Week 5 - Exercises

Oct. 13 & 15, 2015

Example 1.

Find $\dfrac{dy}{dx}$ given that

a. $y = \pi^{e^2x^3}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\pi^{e^2x^3}\right) = \frac{d}{du}\left(\pi^u\right)\frac{du}{dx}, \quad u = e^2x^3$$
$$\frac{dy}{dx} = \ln(\pi)\pi^u \frac{d}{dx}\left(e^2x^3\right)$$
$$\frac{dy}{dx} = \ln(\pi)\pi^u \left(3e^2x^2\right)$$
$$\frac{dy}{dx} = \ln(\pi)\pi^{e^2x^3} \left(3e^2x^2\right)$$

b. $y^e = 2^x$

Solution:

$$\frac{d}{dx}\left(y^e\right) = \frac{d}{dx}\left(2^x\right)$$
$$\frac{d}{du}\left(u^e\right)\frac{du}{dx} = \ln(2) \times 2^x, \quad u = y$$
$$eu^{e-1}\frac{du}{dx} = \ln(2) \times 2^x$$
$$ey^{e-1}\frac{dy}{dx} = \ln(2) \times 2^x$$
$$\frac{dy}{dx} = \frac{\ln(2) \times 2^x}{ey^{e-1}}$$

Alternative approach:

$$\ln(y^e) = \ln(2^x)$$
$$e\ln(y) = x\ln(2)$$
$$\frac{d}{dx}(e\ln(y)) = \frac{d}{dx}(x\ln(2))$$
$$\frac{d}{du}(e\ln(u))\frac{du}{dx} = \ln(2), \quad u = y$$
$$\frac{e}{u}\frac{du}{dx} = \ln(2)$$
$$\frac{e}{y}\frac{dy}{dx} = \ln(2)$$

$$\frac{dy}{dx} = \frac{\ln(2)y}{e}$$

Note that the results of the two approaches are equivalent:

$$\frac{\ln(2) \times 2^x}{ey^{e-1}} = \frac{\ln(2)y^e}{ey^{e-1}} = \frac{\ln(2)y}{e}$$

c. $y^x = x$

Solution:

$$\ln(y^x) = \ln(x)$$
$$x\ln(y) = \ln(x)$$
$$\frac{d}{dx}(x\ln(y) = \ln(x)$$
$$\frac{dx}{dx}\ln(y) + x\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\ln(x))$$
$$1 \times \ln(y) + x\frac{d}{du}(\ln(u))\frac{du}{dx} = \frac{1}{x}, \quad u = y$$
$$\ln(y) + x\frac{1}{u}\frac{du}{dx} = \frac{1}{x}$$
$$\ln(y) + \frac{x}{y}\frac{dy}{dx} = \frac{1}{x}$$
$$\frac{dy}{dx} = \left(\frac{1}{x} - \ln(y)\right)\left(\frac{y}{x}\right)$$

d. $\cos(\pi y) = \sin(3x)$

Solution:

$$\frac{d}{dx}\cos(\pi y) = \frac{d}{dx}(\sin(3x))$$
$$\frac{d}{du}\cos(u)\frac{du}{dx} = 3\cos(3x), \quad u = \pi y$$
$$-\sin(u)\frac{du}{dx} = 3\cos(3x)$$
$$-\sin(\pi y)\frac{d(\pi y)}{dx} = 3\cos(3x)$$
$$-\sin(\pi y)\pi\frac{dy}{dx} = 3\cos(3x)$$
$$\frac{dy}{dx} = -\frac{3\cos(3x)}{\pi\sin(y)}$$

e.

$$\frac{y^{5a}x^2}{\sqrt{x^3+4}} = \frac{b^x\sin(x^4)}{e^x + y}$$

a and b are constants.

Solution:

$$\ln\left(\frac{y^{5a}x^2}{\sqrt{x^3+4}}\right) = \ln\left(\frac{b^x\sin(x^4)}{e^x+y}\right)$$
$$\ln(y^{5a}x^2) - \ln(\sqrt{x^3+4}) = \ln(b^x\sin(x^4)) - \ln(e^x+y)$$
$$\ln(y^{5a}) + \ln(x^2) - \ln((x^3+4)^{1/2}) = \ln(b^x) + \ln(\sin(x^4)) - \ln(e^x+y)$$
$$5a\ln(y) + 2\ln(x) - \frac{1}{2}\ln(x^3+4) = x\ln(b) + \ln(\sin(x^4)) - \ln(e^x+y)$$
$$\frac{d}{dx}(5a\ln(y)) + \frac{d}{dx}(2\ln(x)) - \frac{d}{dx}\left(\frac{1}{2}\ln(x^3+4)\right) =$$
$$\frac{d}{dx}(x\ln(b)) + \frac{d}{dx}(\ln(\sin(x^4))) - \frac{d}{dx}(\ln(e^x+y))$$
$$\frac{d}{du}(5a\ln(u))\frac{du}{dx} + \frac{2}{x} - \frac{d}{dv}\left(\frac{1}{2}\ln(v)\right)\frac{dv}{dx} =$$
$$\ln(b) + \frac{d}{dw}(\ln(w))\frac{dw}{dx} - \frac{d}{dp}(\ln(p))\frac{dp}{dx}$$
$$\frac{5a}{u}\frac{du}{dx} + \frac{2}{x} - \frac{1}{2v}\frac{dv}{dx} =$$
$$\ln(b) + \frac{1}{w}\frac{dw}{dx} - \frac{1}{p}\frac{dp}{dx} \tag{1}$$

Here $u = y, v = x^3+4, w = \sin(x^4), p = e^x + y$.

$$\frac{du}{dx} = \frac{dy}{dx}$$
$$\frac{dv}{dx} = \frac{d}{dx}\left(x^3+4\right) = 3x^2$$
$$\frac{dw}{dx} = \frac{d}{dx}(\sin(x^4)) = \frac{d}{dU}(\sin(U))\frac{dU}{dx}, \quad U = x^4$$
$$\frac{dw}{dx} = \cos(U)\frac{dU}{dx} = \cos(x^4)\frac{dx^4}{dx} = \cos(x^4)4x^3$$
$$\frac{dp}{dx} = \frac{d}{dx}(e^x+y) = \frac{d}{dx}(e^x) + \frac{dy}{dx} = e^x + \frac{dy}{dx}$$

Now we can plug the above equations in equation (1).

$$\frac{5a}{u}\frac{du}{dx} + \frac{2}{x} - \frac{1}{2v}\frac{dv}{dx} =$$
$$\ln(b) + \frac{1}{w}\frac{dw}{dx} - \frac{1}{p}\frac{dp}{dx}$$
$$\frac{5a}{y}\frac{dy}{dx} + \frac{2}{x} - \frac{1}{2(x^3+4)}(3x^2) =$$
$$\ln(b) + \frac{1}{\sin(x^4)}(\cos(x^4)4x^3) - \frac{1}{e^x+y}\left(e^x + \frac{dy}{dx}\right)$$

$$\left(\frac{5a}{y}+\frac{1}{e^x+y}\right)\frac{dy}{dx}=$$
$$\ln(b)+\frac{4x^3\cos(x^4)}{\sin(x^4)}-\frac{e^x}{e^x+y}-\frac{2}{x}+\frac{3x^2}{2(x^3+4)}$$
$$\frac{dy}{dx}=\frac{\ln(b)+\frac{4x^3\cos(x^4)}{\sin(x^4)}-\frac{e^x}{e^x+y}-\frac{2}{x}+\frac{3x^2}{2(x^3+4)}}{\frac{5a}{y}+\frac{1}{e^x+y}}$$

f. $2^{x^2}\log_3(x^2+2x)^5=y\sqrt[3]{x^5+\sin(x)}$

Solution:

$$\ln\left(2^{x^2}\log_3(x^2+2x)^5\right)=\ln\left(y\sqrt[3]{x^5+\sin(x)}\right)$$
$$\ln\left(2^{x^2}\right)+\ln\left(\log_3(x^2+2x)^5\right)=\ln(y)+\ln\left(\sqrt[3]{x^5+\sin(x)}\right)$$
$$x^2\ln(2)+\ln\left(5\log_3(x^2+2x)\right)=\ln(y)+\frac{1}{3}\ln\left(x^5+\sin(x)\right)$$
$$\frac{d}{dx}\left(x^2\ln(2)\right)+\frac{d}{dx}\left(\ln\left(5\log_3(x^2+2x)\right)\right)=\frac{d}{dx}(\ln(y))+\frac{d}{dx}\left(\frac{1}{3}\ln\left(x^5+\sin(x)\right)\right)$$
$$\ln(2)2x+\frac{d}{du}(\ln(u))\frac{du}{dx}=\frac{d}{dv}(\ln(v))\frac{dv}{dx}+\frac{d}{dw}\left(\frac{1}{3}\ln(w)\right)\frac{dw}{dx},$$
$$\ln(2)2x+\frac{1}{u}\frac{du}{dx}=\frac{1}{v}\frac{dv}{dx}+\frac{1}{3}\frac{1}{w}\frac{dw}{dx} \tag{2}$$

Here $u=5\log_3(x^2+2x), v=y, w=x^5+\sin(x)$.

$$\frac{dv}{dx}=\frac{dy}{dx}$$
$$\frac{du}{dx}=\frac{d}{dx}\left(5\log_3(x^2+2x)\right)=\frac{d}{dU}(5U)\frac{dU}{dx},\quad U=\log_3(x^2+2x)$$
$$\frac{du}{dx}=5\frac{dU}{dx}=5\frac{d}{dx}\left(\log_3(x^2+2x)\right)=5\frac{d}{dV}(\log_3(V))\frac{dV}{dx},\quad V=x^2+2x \tag{3}$$

Here we need to find $\frac{d}{dV}(\log_3(V))$. Note that $\frac{d}{dV}(\ln(V))=\frac{1}{V}$. So we need to find what is the derivative of the logarithm function when the base is not e. Remember that we can change the base of the logarithm:

$$\log_3(V)=\frac{\ln(V)}{\ln(3)}$$

Therefore we have

$$\frac{d}{dV}(\log_3(V))=\frac{d}{dV}\left(\frac{\ln(V)}{\ln(3)}\right)=\frac{1}{\ln(3)V}$$

Now we can calculate dv/dx. Plugging the above results in equation (3), we find:

$$\frac{du}{dx} = 5\frac{d}{dV}(\log_3(V))\frac{dV}{dx} = \frac{5}{\ln(3)V}\frac{dV}{dx} = \frac{5}{\ln(3)(x^2+2x)}\frac{d(x^2+2x)}{dx} = \frac{5(2x+2)}{\ln(3)(x^2+2x)}$$
$$\frac{dw}{dx} = \frac{d}{dx}(x^5+\sin(x)) = 5x^4+\cos(x)$$

Now we can plug everything back in equation (2).

$$\ln(2)2x + \frac{1}{u}\frac{du}{dx} = \frac{1}{v}\frac{dv}{dx} + \frac{1}{3}\frac{1}{w}\frac{dw}{dx}$$
$$\ln(2)2x + \frac{1}{5\log_3(x^2+2x)}\frac{5(2x+2)}{\ln(3)(x^2+2x)} = \frac{1}{y}\frac{dy}{dx} + \frac{(5x^4+\cos(x))}{3(x^5+\sin(x))}$$
$$\frac{dy}{dx} = \left(\frac{1}{5\log_3(x^2+2x)}\frac{5(2x+2)}{\ln(3)(x^2+2x)} - \frac{(5x^4+\cos(x))}{3(x^5+\sin(x))} + \ln(2)2x\right)y$$

Example 2.

Given $F(x) = \dfrac{2\sin(\frac{\pi}{2}F(G(x)))}{x}$, find an equation of the tangent line to $F(x)$ at $x = 2$. Assume

$$\begin{array}{lll} F(1)=3, & F(2)=1, & F(3)=2,\\ G(1)=3, & G(2)=2, & G(3)=1,\\ G'(1)=2, & G'(2)=1, & G'(3)=3. \end{array}$$

Solution: First let's find the slope of the tangent line.

$$\begin{aligned}\frac{dF}{dx} &= 2\frac{d}{dx}\left(\frac{\sin(\frac{\pi}{2}F(G(x)))}{x}\right)\\ &= 2\frac{\frac{d}{dx}(\sin(\frac{\pi}{2}F(G(x))))x - \frac{dx}{dx}\sin(\frac{\pi}{2}F(G(x)))}{x^2}\\ &= 2\frac{\frac{d}{dx}(\sin(\frac{\pi}{2}F(G(x))))x - \sin(\frac{\pi}{2}F(G(x)))}{x^2}\end{aligned}$$

Now we need to find $\frac{d}{dx}(\sin(\frac{\pi}{2}F(G(x))))$.

$$\begin{aligned}\frac{d}{dx}(\sin(\frac{\pi}{2}F(G(x)))) &= \frac{d}{dU}(\sin(U))\frac{dU}{dx}, \quad U = \frac{\pi}{2}F(G(x)))\\ &= \cos(U)\frac{\pi}{2}\frac{d}{dx}(F(G(x)))\\ &= \cos\left(\frac{\pi}{2}F(G(x))\right)\frac{\pi}{2}G'(x)F'(G(x))\end{aligned}$$

So we have found an equation for dF/dx.

$$\begin{aligned}
\frac{dF}{dx} &= 2\frac{\frac{d}{dx}(\sin(\frac{\pi}{2}F(G(x))))x - \sin(\frac{\pi}{2}F(G(x)))}{x^2} \\
&= 2\frac{\frac{\pi}{2}\cos(\frac{\pi}{2}F(G(x)))G'(x)F'(G(x))x - \sin(\frac{\pi}{2}F(G(x)))}{x^2}
\end{aligned}$$

When $x = 2$ we have

$$\begin{aligned}
F'(2) &= 2\frac{\frac{\pi}{2}\cos(\frac{\pi}{2}F(G(2)))G'(2)F'(G(2)) \times 2 - \sin(\frac{\pi}{2}F(G(2)))}{2^2}, \quad G(2) = 2, G'(2) = 1 \\
&= \frac{\frac{\pi}{2}\cos(\frac{\pi}{2}F(2)) \times 1 \times F'(2) \times 2 - \sin(\frac{\pi}{2}F(2))}{2}, \quad F(2) = 1 \\
&= \frac{\frac{\pi}{2}\cos(\frac{\pi}{2}) \times F'(2) - \sin(\frac{\pi}{2})}{2}, \\
&= \frac{-\sin(\frac{\pi}{2})}{2} = -\frac{1}{2}
\end{aligned}$$

So the slope of the tangent line is $-1/2$ and the point where we are finding the tangent line is $(2, F(2)) = (2, 1)$. So the equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$