① MATH 110
Jan. 4, 2016

Week 1

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Lectures: MWF 8-9 am
Office hours: MV 9_10 am

clickers
interrupt me to ask questions.

biweekly quizzes
weekly online webwork
biweekly homework (written)
weekly workshops

Today: final review
implicit diff.

Overall course average: 50

final, Q5: A ball is floating on the surface of the sea. The ball's height h (in m) above the sea floor at time m (in s) is given by

$$h(t) = 5 + 0.1 \sin(2t)$$

Answer the questions below. Evaluate all func. values and provide justification for your claims. Include units.

a) What is the range of possible values for the ball's height?
$h(t)$

looking for the range of $h(t)$

$h(t) = 5 + 0.1 \sin(2t)$ $\qquad t \geq 0$

$\min(\sin(2t)) = \sin(2 * \frac{3\pi}{4}) = \sin(\frac{3\pi}{2}) = -1$

$\max(\sin(2t)) = \sin(2 * \frac{\pi}{4}) = \sin(\frac{\pi}{2}) = 1$

$$\min(h(t)) = 5 + \cancel{\min}\ 0.1 * \min(\sin(2t)) = 5 + 0.1 * (-1)$$
$$= 4.9$$

$$\max(h(t)) = 5 + 0.1 * \max(\sin(2t)) = 5 + 0.1 * (1) = 5.1$$

$$\Rightarrow \quad 4.9 \leqslant h(t) \leqslant 5.1$$

b) What is the (vertical) velocity of the ball at $t = \frac{\pi}{6}$

Velocity: $v = \frac{dh}{dt} = \frac{d}{dt}(5 + 0.1\ \sin(2t))$

$$= \frac{d}{dt}(5) + \frac{d}{dt}(0.1 \sin(2t))$$

$$= 0 + 0.1 \frac{d}{dt} \sin(2t)$$

chain rule

$$= 0.1 * \frac{du}{dt} \frac{d \sin(u)}{du}$$

$u = 2t$

$$= 0.1 * \frac{d(2t)}{dt} * \cos(u)$$

$$= 0.1 * 2 * \cos(2t) = 0.2 \cos(2t)$$

$$\Rightarrow v(t) = 0.2 \cos(2t) \Rightarrow v\left(\frac{\pi}{6}\right) = 0.2 \cos\left(2 * \frac{\pi}{6}\right)$$

$$= 0.2 \cos\left(\frac{\pi}{3}\right)$$

$$= 0.2 * \frac{1}{2} = 0.1$$

$$\Rightarrow \boxed{v\left(\frac{\pi}{6}\right) = 0.1}$$

c) Is the ball moving up or down at $t = \frac{\pi}{6}$? Explain

Moving up $\Rightarrow$ h is increasing $\Rightarrow$ $\frac{dh}{dt} > 0$

Moving down $\Rightarrow$ h is decreasing $\Rightarrow$ $\frac{dh}{dt} < 0$

$$V\left(\frac{\pi}{6}\right) = \left.\frac{dh}{dt}\right|_{t=\frac{\pi}{6}} = 0.1 > 0 \Rightarrow \text{The ball is moving up}$$

d) Is the ball speeding up or down at $t = \frac{\pi}{6}$? Explain.

Speed up $\Rightarrow$ V is increasing $\Rightarrow$ $\frac{dV}{dt} > 0$

Speed down $\Rightarrow$ V is dereasing $\Rightarrow$ $\frac{dV}{dt} < 0$

$$V = 0.2\cos(2t) \Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(0.2\cos(2t)\right)$$

$$= 0.2\frac{d}{dt}\left(\cos(2t)\right)$$

Chain rule

$$= 0.2 \frac{du}{dt}\frac{d\cos u}{dt} \qquad u = 2t$$

$$= 0.2 * \frac{d(2t)}{dt} * (-\sin(u))$$

$$= -0.4\sin(2t)$$

Reminder: acceleration: ~~dV~~ $a = \frac{dV}{dt}$

$$\left.\frac{dV}{dt}\right|_{t=\frac{\pi}{6}} = -0.4\sin\left(\frac{2\pi}{6}\right) = -0.4 * \frac{\sqrt{3}}{2} = -0.2\sqrt{3} < 0$$

$\Rightarrow$ The ball is speeding down

e) find the initial position of the ball and then determine what is the earliest time $t>0$ at which the ball is back in its initial position.

initial position $\Rightarrow h(0) = ?$ $\Rightarrow h(0) = 5$

$\Rightarrow$ $5 = 5 + 0.1 \sin(2t)$

$0 = 0.1 \sin(2t)$ find smallest $t>0$ s.t. $\sin(2t) = 0$

$\Rightarrow 2t = \pi$

$\Rightarrow \boxed{t = \frac{\pi}{2}}$: earliest time $t>0$ at which the ball is back in its initial position.

Final, Q 8: If a spherical tank of radius 2 meters is filled up to h meters, the the volume of water V in the tank is given by the following function

$$V(h) = \pi h^2 \left(2 - \frac{h}{3}\right)$$

where h is measured from the deepest point in the tank.

a) What is the domain of the function $V(h)$ in this context?

What values can h have?

h varies from 0 to 2R $\Rightarrow$ Domain: $0 \leq h \leq 2 \times 2$ m

R: Radius of sphere

$0 \leq h \leq 4$m

b) As the water is poured into the tank, at what rate is the volume of water in the tank changing with respect to the height of the water when the water is 1m deep.

$h = 1$

What are the units of this quantity?

rate of change of V wrt h $\Rightarrow \frac{dV}{dh} = ?$

product rule

$$\frac{dV}{dh} = \frac{d}{dh}\left(\pi h^2 \left(2 - \frac{h}{3}\right)\right) = \frac{d}{dh}(\pi h^2)\left(2 - \frac{h}{3}\right) + \pi h^2 \frac{d}{dh}\left(2 - \frac{h}{3}\right)$$

$$= 2\pi h \left(2 - \frac{h}{3}\right) - \frac{\pi h^2}{3}$$

Today: finish Q8 from the final
Implicit differentiation, from the ref book: 2.9

section website: blogs.ubc.ca/idak/math110-001

Workshops start next week

Recap: Chain rule

$$\frac{d}{dx}(f \circ g(x)) = \frac{d}{dx}(f(g(x))) = g'(x)\, f'(g(x)) \quad (1)$$

OR

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad (2)$$

$F(x) = \sin(2x)$

(1)

$g(x) = 2x \Rightarrow g' = 2$

$f(u) = \sin(u) \Rightarrow f' = \cos(u)$

$F(x) = f(g(x))$

$$\frac{dF}{dx} = g'(x)\, f'(g(x))$$

$$= 2 * \cos(2x)$$

(2)

$u = 2x$

$$\frac{dF}{dx} = \frac{d\sin(u)}{du}\frac{du}{dx}$$

$$= \cos(u) * \frac{d(2x)}{dx}$$

$$= \cos(2x) * 2$$

$$\left.\frac{dV}{dh}\right|_{h=1} = 2\pi*1\,\overbrace{\left(2-\frac{1}{3}\right)}^{\frac{5}{3}} - \frac{\pi*1}{3} = \frac{10\pi}{3} - \frac{\pi}{3} = 3\pi$$

$$\boxed{\left.\frac{dV}{dh}\right|_{h=1} = 3\pi}$$

Mon. class ended here

c) Over time, both the volume of water and the height of the water change. Suppose you observe that the height of water is increasing at a rate of 0.1 m/min when the water in the tank is 1m deep. How fast (in m^3/min) is the volume increasing at that instant?

$$\frac{dh}{dt} = 0.1 \text{ m/min} \qquad h = 1\text{ m}$$

chain rule

$$\left.\frac{dV}{dt}\right|_{h=1m} = \frac{dV}{dh} * \frac{dh}{dt} = \left.\left(2\pi h\left(2-\frac{h}{3}\right) - \frac{\pi h^2}{3}\right)\right|_{h=0.1} * (0.1 \text{ m/min})$$

$$= 3\pi * 0.1 = 0.3\pi \ \frac{m^3}{min}$$

Q, a Find ~~the~~ an equation of the tangent line to

$f(x) = x^2$ at $x = 2$

Slope of the tangent line: $f'(2)$

$$\frac{df}{dx} = 2x \Rightarrow \left.\frac{df}{dx}\right|_{x=2} = 2*2 = 4$$

$x = 2 \Rightarrow f(2) = 4 \Rightarrow$ eq. of the tangent line

$y - f(a) = f'(a)(x-a)$ $\qquad y - 4 = 4(x-2)$

Q, b Find ~~the~~ an equation of the tangent line to $f^{-1}(x)$

at $x = 2$.

① Find inververse $\quad f(x) = x^2$

$$x = y^2$$

$\mp\sqrt{x} = y$ $\qquad$ is not a function

$f^{-1}(x) = \sqrt{x}$ $\quad x \geq 0$, range:

$f^{-1}(x) = -\sqrt{x}$ $\quad x \geq 0$, range: $(-\infty, 0]$

$$\frac{df^{-1}}{dx} = \begin{cases} \dfrac{d\sqrt{x}}{dx} = \dfrac{1}{2}\dfrac{1}{\sqrt{x}} \Rightarrow \left.\dfrac{df^{-1}}{dx}\right|_{x=2} = \dfrac{1}{2\sqrt{2}} \\ \dfrac{d(-\sqrt{x})}{dx} = \dfrac{-1}{2}\dfrac{1}{\sqrt{x}} \Rightarrow \left.\dfrac{df^{-1}}{dx}\right|_{x=2} = \dfrac{-1}{2\sqrt{2}} \end{cases}$$

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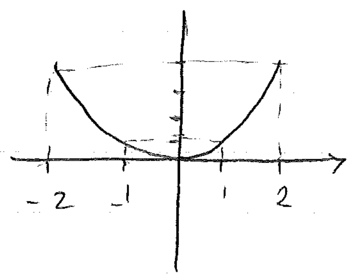
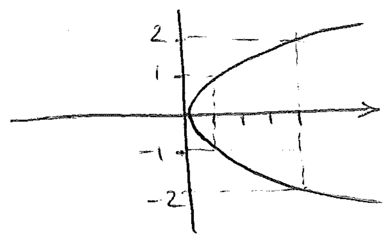
Could I have accomplished this w/o finding

$f^{-1}(x) = \sqrt{x}$ and $f^{-1}(x) = -\sqrt{x}$?

$f(x)$ is not invertible on $(-\infty, \infty)$. i.e. $y^2 = x$ is not a function. But, can we find tangents to this curve for $-\infty < y < \infty$?



$\Rightarrow$



think chain rule $\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$

$$\frac{du^2}{du}\frac{du}{dx} = \frac{dx}{dx} \qquad u = y$$

$$2u\frac{du}{dx} = 1$$

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Exercise: Verify this is consistent w/ what we did before.

Example: Find ~~the~~ an equation of the tangent line to the curve $y^2 + 2y - 3 = x$ at $x = 0$.

⇒ Can't isolate y ⇒ need to do implicit differentiation.

$$\frac{d}{dx}\left(y^2 + 2y - 3\right) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(2y) - \frac{d}{dx}(3) = 1$$

(chain rule)

$$\frac{dy^2}{dy}\frac{dy}{dx} + 2\frac{dy}{dx} - 0 = 1$$

$$\frac{dy}{dx}\left(2y + 2\right) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y+2}$$

Now we need to find y when $x=0$

$$x=0 \Rightarrow y^2 + 2y - 3 = 0 \Rightarrow y = \frac{-2 \mp \sqrt{4 + 4 * 3}}{2} = \frac{-2 \mp \sqrt{16}}{2}$$

$$y = \frac{-2 \mp}{2} = \begin{cases} -3 \\ 1 \end{cases}$$

$x = 0,\ y = -3 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2*(-3)+2} = \dfrac{-1}{4} \Rightarrow$ tangent line: $y+3 = \dfrac{-1}{4}(x-0)$

$x = 0,\ y = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2*1+2} = \dfrac{1}{4} \Rightarrow$ tangent line: $y-1 = \dfrac{1}{4}(x-0)$

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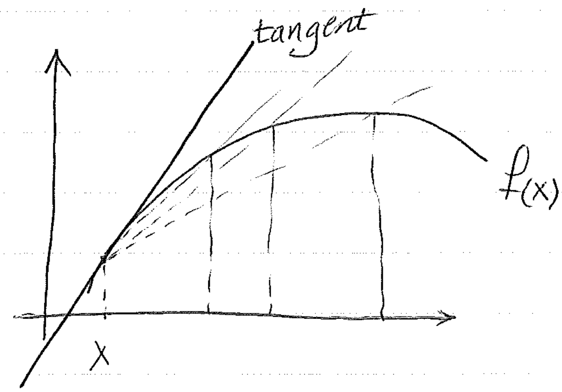
Jan 8, 2015

Recap:

$$\frac{df}{dx} = f'(x)$$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$



example: volume of spherical balloon

$$V = \frac{4}{3}\pi r^3 \quad (m^3)$$

$$\frac{m^3}{m} \quad \frac{dV}{dr} = 4\pi r^2 = V' \quad (m^2)$$

Now what if I tell you r is changing w/ time: $r = 0.1t$

How fast is the volume changing w/ time?

$$\frac{dV}{dt} = \frac{dV}{dr} * \frac{dr}{dt} = 4\pi r^2 * \frac{d(0.1t)}{dt} = 4\pi r^2 * 0.1 = 0.4\pi r^2 \ \frac{m^3}{s}$$

$$= V' \ !?$$

V' doesn't show ~~about~~ the independent variable of the derivative.

<u>Exercise:</u> Assuming that the volume of the balloon is increasing at a rate of $0.1\ m^3/s$, Can you find the rate of change of balloon's radius w/ time when the volume of the balloon is $\frac{\pi}{6} m^3$

Wed. lecture ended here

Example: find $\frac{dy}{dx}$ given that $x\,\mathrm{Sin}(y) = x^2 + y$.

Can't isolate $y \Rightarrow$ implicit diff.

$$\frac{d}{dx}\left(x\,\mathrm{Sin}(y)\right) = \frac{d}{dx}\left(x^2+y\right)$$

product rule

$$\frac{dx}{dx} * \mathrm{Sin}(y) + x\,\frac{d\,\mathrm{Sin}(y)}{dx} = \frac{d}{dx}(x^2) + \frac{dy}{dx}$$

chain rule

$$1 * \mathrm{Sin}(y) + x\,\frac{d\,\mathrm{Sin}(y)}{dy}\frac{dy}{dx} = 2x + \frac{dy}{dx}$$

$$\mathrm{Sin}(y) + x\,\mathrm{Cos}(y)\frac{dy}{dx} = 2x + \frac{dy}{dx}$$

$$\mathrm{Sin}(y) - 2x = \frac{dy}{dx} - x\,\mathrm{Cos}(y)\frac{dy}{dx}$$

$$\mathrm{Sin}(y) - 2x = \frac{dy}{dx}\left(1 - x\,\mathrm{Cos}(y)\right)$$

$$\frac{dy}{dx} = \frac{\mathrm{Sin}(y) - 2x}{1 - x\,\mathrm{Cos}(y)}$$

$\boxed{\text{Example}}$: Find the slope of the tangent line to the circle $x^2 + (y-2)^2 = 4$ at $x=0,\ y=4$.

Choices

a) 0   b) ∞ c) 1   d) none of the above

e) I don't know!

$$\frac{d}{dx}\left(x^2 + (y-2)^2\right) = \frac{d}{dx}(4)$$

$$\frac{dx^2}{dx} + \frac{d}{dx}(y-2)^2 = 0$$

$$2x + \frac{du^2}{du}\frac{du}{dx} = 0,\quad u = y-2$$

$$2x + 2u\,\frac{d}{dx}(y-2) = 0$$

$$2x + 2(y-2)\left[\frac{dy}{dx} - \cancelto{0}{\frac{d(2)}{dx}}\right] = 0$$

$$2x + 2(y-2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2(y-2)} = \frac{-x}{y-2}$$

$$x = 0,\ y = 4 \Longrightarrow \frac{dy}{dx} = \frac{0}{4-2} = 0$$

Question: Assuming that $x^2 + (y-2)^2 = 4$, find $\frac{d^2y}{dx^2}$

quotient rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-x}{y-2}\right) = \frac{-1 * (y-2) - (-x) * \frac{d}{dx}(y-2)}{(y-2)^2}$$

$$= \frac{-y+2+x\left(\frac{dy}{dx} - \frac{d2}{dx}\right)}{(y-2)^2} = \frac{-y+2+x\frac{dy}{dx}}{(y-2)^2}$$

$$= \frac{-y+2+x\left(\frac{-x}{y-2}\right)}{(y-2)^2} = \frac{\frac{(-y+2)(y-2)-x^2}{y-2}}{\frac{(y-2)^2}{1}}$$

$$= \frac{-(y-2)^2 - x^2}{(y-2)^3}$$

Example: Find the equation(s) of the tangent line(s) to $xy=6$ that pass through the point $(6,-3)$.

$$\frac{d}{dx}(xy) = \frac{d6}{dx}$$

$$\frac{dx}{dx}y + x\frac{dy}{dx} = 0$$

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

Now let's assume that the tangent line at the point (a,b) on the curve & $xy=6$ passes through the point $(6,-3)$

eq. of the tangent line at (a,b)

$$y-b = \frac{-b}{a}(x-a)$$

$$-3-b = \frac{-b}{a}(6-a)$$

Since (a,b) is a point on the curve

$$\Rightarrow ab = 6 \Rightarrow b = \frac{6}{a}$$

$$\Rightarrow -3-\frac{6}{a} = \frac{\frac{-6}{a}}{\frac{a}{1}}(6-a)$$

$$-3-\frac{6}{a} = \frac{-6}{a^2}(6-a)$$

$$-3a^2-6a = -36+6a$$

$$0 = 3a^2+12a-36$$

$$0 = a^2+4a-12$$

$$\Rightarrow a = \frac{-4 \mp \sqrt{16+48}}{2} = \frac{-4 \mp \sqrt{64}}{2}$$

$$\Rightarrow a = \frac{-4 \mp 8}{2} = \begin{cases} 2 \Rightarrow b=3 \\ -6 \Rightarrow b=-1 \end{cases}$$

$F(3)=2$

$F'(3) = \cancel{2}\,1 \qquad G'(2)=2$

Example: Given that $\quad F'(2)=0, \quad G'(3) = 4$

$F(2)=3 \qquad G(3)=2$

find the slope of the tangent line to $2+\sqrt{y} = F(G(x))$

at $x=3$

Choices:

a) 8

b) 0

c) $\frac{1}{2}$

d) none of the above

e) I don't know.

$$\frac{d}{dx}\left(2+\sqrt{y}\right) = \frac{d}{dx}\left(F(G(x))\right)$$

$$\frac{d}{dx}(2) + \frac{d}{dx}\left(\sqrt{y}\right) = G'(x)\,F'(G(x))$$

$$0 + \frac{d\sqrt{y}}{dy} * \frac{dy}{dx} = G'(x)\,F'(G(x))$$

$$\frac{1}{2}\frac{1}{\sqrt{y}} * \frac{dy}{dx} = G'(x)\,F'(G(x))$$

$$\frac{dy}{dx} = 2\sqrt{y}\,G'(x)\,F'(G(x))$$

$$x=3 \Rightarrow G(3)=2 \Rightarrow F(G(3)) = F(2) = 3$$

$$\Rightarrow 2+\sqrt{y} = 3 \Rightarrow \sqrt{y}=1 \Rightarrow y=1$$

$$\frac{dy}{dx} = 2\sqrt{1} * G'(3) * F'(2) = 2*4*0 = 0$$