① MATH 110
March 14_18
Week 11

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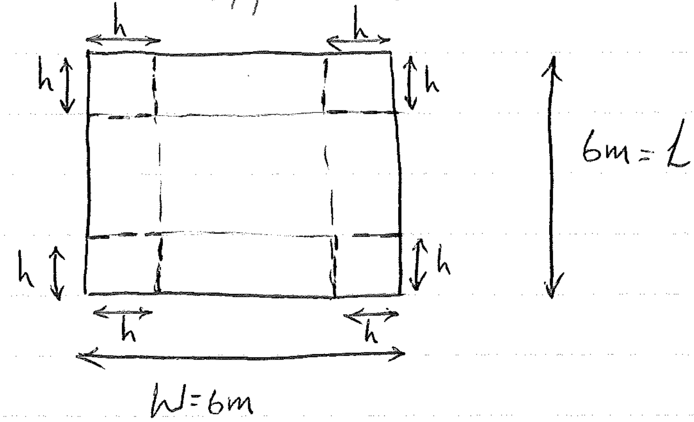
Last week: Curve sketching, abs. min and max

This week: Optimization (Ref. book section 3.5)

Example: I have a sheet of cardboard of size $6 * 6 m^2$. I need to cut out squares from the corners of the card board so that I may fold it into a box.* How large should this cut out squares be to maximize the volume of the box?

*with an open top

Step 0: Make a sketch and label variables/parameters



Step 1: Identify and write the known information

$W = L = 6m$

Step 2: Write the question in a mathematical form

Step 2.1 ⇒ Identify the variable to be optimized (V here)

Volume the box ⇒ V

Step 2.2 ⇒ Identify the independent variable(s) with respect to which you should optimize (h here)

of cut out squares ⇒ h ✓

[A x makes it difficult

Step 3: Find the equation(s) connecting the two

$$V = (6-2h)(6-2h)h$$

Step 4: Identify which parameters/variables change w/ h

(e.g. say "I have a cardboard of size $. * W m^2$ ..."

But W and L are not going to change w/ h)

Step 5: differentiate (with respect to h) to identify the critical numbers

$$\frac{dV}{dh} = \frac{d}{dh}\left((6-2h)(6-2h)h\right) = \frac{d}{dh}\left((6-2h)^2 h\right)$$

$$= \frac{d}{dh}\left((6-2h)^2\right) * h + (6-2h)^2 \frac{dh}{dh}$$

$$= 2*(-2)*(6-2h)h + (6-2h)^2$$

$$= -24h + 8h^2 + 36 + 4h^2 - 24h$$

$$= 12h^2 - 48h + 36$$

critical numbers $\Rightarrow$

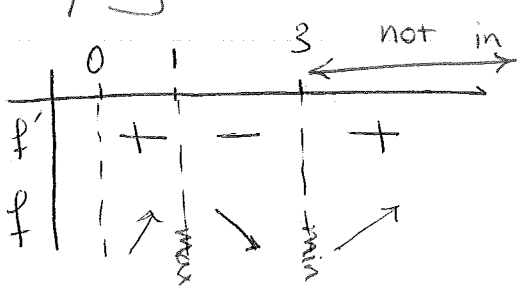
- $\frac{dV}{dh} = 0 \Rightarrow h = \frac{48 \mp \sqrt{48^2 - 4*12*36}}{2*12}$
- $\frac{dV}{dh}$ DNE $\quad \frac{dV}{dh}$ exists everywhere
- end point? $\Rightarrow$ What is the domain? $\quad 0 \leqslant h \leqslant 3$

$$\Rightarrow \begin{matrix} h = 0 \\ h = 3 \end{matrix}$$

$$h = \frac{48 \mp \sqrt{48^2 - 4*12*36}}{2*12} = \frac{4 \mp \sqrt{4^2 - 4*3}}{2} = \frac{4 \mp \sqrt{16-12}}{2} = \frac{4 \mp 2}{2}$$

$$= \begin{cases} 3 \\ 1 \end{cases}$$

Step 6: plug all the known values and "optimize"!



$V(0) = 0$

$\boxed{V(1) = 16\ m^2} \Rightarrow$ The abs. max. of $V(h)$ is $V(1) = 16m^2$

$V(3) = 0$

Quiz on Fri, March 18th

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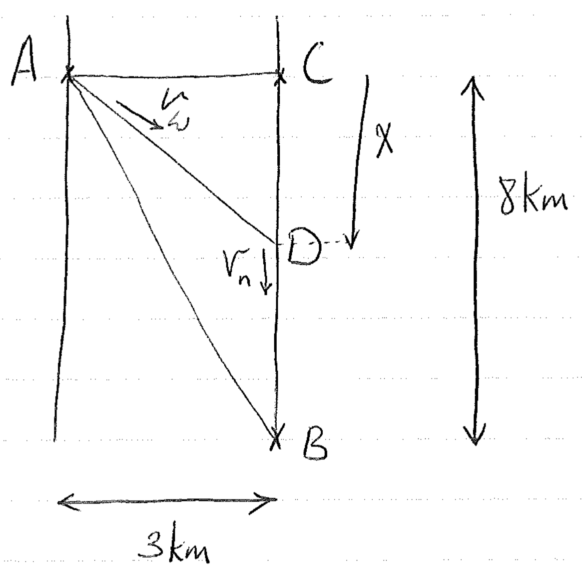
Example: A man launches his boat from point A on a bank of a straight river, 3km wide and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between B and C and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible com with the speed at which the man rows.)



Step 0 : Given (label x)

Step 1 :

V_w : rowing speed $= 6$ km/h

V_n : running speed $= 8$ km/h

Step 2.1

variable to be optimized: t

a) x b) t c) AD+DB d) none of the above

Step 2.2 independent variable: x

a) x b) t c) AD+DB d) none of the above

Step 3: Reminder: $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$$t = t_{AB} = t_{AD} + t_{DB} = \frac{AD}{V_w} + \frac{DB}{V_n} = \frac{\sqrt{3^2+x^2}}{6} + \frac{8-x}{8} \quad h$$

Step 4: river width, BC, V_w and V_n do not change w/ time.

Step 5:

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{\sqrt{3^2+x^2}}{6}\right) + \frac{d}{dx}\left(\frac{8-x}{8}\right)$$

$$\frac{dt}{dx} = \frac{d}{du}\left(\frac{\sqrt{u}}{6}\right)\frac{du}{dx} - \frac{1}{8}, \quad u = 9+x^2$$

$$\frac{dt}{dx} = \frac{1}{12}u^{-1/2} * \frac{d}{dx}(9+x^2) - \frac{1}{8}$$

$$\frac{dt}{dx} = \frac{1}{12\sqrt{9+x^2}} * 2x - \frac{1}{8}$$

$$\frac{dt}{dx} = \frac{4x}{24\sqrt{9+x^2}} - \frac{3\sqrt{9+x^2}}{24\sqrt{9+x^2}} = \frac{4x - 3\sqrt{9+x^2}}{24\sqrt{x^2+9}}$$

Critical numbers $\Rightarrow$

$\frac{dt}{dx} = 0 \Rightarrow 4x - 3\sqrt{9+x^2} = 0$

$4x = 3\sqrt{9+x^2}$

$16x^2 = 9(9+x^2) \Rightarrow 7x^2 = 81$

$\Rightarrow x = \mp\sqrt{\frac{81}{7}} = \mp\frac{9}{\sqrt{7}}$

$\frac{dt}{dx}$ DNE $\Rightarrow \sqrt{x^2+9} = 0 \Rightarrow x^2+9 = 0$ does not have a solution

end points? $\Rightarrow$ What is the domain?

$0 \leq x \leq 8 \Rightarrow \begin{cases} x = 0 \\ x = 8 \end{cases}$

Step 6 Note that $x = -\frac{9}{\sqrt{7}}$ is not in the domain.

$t(0) = \frac{\sqrt{9}}{6} + \frac{8}{8} = 1.5\,h$

$t(8) = \frac{\sqrt{9+64}}{6} + \frac{8-8}{8} = \frac{\sqrt{73}}{6}\,h \simeq 1.42\,h$

$t\left(\frac{9}{\sqrt{7}}\right) = \frac{\sqrt{9+\frac{9^2}{7}}}{6} + \frac{8-\frac{9}{\sqrt{7}}}{8} = \frac{\sqrt{\frac{63+81}{7}}}{6} + 1 - \frac{9}{8\sqrt{7}}$

$= \frac{\sqrt{\frac{144}{7}}}{6} + 1 - \frac{9}{8\sqrt{7}} = \frac{\cancel{12}^{\,2}}{\cancel{6}\sqrt{7}} + 1 - \frac{9}{8\sqrt{7}}$

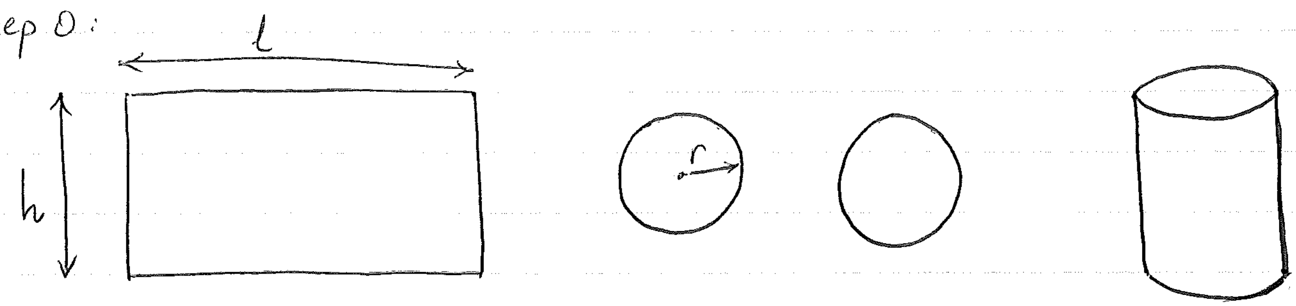
$= \frac{16-9}{8\sqrt{7}} + 1 = \frac{7}{8\sqrt{7}} + 1 = \frac{\sqrt{7}}{8} + 1 \simeq 1.33\,h$

$1.33 < 1.42 < 1.5$

$\Rightarrow t\left(\frac{9}{\sqrt{7}}\right) \simeq 1.33$ is the abs min of $t(x)$ on $0 \leq x \leq 8$

Example: Suppose we are in charge of designing a 250 ml cylindrical pop can for drink Inc. What dimensions should we choose to minimize the surface area?

Step 0:



Step 1. $l = 2\pi r$, $V = \pi r^2 h = 250\text{ ml} = 250\text{ cm}^3$

Step 2.1: variable to be optimized A: total area of the can

Step 2.2: identify the independent variables: h, r

a) r b) h c) r, h d) none of the above e) I don't know

Remember $\pi r^2 h = 250 \Rightarrow \begin{cases} h = \dfrac{250}{\pi r^2} \\ r = \sqrt{250/\pi h} \end{cases}$

Step 3: Find the equation connecting A, r (and h)

$$A = hl + 2\pi r^2 = 2\pi r h + 2\pi r^2$$

Step 4: Identify which parameters / variables change w/ r and l

$h = h(r)$

Step 5:
$$\frac{dA}{dr} = \frac{d}{dr}(2\pi r h) + \frac{d}{dr}(2\pi r^2) = \frac{d}{dr}(2\pi r)h + 2\pi r\frac{dh}{dr} + 4\pi r$$

$$\frac{dA}{dr} = 2\pi h + 2\cancel{\pi}\cancel{r} * \frac{-2*250}{\cancel{\pi} r^{\cancel{3}2}} + 4\pi r$$

$$\frac{dA}{dr} = 2\pi \frac{250}{\pi r^2} + \frac{-4 * 250}{r^2} + 4\pi r = \frac{500}{r^2} - \frac{1000}{r^2} + 4\pi r$$

$$\frac{dA}{dr} = \frac{-500}{r^2} + 4\pi r$$

Critical numbers :

- $\frac{dA}{dr} = 0 \Rightarrow \frac{-500}{r^2} + 4\pi r = 0 \Rightarrow \frac{500}{r^2} = 4\pi r$
  $\Rightarrow r^3 = \frac{500}{4\pi} \Rightarrow r = \sqrt[3]{\frac{500}{4\pi}}$
- $\frac{dA}{dr}$ DNE $\Rightarrow r = 0$
- end points? $r > 0$

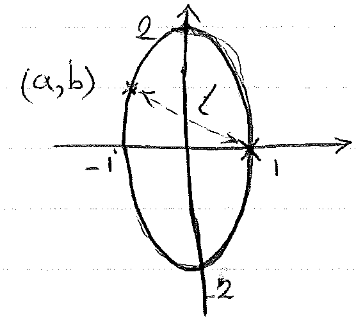
Step 6 : $A\left(\sqrt[3]{\frac{500}{4\pi}}\right) = 2\pi \sqrt[3]{\frac{500}{4\pi}} * \frac{250}{\pi \sqrt[3]{\left(\frac{500}{4\pi}\right)^2}} + 2\pi \sqrt[3]{\left(\frac{500}{4\pi}\right)^2}$

$$= \frac{500}{\sqrt{\frac{500}{4\pi}}} + 2\pi \sqrt[3]{\frac{500^2}{(4\pi)^2}}$$

		$(500/(4\pi))^{1/3}$	
f'	—	0	+
f	$\searrow$	min	$\nearrow$

The abs. min of A is at $r = \left(500/4\pi\right)^{1/3}$

Example: Find the points on the ellipse $4x^2+y^2=4$ that are farthest away from the point $(1,0)$.



Step 0: ~~not necessary~~

Step 1: $4a^2+b^2=4$

Step 2.1 maximize l

Step 2.2 independent variable

a) a b) b c) a, b d) none of the above

e) I don't know

Step 3: $l^2=(a-1)^2+(b-0)^2=(a-1)^2+b^2$

$$\left.\begin{array}{l} l=\sqrt{(a-1)^2+b^2} \\ b^2=4-4a^2 \end{array}\right\} \quad \begin{array}{l} l=\sqrt{(a-1)^2+4-4a^2}=\sqrt{a^2-2a+1+4-4a^2} \\ l=\sqrt{-3a^2-2a+5} \end{array}$$

Step 4: l is given in terms of a

Step 5: $\dfrac{dl}{da}=\dfrac{d}{da}\left(\sqrt{-3a^2-2a+5}\right)=\dfrac{-6a-2}{2\sqrt{-3a^2-2a+5}}$

Critical numbers $\Rightarrow$

$$\frac{dl}{da} = 0 \Rightarrow -6a - 2 = 0 \Rightarrow a = \frac{+2}{-6} = \frac{-1}{3}$$

$$\frac{dl}{da} \text{ DNE} \Rightarrow \sqrt{-3a^2 - 2a + 5} = 0 \Rightarrow -3a^2 - 2a + 5 = 0$$

$$\Rightarrow 3a^2 + 2a - 5 = 0 \Rightarrow a = \frac{-2 \mp \sqrt{4 + 60}}{6}$$

$$\Rightarrow a = \frac{-2 \mp \sqrt{64}}{6} = \frac{-2 \mp 8}{6} = \begin{cases} 1 \\ \frac{-10}{6} \end{cases}$$

endpoints: $-1 \leq a \leq 1$

$\Rightarrow$ Note that $a = \frac{-10}{6}$ is not in the domain.

Step 6:

$$l\left(\frac{-1}{3}\right) = \sqrt{\left(\frac{-1}{3} - 1\right)^2 + 4 - 4 \times \frac{1}{9}} = \sqrt{\frac{16}{9} + 4 - \frac{4}{9}} = \sqrt{\frac{16 + 36 - 4}{9}}$$

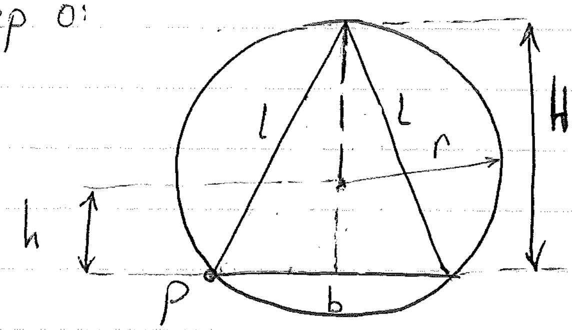
$$= \sqrt{\frac{48}{9}} = \frac{4}{3}\sqrt{3} \simeq 2.31$$

$$l(1) = \sqrt{(1-1)^2 + 4 - 4} = 0 \qquad l(-1) = \sqrt{(-1-1)^2 + 4 - 4} = 2$$

abs max of l is $l\left(\frac{-1}{3}\right) \simeq 2.31$

Example: Find the dimensions of the isosceles triangle of the largest area that can be inscribed in a circle of radius $r = 2$ cm

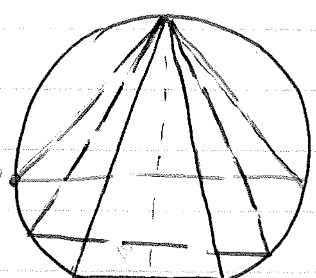
Step 0:



Step 1: $r = 2cm$

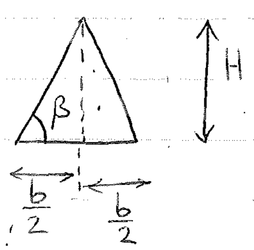
Step2: Step2.1 var. to be optimized A

Step2.2 identify the independent var?



Step 3: $A = \frac{1}{2} Hb$

I can write b as a function of L or H



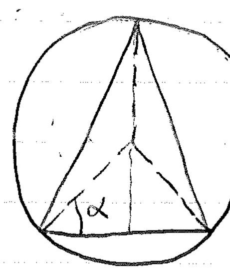
$\frac{H}{l} = \sin\beta$

$\frac{\frac{b}{2}}{H} = \cot\beta$

But b, H, L and β all change as I move P on the circle.

let's draw the radiuses

$\frac{\frac{b}{2}}{r} = \cos\alpha \Rightarrow b = 2r\cos\alpha$



$H = r + h$

$\frac{h}{r} = \sin\alpha \Rightarrow h = r\sin\alpha$

$H = r + r\sin\alpha$

$$b = 2r\cos\alpha = 4\cos\alpha$$

$$H = r + r\sin\alpha = 2 + 2\sin\alpha$$

$$A = \frac{1}{2}bH = \frac{1}{\cancel{2}}4\cos\alpha\left(\cancel{2}^{1} + \cancel{2}^{1}\sin\alpha\right) = 4\cos\alpha\,(1+\sin\alpha)$$

Step 4: Now we've written A in terms of α. So the independen variable is α.

Step 5: $\frac{dA}{d\alpha} = \frac{d}{d\alpha}\Big(4\cos(\alpha)\,(1+\sin(\alpha))\Big)$

$$= \frac{d}{d\alpha}\left(4\cos(\alpha)\right)\left(1+\sin(\alpha)\right) + 4\cos(\alpha)\,\frac{d}{d\alpha}\left(1+\sin\alpha\right)$$

$$= -4\sin\alpha\,\left(1+\sin(\alpha)\right) + 4\cos^2\alpha$$

$$= -4\sin\alpha - 4\sin^2\alpha + 4\cos^2\alpha$$

Remember: $\sin^2\alpha + \cos^2\alpha = 1$ It's a lot easier to use $\cos^2\alpha = 1 - \sin^2\alpha$ instead of $\sin\alpha = \sqrt{1-\cos^2\alpha}$

$$\frac{dA}{d\alpha} = -4\sin\alpha - 4\sin^2\alpha + 4 - 4\sin^2\alpha$$

$$= -8\sin^2\alpha - 4\sin\alpha + 4$$

Critical numbers $\left\{ \begin{array}{l} \frac{dA}{d\alpha} = 0 \Rightarrow -8\sin^2\alpha - 4\sin\alpha + 4 = 0 \qquad y = \sin\alpha \\ -8y^2 - 4y + 4 = 0 \\ -2y^2 - y + 1 = 0 \\ y = \frac{1 \mp \sqrt{1+8}}{-4} = \frac{1 \mp 3}{-4} = \begin{cases} -1 \Rightarrow \alpha = -\pi/2 \\ \frac{1}{2} \Rightarrow \alpha = \pi/6 \end{cases} \end{array} \right.$

$\frac{dA}{d\alpha}$ is defined everywhere

end points : $0 \leq \alpha \leq \frac{\pi}{4}$

$A(\pi/6) = 4\cos\frac{\pi}{6}\left(1+\sin\frac{\pi}{6}\right) = 4 * \frac{\sqrt{3}}{2} * \frac{3}{2} = 3\sqrt{3} \simeq 5.20$

$A(0) = 4\cos(0)\left(1+\sin(0)\right) = 4$

$A\left(\frac{\pi}{4}\right) = 4\cos\left(\frac{\pi}{4}\right)\left(1+\sin\frac{\pi}{4}\right) = 4 * \frac{\sqrt{2}}{2} * \left(1+\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}+2 \simeq 4.8$

So the abs. max of A is $A\left(\frac{\pi}{6}\right) \simeq 5.2$