Week 1 - an example on limits

Sept. 17, 2015

Example 1.

Examine $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Solution: Notice that $f(x) = \sin(1/x)$ is not defined at $x = 0$. Does this mean that the limit does not exist? Let's graph $f(x) = \sin(1/x)$ to see how it changes around $x = 0$

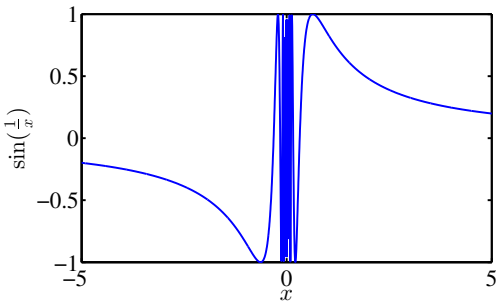
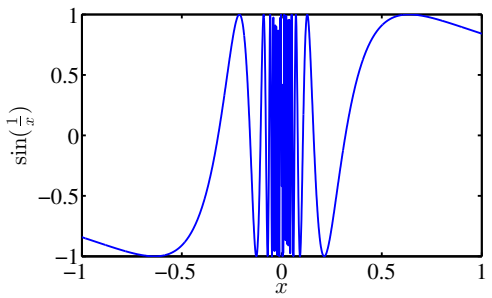
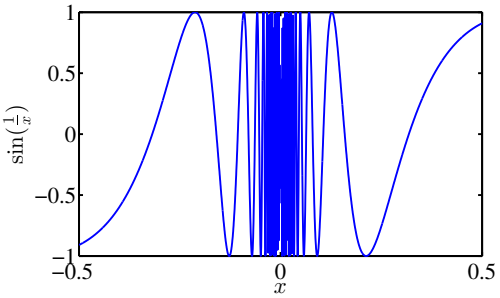


Figure 1: $-5 < x < 5$

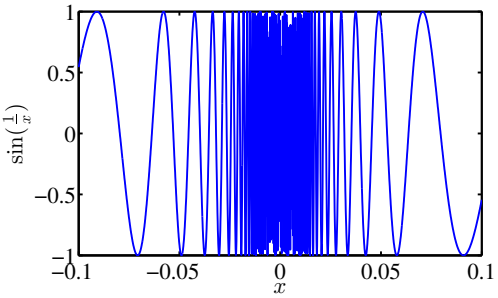
It is not quite obvious in figure 1 what is happening around $x = 0$. Let's zoom in to take a closer look ...



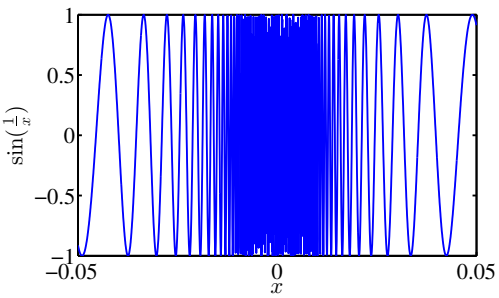
(a) $-1 < x < 1$



(b) $-0.5 < x < 0.5$



(c) $-0.1 < x < 0.1$



(d) $-.05 < x < .05$

Figure 2

Notice in figure 2 that as we zoom closer to $x = 0$ the function $f(x)$ oscillates progressively faster. Alternatively, think about how many times this function crosses $y = 0$ between $x = 0$ and $x = 1$:

$$\begin{aligned} &\sin(1/x) = 0 \\ &\Rightarrow 1/x = k\pi, \quad k = 1, 2, 3, \ldots \\ &\Rightarrow x = \frac{1}{k\pi} \end{aligned}$$

So we have

$$\sin(1/x) = 0 \quad \text{if} \quad \left(x = \frac{1}{k\pi} \text{ and } k \in N \right)$$

Notice that for $k \geq 1$ we have $1/(k\pi) < 1$; i.e. there are infinitely many points between $x = 0$ and $x = 1$ where $\sin(1/x) = 0$. So $f(x)$ continuously oscillates and does not approach a certain limit. So $\lim_{x \to 0} f(x)$ does not exist. Indeed the left-hand and right-hand limits don't exist either.