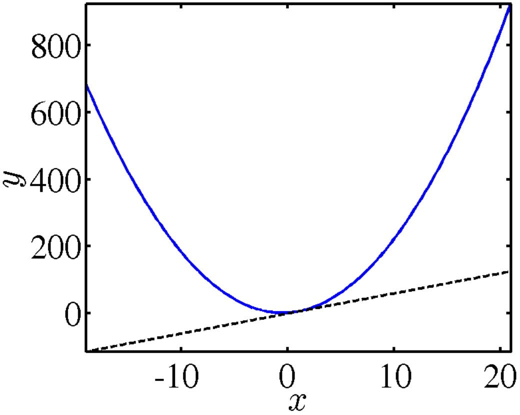
Linear approximation



$$y = 2x^2 + 2x + 1,$$
$$-0.01 < x - 1 < 0.01$$

dashed line is the tangent line at $x = 1$

Linear approximation



$$y = 2x^2 + 2x + 1,$$

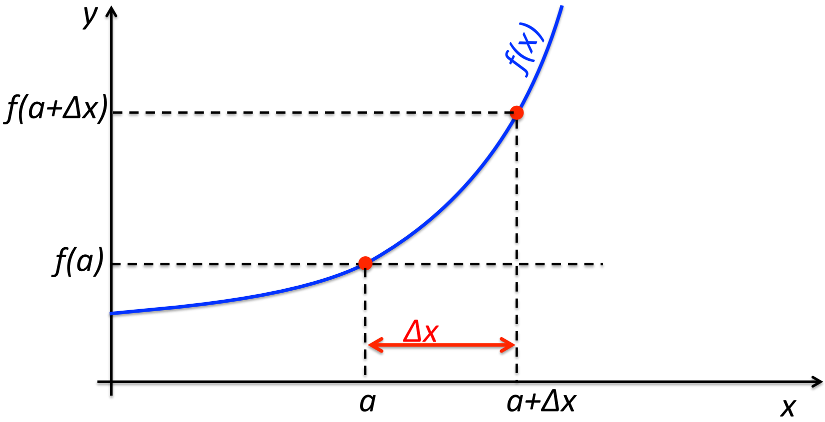
$$\frac{dy}{dx} = 4x + 2$$

$$x = 1 \Rightarrow y = 5, y' = 6$$

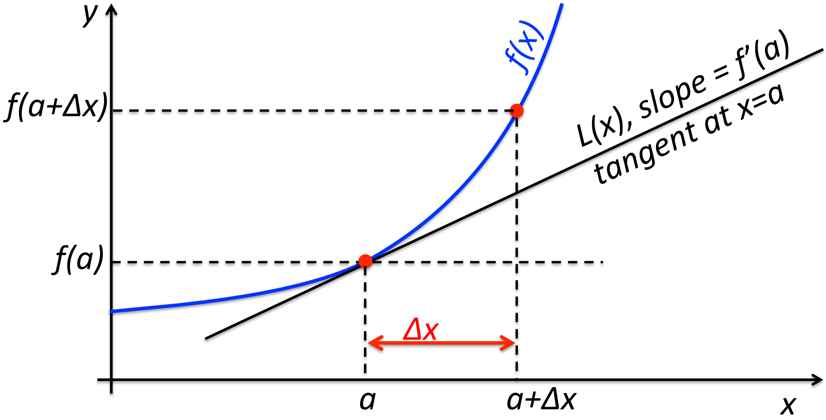
So the equation of the tangent line is:

$$y - 5 = 6(x - 1)$$

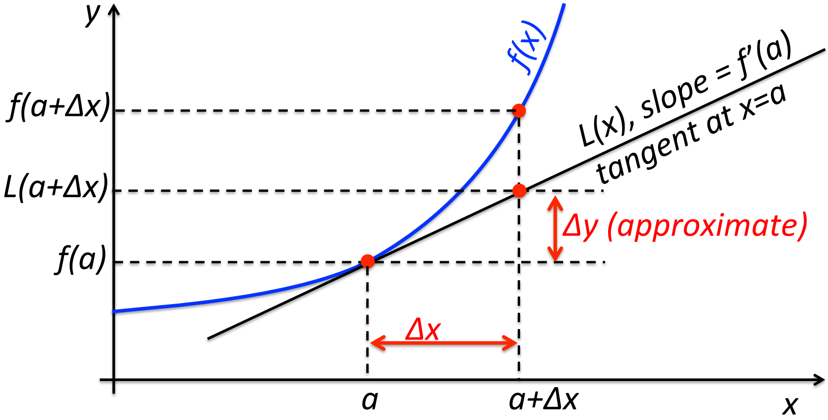
Relationship between Δx and Δy



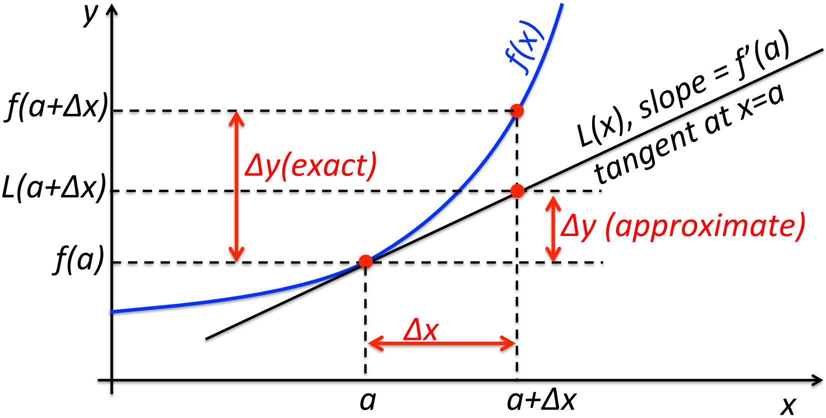
Relationship between Δx and Δy



Relationship between Δx and Δy



Relationship between Δx and Δy



Relationship between Δx and Δy

Relationship between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two point a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a)\,\Delta x$$

where $a + \Delta x$ is in I.

Note:

If f is concave up on I, using linear approximation of f at a we underestimate the value of the function; i.e. $L(a + \Delta x) < f(a + \Delta x)$. Alternatively, if f is concave down on I, using linear approximation of f at a we overestimate the value of the function; i.e. $L(a + \Delta x) > f(a + \Delta x)$.

Linear approximation and concavity

$y = 2x^2 + 2x + 1$

$y' = \frac{dy}{dx} = 4x + 2$

$y = x^2 + 4x$

$y' = \frac{dy}{dx} = 2x + 4$

At $x = 1$:

$y = 5$

$y' = 6$

$y = 5$

$y' = 6$

Linear approximation at $x = 1$:

$L(x) = 5 + 6(x - 1)$

$L(x) = 5 + 6(x - 1)$

Question: For which function does $L(1.1)$ give a more accurate estimate of y at $x = 1.1$?

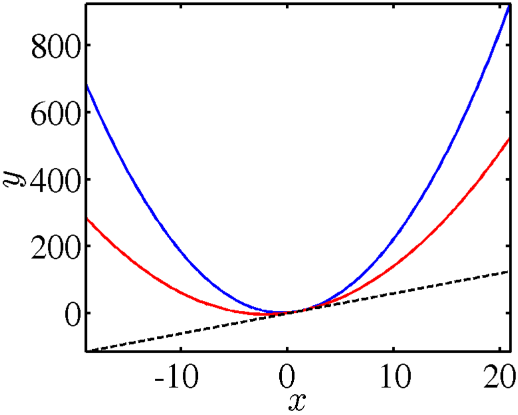
a) $y = 2x^2 + 2x + 1$

b) $y = x^2 + 4x$

c) It depends!

d) I don't know!

Linear approximation and concavity



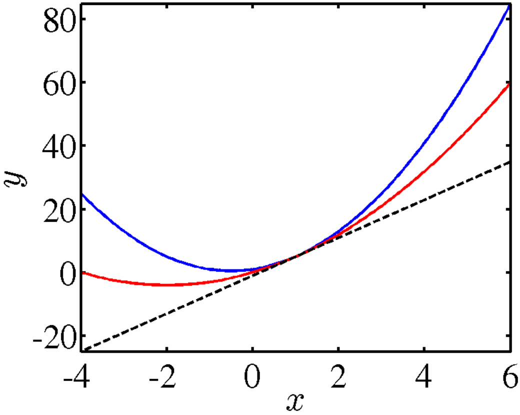
$y = 2x^2 + 2x + 1,$

$y = x^2 + 4x,$

$-20 < x - 1 < 20$

dashed line is the tangent line at $x = 1$

Linear approximation and concavity



$y = 2x^2 + 2x + 1,$

$y = x^2 + 4x,$

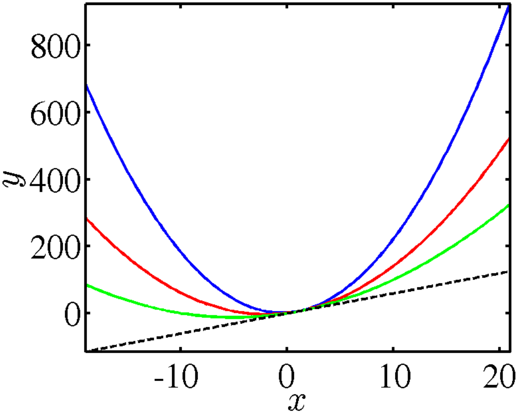
$-5 < x - 1 < 5$

dashed line is the tangent line at $x = 1$

$y'' = 4, \quad y'' = 2,$

The function with the smaller $|y''|$ gives a better estimate.

Linear approximation and concavity



$y = 2x^2 + 2x + 1,$

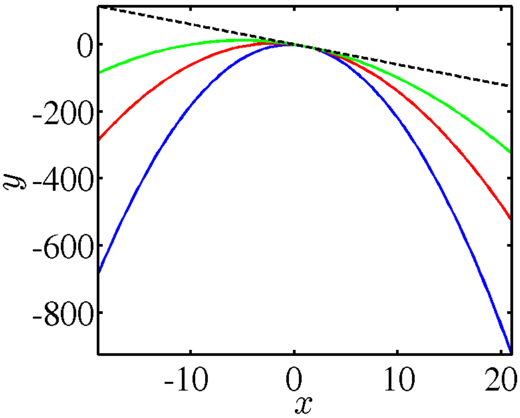
$y = x^2 + 4x,$

$y = \frac{1}{2}x^2 + 5x - \frac{1}{2},$

dashed line is the tangent line at $x = 1$

$y'' = 4, \quad y'' = 2, \quad y'' = 1$

Linear approximation and concavity



$y = -2x^2 - 2x - 1,$

$y = -x^2 - 4x,$

$y = -\frac{1}{2}x^2 - 5x + \frac{1}{2},$

dashed line is the tangent line at $x = 1$

$y'' = -4, \quad y'' = -2,$

$y'' = -1$

Quadratic approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Exercise: Find the quadratic approximation of the following functions about $x = 0$:

1. $f(x) = sin(x)$
2. $f(x) = cos(x)$
3. $f(x) = x^2 + 3x + 1$

Taylor polynomials

Let f be a function with $f', f'', ..., f^{(n)}$ defined at $x = a$. The n^{th}–order Taylor polynomial for f with its center at $x = a$, denoted p_n, has the property that it matches f in value, slope and all derivatives up to the n^{th} derivative at a. The n^{th}–order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-1) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Remainder

Definition: Let P_n be the Taylor polynomial of order n for f. The remainder in using p_n to approximate f at point x is $R_n(x) = f(x) - p_n(x)$.

Note: Error $= |R_n(x)|$, Relative error $= \dfrac{|R_n(x)|}{|f(x)|}$

Theorem: Suppose there exists a number M such that $f''(c) \leq M$ for all c between a and x inclusive. The remainder in the 1^{st} order Taylor polynomial (i.e. linear approximation) for f centered at a satisfies

$$|R_1(x)| \leq M\frac{(x-a)^2}{2}$$

Note: $M\dfrac{(x-a)^2}{2}$ is our estimate of maximum error, *error bound* or *the worst-case error.*

Exercises

- Assuming that $f(2) = 0.2$, $f'(2) = 0.3$ and $f''(2) = 0.5$, estimate $f(1.8)$ and $f(2.1)$.
  Bonus: estimate $f'(2.1)$.

- Use linear approximation to estimate the following quantities. Choose a value of a that produces a small error. Without using a calculator figure out if you have overestimated or underestimated the value of the function. Give the worst-case error.

$a)\dfrac{1}{\sqrt{112}}$ $\qquad b)\ln(.95)$ $\qquad c)\tan^{-1}(0.9)$

$d)e^{0.1}$ $\qquad e)\sqrt{1.7}$

Exercises

- Find the linear approximation of $\sin(x)$ about $a = 0$. How close should x be to a to ensure that error is less than 0.02?