MATH 110
March 7-11, 2016
Week 10

Last week: L'Hopital's rule, Curve sketching

This week: Curve sketching, abs. min and max (Ref. book, section 3.1)

L'Hopitals rule: "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" or "$0 * \infty$"

Curve sketching
- * Domain
- Intercepts
- Symmetry?
- * Asymptotes?
- * Intervals where f increases/decreases
- * local min and max
- * Concavity and inflection points

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Example from last time:

$f(x) = x^{2/3} e^x$

Domain: $(-\infty, \infty)$, no symmetry, Horizontal asymptotes? $y=0$
Vertical asymptotes? none

Step 5: Intervals where f increases / decreases?

increases → $f' > 0$; decreases → $f' < 0$

product rule

$$\frac{df}{dx} = \frac{d}{dx}\left(x^{2/3} e^x\right) = \frac{2}{3} x^{\frac{2}{3}-1} e^x + x^{2/3} e^x = e^x\left(\frac{2}{3} x^{-1/3} + x^{2/3}\right)$$

$$= e^x\left(\frac{2}{3}\frac{1}{x^{1/3}} + x^{2/3}\right) = e^x\left(\frac{2}{3x^{1/3}} + \frac{x^{2/3} * 3x^{1/3}}{3x^{1/3}}\right)$$

$$= e^x\left(\frac{2+3x}{3x^{1/3}}\right)$$

$e^x > 0$ for all $-\infty < x < \infty$

$2 + 3x = 0 \Rightarrow x = \frac{-2}{3} \Rightarrow f' = 0$ at $x = -\frac{2}{3}$

$3x^{1/3} = 0 \Rightarrow x^{1/3} = 0 \Rightarrow x = 0 \Rightarrow f'$ is not defined at $x = 0$

		$-2/3$		0	
e^x	+		+		+
$3x^{1/3}$	—		—	0	+
$2+3x$	—	0	+		+
f'	+	0	—	DNE	

	$-\infty$	$-2/3$		0	$+\infty$
f'	+	0	—		+

$\lim_{x \to 0^+} f' = \lim_{x \to 0^+} e^x \frac{2+3x}{3x^{1/3}} = ?$

a) 0 b) $\frac{2}{0}$

c) $+\infty$ d) $-\infty$

e) None of the above

$$\lim_{x \to 0^+} \frac{\overset{1}{e^x}\,\overset{2}{(2+3x)}}{\underset{0^+}{3x^{1/3}}} = +\infty$$

$\rightarrow x > 0$ (e.g $x = 0.1$) $\Rightarrow x^{1/3} > 0$

$$\lim_{x \to 0^-} f' = \lim_{x \to 0^-} \frac{\overset{1}{e^x}\,\overset{2}{(2+3x)}}{\underset{0^-}{3x^{1/3}}} = -\infty$$

$\rightarrow x < 0$ (e.g $x = -0.1$) $\Rightarrow x^{1/3} < 0$

$\boxed{\text{Step 6}}$ Find local min and max.

Note that $x = \frac{-2}{3}$ and $x = 0$ are both in the domain of f.

$\Rightarrow \begin{cases} f \text{ has a local max at } x = -\frac{2}{3}\ ; & f(\frac{-2}{3}) = (\frac{-2}{3})^{2/3} e^{-2/3} \simeq 0.23 \\ f \text{ has a local min at } x = 0\ ; & f(0) = 0 \end{cases}$

$\boxed{\text{Step 7}}$ Concavity and inflection points

product rule

$$f'' = \frac{d}{dx}(f') = \frac{d}{dx}\left(e^x\left(\frac{2}{3}x^{-1/3} + x^{2/3}\right)\right) = e^x\left(\frac{2}{3}x^{-1/3} + x^{2/3}\right) + e^x\left(\frac{-2}{9}x^{\frac{-4}{3}} + \frac{2}{3}x^{\frac{-1}{3}}\right)$$

$$= e^x\left[\frac{2}{3}x^{-1/3} + x^{2/3} - \frac{2}{9}x^{-4/3} + \frac{2}{3}x^{-1/3}\right]$$

$$f'' = e^x\left[x^{2/3} + \frac{4}{3}\frac{1}{x^{1/3}} - \frac{2}{9}\frac{1}{x^{4/3}}\right] = e^x\left[\frac{9x^{2/3+4/3}}{9x^{4/3}} + \frac{4*3x}{9x^{-1/3+1}} - \frac{2}{9x^{4/3}}\right]$$

$$= e^x\left[\frac{9x^2+12x-2}{9x^{4/3}}\right]$$

$e^x > 0$ for all x

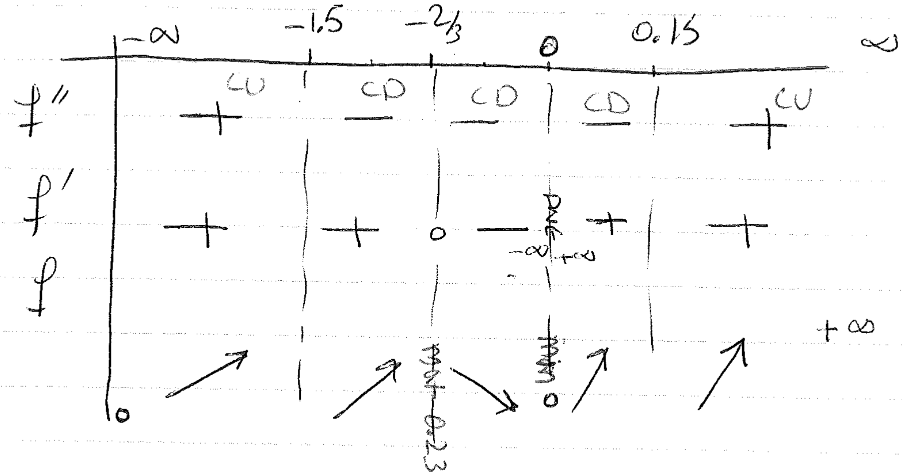
$$9x^{4/3} = 0 \Rightarrow x = 0$$

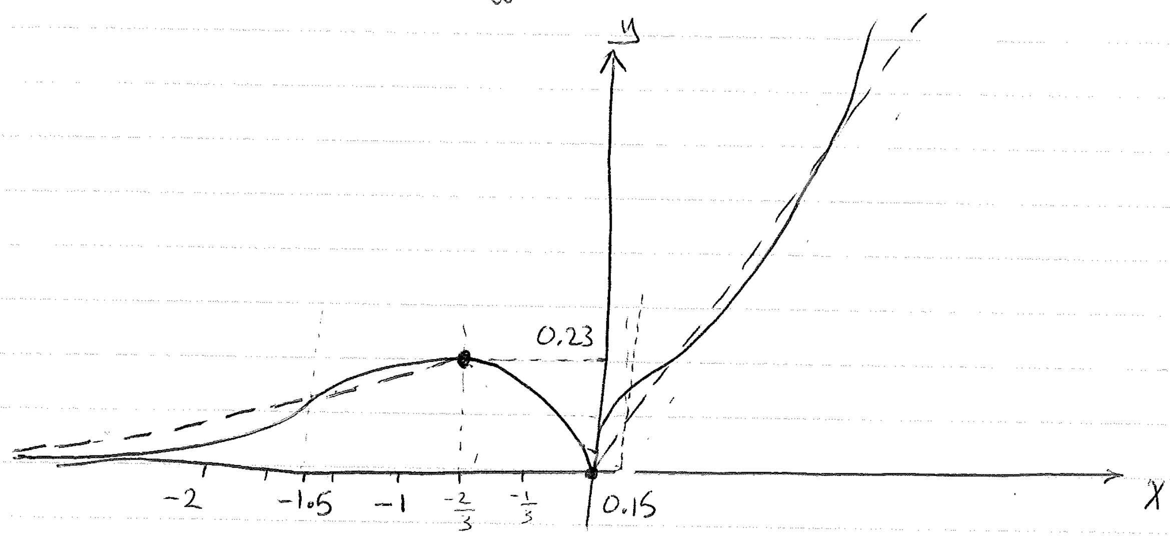
$$9x^2+12x-2 = 0 \Rightarrow x = \frac{-12 \mp\sqrt{12^2+72}}{18} = \frac{-6\mp\sqrt{6^2+18}}{9}$$

$$x = \frac{-2\mp\sqrt{2^2+2}}{3} = \frac{-2\mp\sqrt{6}}{3} \simeq \begin{cases} 0.15 \\ -1.5 \end{cases}$$

		-1.5		0		0.15	
e^x	$+$		$+$		$+$		$+$
$9x^{4/3}$	$+$		$+$	0	$+$		$+$
$9x^2+12x-2$	$+$		$-$		$-$		$+$
f''	$+$		$-$		$-$		$+$

	-2		-1.5		-1	0	0.1	0.15		1
f''		$+$	0	$-$			$-$	0	$+$	





What is the abs. $\underbrace{\text{max}}_{}$ of $f_{(x)}$ on the interval $[-1, 1]$?

① find critical points $\Rightarrow$
- end points: $x = -1 \rightarrow f(-1) = e^{-1} \simeq \frac{1}{2.72}$
- $x = 1 \rightarrow f_{(1)} = e \simeq 2.72$
- $f' = 0 \Rightarrow x = -\frac{2}{3} \Rightarrow f_{(-\frac{2}{3})} \simeq 0.23$
- f' DNE $\Rightarrow x = 0 \Rightarrow f_{(0)} = 0$

Choices: a) 0.23 e) I don't know
b) $-\frac{2}{3}$
c) e
d) none of the above

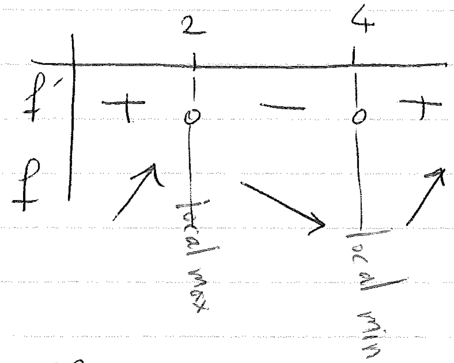
$\Rightarrow$ the abs min of f on $[-1, 1]$ is $f(0) = 0$
the abs max of f on $[-1, 1]$ is $f(1) = e$

Mon. class ended here

Example: $f(x) = x^3 - 9x^2 + 24x + 3$

a) Find the abs. max of f on $[1, 3]$

$f'(x) = 3x^2 - 18x + 24 = 3(x-2)(x-4)$



Choices

a) 2

b) 23

c) 21

d) 19

e) None of the above

$f(2) = 23$

$f(1) = 19$

$f(3) = 21$

b) Find the abs. max of f on $[-1, 7]$

Choices

a) 23

b) 71

c) 73

d) 7

e) None of the above

$f(-1) = -31$

$f(2) = 23$

$f(7) = 73$

$f(4) = 19$

Find the abs. max and abs. min of $f(x) = |x^2-6x-7|$ on $[-2, 9]$

$$f(x) = |x^2-6x-7| = |(x+1)(x-7)|$$

		-1		7	
$(x+1)(x-7)$	$+$	0	$-$	0	$+$

$$f(x) = \begin{cases} x^2-6x-7 & x \leq -1 \text{ or } x \geq 7 \\ -x^2+6x+7 & -1 < x < 7 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x-6 & x \leq -1 \text{ or } x \geq 7 \\ -2x+6 & -1 < x < 7 \end{cases}$$

$$2x-6 = 0 \Rightarrow x = 3 \qquad -1 \leq x=3 \leq 7$$

What are the critical numbers on the interval $[-2, 9]$

Choices

a) $x = 3$

b) $x=-1, x=7$

c) $x=3, x=-1, x=7$

d) none of the above

e) I don't know

Critical numbers:

$f'=0 \Rightarrow x = 3$

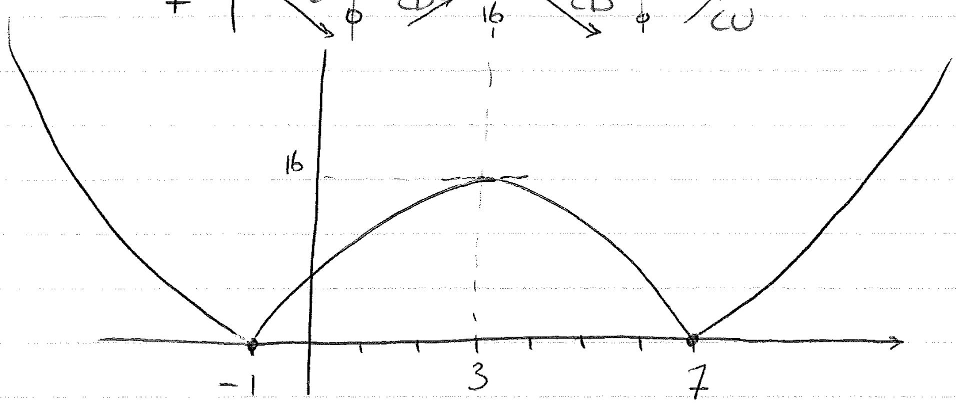
f' is not defined $\Rightarrow x=-1, x=7$

end points $\Rightarrow x=-2, x=9$

$$\left.\begin{aligned} f(3) &= |4*(-4)| = 16 \\ f(-1) &= |0*(-7)| = 0 \\ f(7) &= |8*0| = 0 \\ f(-2) &= |-1*(-9)| = 9 \\ f(9) &= |10*2| = 20 \end{aligned}\right\} \Rightarrow \begin{cases} \text{abs min at } x=-1 \text{ or } x=7,\ f(-1)=f(7)=0 \\ \text{abs max at } x=9,\ f(9)=20 \end{cases}$$

		-1		3		7	
$2x-6$	$-$		$-$	0	$+$		$+$
f'	$-$		$+$	0	$-$		$+$
f''	$+$		$-$		$-$		$+$
f	$\searrow$ CU	0	CD $\nearrow$	16	$\searrow$ CD	0	$\nearrow$ CU

$$f''(x) = \begin{cases} 2 & x \leq -1,\ x \geq 7 \\ -2 & -1 < x < 7 \end{cases}$$



Example: Sketch $f(x) = \dfrac{x^2-4}{x\,|x+1|}$

Reminder: $\begin{cases} a \geq 0 \Rightarrow |a| = a \\ a < 0 \Rightarrow |a| = -a \end{cases}$

$$x+1=0 \Rightarrow x=-1 \Rightarrow \begin{cases} x<-1 \Rightarrow x+1<0 \Rightarrow |x+1| = -x-1 \\ x \geq -1 \Rightarrow x+1>0 \Rightarrow |x+1| = x+1 \end{cases}$$

$$\Rightarrow \begin{cases} x<-1, & f(x) = \dfrac{x^2-4}{x(-x-1)} \\ x \geq -1, & f(x) = \dfrac{x^2-4}{x(x+1)} \end{cases}$$

Step1: Domain: $x \neq 0$ and $x \neq -1$

Step2: $x=0$ is not in the domain of f $\Rightarrow$ no y intercepts

$y=0 \Rightarrow x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\mp 2$

Step3: Symmetry $f(-x) = \dfrac{(-x)^2-4}{(-x)|-x+1|} = \dfrac{x^2-4}{-x\,|x-1|}$

$f(-x) \neq f(x)$ not symmetric about y axis

$f(-x) \neq -f(+x)$ not symmetric about origin

Step 4: Asymptotes

$x < -1 \Rightarrow f(x) = \dfrac{x^2-4}{x(-x-1)} \Rightarrow \lim\limits_{x\to-\infty} \dfrac{(x^2-4)}{x(-x-1)} \overset{?}{=} \dfrac{\infty}{-\infty} \Rightarrow$ Use L'Hopital's rule

∞^2, $-\infty$, ∞

$$\lim_{x\to-\infty} \frac{x^2-4}{x(-x-1)} = \lim_{x\to-\infty} \frac{(x^2-4)'}{(-x^2-x)'} = \lim_{x\to-\infty} \frac{2x}{-2x-1} \overset{?}{=} \frac{-\infty}{\infty}$$ Use L'Hopital's rule again

$$= \lim_{x\to-\infty} \frac{(2x)'}{(-2x-1)'} = \lim_{x\to-\infty} \frac{2}{-2} = -1$$

Alternatively: $$\lim_{x\to-\infty} \frac{x^2-4}{x(-x-1)} = \lim_{x\to\infty} \frac{x^2\left(1-\frac{4}{x^2}\right)}{x^2\left(-1-\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{1-\frac{4}{x^2}}{-1-\frac{1}{x}} = -1$$

Alternatively notice that the largest power of x in the numerator and the denominator is the same (i.e. 2). So as $x\to\infty$ the limit is the ratio of the coeficients of x^2 in the numerator and denominator; i.e. $\frac{+1}{-1} = -1$

$x \geq -1 \Rightarrow f(x) = \dfrac{x^2-4}{x(x+1)} \Rightarrow \lim\limits_{x\to\infty} f(x) = ?$

Choices:
- a) 1
- b) −1
- c) 0
- d) none of the above
- e) I don't know

Horizontal asymptotes: $\begin{cases} y = 1 & \text{as } x \to \infty \\ y = -1 & \text{as } x \to -\infty \end{cases}$

Vertical asymptotes?

Choices
a) None
b) $x = \mp 2$
c) $x = 0$, $x = -1$
d) None of the above
e) I don't know

$x|x+1| = 0 \Rightarrow \begin{cases} x = 0 \\ x = -1 \end{cases}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2-4}{x|x+1|} \overset{?}{=} \frac{-4}{0 * 1} = \infty$$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2-4}{x|x+1|} \overset{?}{=} \frac{-3}{-1 * 0} = \infty$$

$x = 0$ and $x = -1$ are vertical asymptotes

Question: $\lim_{x \to 0^+} f(x) =$ ____ and $\lim_{x \to 0^-} f(x) =$ ____

a) $-\infty$, $-\infty$
b) $+\infty$, $+\infty$
c) $-\infty$, $+\infty$
d) $+\infty$, $-\infty$

$$\lim_{x \to 0^+} \frac{\overset{-4}{(x^2-4)}}{\underset{+}{x}\,\underset{+}{|x+1|}} = -\infty \qquad \lim_{x \to 0^-} \frac{\overset{-4}{(x^2-4)}}{\underset{-}{x}\,\underset{+}{|x+1|}} = +\infty$$

Question: $\lim\limits_{x\to -1^+} f(x) = \_\_\_\_$ and $\lim\limits_{x\to -1^-} f(x) = \_\_\_\_$

a) $-\infty\ ; -\infty$ c) $-\infty\ , +\infty$

b) $+\infty\ , +\infty$ d) $+\infty\ , -\infty$

$$\lim_{x\to -1^+} \frac{(x^2-4)^{\,-3}}{\underset{-1}{(x)}\underset{+}{|x+1|}} = +\infty \qquad \lim_{x\to -1^-} \frac{(x^2-4)^{\,-3}}{\underset{-1}{(x)}\underset{+}{|x+1|}} = +\infty$$

Step 5

$$x<-1,\ \frac{df}{dx} = \frac{d}{dx}\left(\frac{x^2-4}{x(-x-1)}\right) = \frac{d}{dx}\left(\frac{x^2-4}{-x^2-x}\right) = \frac{\overbrace{2x(-x^2-x)-(x^2-4)(-2x-1)}^{-2x^3-x^2+8x+4}}{(-x^2-x)^2}$$

$$= \frac{\cancel{-2x^3}-2x^2+\cancel{2x^3}+x^2-8x-4}{(-x^2-x)^2} = \frac{-x^2-8x-4}{(x^2+x)^2} = -1\times\frac{x^2+8x+4}{(x^2+x)^2}$$

$$x^2+8x+4=0 \Rightarrow x = \frac{-8\mp\sqrt{64-16}}{2} = \frac{-8\mp\sqrt{48}}{2} = -4\mp\sqrt{12}$$
$$= -4\mp 3.\ldots$$

$$(x^2+x)^2=0 \Rightarrow x^2+x=0 \Rightarrow x(x+1)=0 \Rightarrow \begin{cases} x=0 \\ x=-1 \end{cases}$$ Remember $x<-1$

$$x\geq -1 \Rightarrow f = \frac{x^2-4}{x(x+1)} = -\frac{x^2-4}{x(-x-1)} \Rightarrow \frac{df}{dx} = \frac{x^2+8x+4}{(x^2+x)^2}$$

		$-4-\sqrt{12}$		-1		$-4+\sqrt{12}$		0	
x^2+8x+4	+	0	—		—	0	+		+
$(..)+x)^2$	+		+	0	+		+	0	+
f'	—	0	+	DNE	—	0	+	DNE	+
f	↘		↗		↘		↗		↗

Step 6:

local min: $\begin{cases} f(-4-\sqrt{12}) \simeq -1.07 \\ f(-4+\sqrt{12}) \simeq 15 \end{cases}$

Step 7: Concavity and points of inflection

$x < -1$

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{-x^2-8x-4}{(x^2+x)^2}\right) = \frac{(-2x-8)(x^2+x)^2 - (-x^2-8x-4)2(2x+1)(x^2+x)}{(x^2+x)^4}$$

$$= \frac{(x^2+x)\left[(-2x-8)(x^2+x) - (-x^2-8x-4)2(2x+1)\right]}{(x^2+x)^4}$$

$$= \frac{-2x^3 - 2x^2 - 8x^2 - 8x + 4x^3 + 2x^2 + 32x^2 + 16x + 16x + 8}{(x^2+x)^3}$$

$$= \frac{2x^3 + 24x^2 + 24x + 8}{(x^2+x)^3}$$

Exercise: Show that $2x^3 + 24x^2 + 24x + 8 = 0$ has only one solution.

$x \approx -11$

$x \geq -1$

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{x^2+8x+4}{(x^2+x)^2}\right) = \frac{-2x^3 - 8x^2 - 24x - 8}{(x^2+x)^3}$$

$(x^2+x)^3 = 0$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x = 0, \; x = -1$

		-11		-1		0	
$(x^2+x)^3$	$+$		$+$	0	$-$	0	$+$
$2x^3+8x^2+24x+8$	$-$		$+$		$+$		$+$
f''	$-$	φ	$+$		$+$		$-$

Note that $x = 0$ and $x = 1$ are not in the domain of f. So there are not inflection points.

		-11		$-4-\sqrt{12}$ (-7.5)		-1		$-4+\sqrt{12}$ (-0.5)		0	
f''	$-$		$+$		$+$		$+$		$+$		$-$
f'	$-$		$-$	0	$+$		$-$	0	$+$		$+$
f	-1 CD $\searrow$		CU $\searrow$	-1.07	CU $\nearrow$ $+\infty$		$+\infty$ CU $\searrow$	15	CU $\nearrow$ $+\infty$		$-\infty$ CD $\nearrow$ $+1$