Exercise: Find the (general) antiderivative of the following functions:

a) $f(x) = e^{3x-2}$

b) $f(x) = x\cos(3x^2+5)$

c) $f(x) = 0$

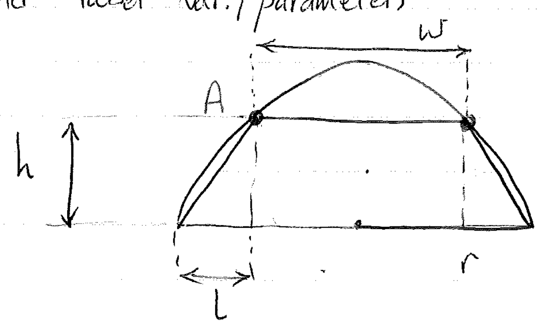
d) $f(x) = xe^x + e^x$

Office hours: Tuesday April 12, 9:30 - 10:30 am

Review of optimization

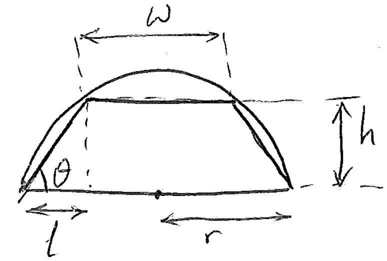
Example: Find the area of the largest trapezoid that can be inscribed in a circle of radius r and whose base is a diameter of the circle.

Step 0: Make a sketch and label var./parameters



Step 1: Identify and write the known information: ;-?

Step 2.1 var. to be optimized: A



Step 2.2 independent var? ..

Choices: h, θ, w, l.

Which one should I pick? We already discussed how we would solve the problem if we pick h as the independent var. The solution would be quite similar (and the final answer will be identical) if we pick w or l instead...

what if we pick θ though?

Step 3: Equation for A?

$$A = \frac{1}{2} h (w + 2r)$$

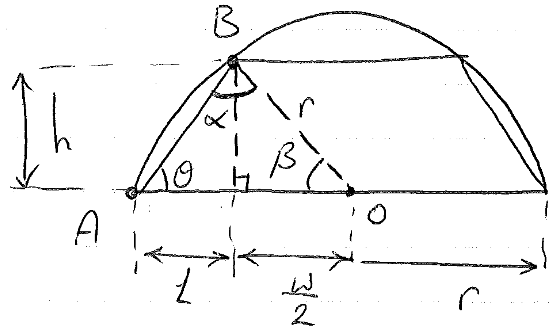
Step 4: Identify which variables change ~~with~~ as the independent var. changes?

h and w change as θ changes

r is a constant

Step 5: diff. A with respect to θ and find critical numbers.

To accomplish this, i.e. to find $\frac{dA}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2} h(\theta)\left(w(\theta) + 2r\right)\right)$ I need to find equations that give h and w as functions of θ:



I think the easiest way to go about this is to notice that ~~the~~ since $OA = OB = r$, $\overset{\triangle}{OAB}$ is an isosceles triangle. Therefore $\alpha = \theta$. We also know that

$$\left.\begin{array}{l}\alpha + \beta + \theta = \pi \\ \alpha = \theta\end{array}\right\} \Rightarrow \beta = \pi - 2\theta$$

$$\frac{w}{2} = r \cos\beta = r\cos(\pi - 2\theta) \Rightarrow w = 2r\cos(\pi - 2\theta)$$

$$h = r\sin\beta = r\sin(\pi - 2\theta)$$

$$\Rightarrow \quad A = \frac{1}{2} r \sin(\pi - 2\theta) * \left(2r\cos(\pi - 2\theta) + 2r\right)$$

$$A = \frac{r^2}{2}\sin(\pi - 2\theta) * \left(2\cos(\pi - 2\theta) + 2\right)$$

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left(\frac{r^2}{2} \operatorname{Sin}(\pi - 2\theta)\left(2 \operatorname{Cos}(\pi - 2\theta) + 2\right)\right)$$

(product rule)

$$\frac{dA}{d\theta} = \frac{1}{2}\frac{dr^2}{d\theta} * \operatorname{Sin}(\pi - 2\theta)\left(2\operatorname{Cos}(\pi - 2\theta) + 2\right)$$

$$+ \frac{r^2}{2} * \frac{d\left(\operatorname{Sin}(\pi - 2\theta)\right)}{d\theta} * \left(2\operatorname{Cos}(\pi - 2\theta) + 2\right)$$

$$+ \frac{r^2}{2} * \operatorname{Sin}(\pi - 2\theta) * \frac{d}{d\theta}\left(2\operatorname{Cos}(\pi - 2\theta) + 2\right)$$

Reminder $\frac{dr}{d\theta} = 0$

$$\frac{dA}{d\theta} = \frac{r^2}{\cancel{2}} * (-\cancel{2}) * \operatorname{Cos}(\pi - 2\theta) * \left(2\operatorname{Cos}(\pi - 2\theta) + 2\right)$$

$$+ \frac{r^2}{\cancel{2}} * \operatorname{Sin}(\pi - 2\theta) * (-\overset{2}{\cancel{4}}) * \left(-\operatorname{Sin}(\pi - 2\theta)\right)$$

$$\frac{dA}{d\theta} = -2r^2 \operatorname{Cos}(\pi - 2\theta) * \left(1 * \operatorname{Cos}(\pi - 2\theta) + 1\right)$$

$$+ 2r^2 \operatorname{Sin}^2(\pi - 2\theta)$$

$$= 2r^2\left[-\operatorname{Cos}^2(\pi - 2\theta) - \operatorname{Cos}(\pi - 2\theta) + \operatorname{Sin}^2(\pi - 2\theta)\right]$$

Reminder: $\operatorname{Sin}^2(\pi - 2\theta) + \operatorname{Cos}^2(\pi - 2\theta) = 1$

$$\operatorname{Sin}^2(\pi - 2\theta) = 1 - \operatorname{Cos}^2(\pi - 2\theta)$$

$$\frac{dA}{d\theta} = 2r^2\left[-\operatorname{Cos}^2(\pi - 2\theta) - \operatorname{Cos}(\pi - 2\theta) + 1 - \operatorname{Cos}^2(\pi - 2\theta)\right]$$

$$\frac{dA}{d\theta} = 2r^2\left[-2\operatorname{Cos}^2(\pi - 2\theta) - \operatorname{Cos}(\pi - 2\theta) + 1\right]$$

Critical numbers ?

* Endpoints? $\Rightarrow$ Domain?

θ is the smallest when $w = 0 \Rightarrow 2r\cos(\pi - 2\theta) = 0$

$\Rightarrow \cos(\pi - 2\theta) = 0 \Rightarrow \pi - 2\theta = \frac{\pi}{2} \Rightarrow 2\theta = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$

θ is the largest when $w = 2r \Rightarrow 2r\cos(\pi - 2\theta) = 2r$

$\Rightarrow \cos(\pi - 2\theta) = 1 \Rightarrow \pi - 2\theta = 0 \Rightarrow 2\theta = \pi \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

Domain : $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$\Rightarrow$ Endpoints: $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$

* $\frac{dA}{d\theta}$ is defined everywhere in the domain

* $\frac{dA}{d\theta} = 0 \Rightarrow -2\cos^2(\pi - 2\theta) - \cos(\pi - 2\theta) + 1 = 0$

lets assume $y = \cos(\pi - 2\theta)$. The above eq. then becomes $-2y^2 - y + 1 = 0$. This is a quadratic equation that we can solve for y

$$y = \frac{1 \mp \sqrt{1+8}}{-4} = \frac{1 \mp \sqrt{9}}{-4} = \frac{1 \mp 3}{-4} = \begin{cases} y = \frac{-2}{-4} = \frac{1}{2} \\ y = \frac{4}{-4} = -1 \end{cases}$$

$y = -1 \Rightarrow \cos(\pi - 2\theta) = -1 \Rightarrow \pi - 2\theta = \pi \Rightarrow \theta = 0$

But $\theta = 0$ is not in the domain. So this is not a critical number.

$y = \frac{1}{2} \Rightarrow \cos(\pi - 2\theta) = \frac{1}{2} \Rightarrow \pi - 2\theta = \frac{\pi}{3} \Rightarrow \frac{2\pi}{3} = 2\theta$

$\Rightarrow \theta = \frac{\pi}{3}$ This is in the domain. So $\theta = \frac{\pi}{3}$ is a critical number.

$$\theta = \frac{\pi}{4} \Rightarrow \begin{cases} w = 2r\cos(\pi - 2 * \frac{\pi}{4}) = 2r\cos(\frac{\pi}{2}) = 0 \\ h = r\sin(\pi - 2 * \frac{\pi}{4}) = r \end{cases}$$

$$\Rightarrow A = \frac{1}{2} h * (2r + w) = \frac{1}{2} * r * (2r + 0) = r^2$$

$$\theta = \frac{\pi}{2} \Rightarrow \begin{cases} w = 2r\cos(\pi - 2 * \frac{\pi}{2}) = 2r\cos(0) = 2r \\ h = r\sin(\pi - 2 * \frac{\pi}{2}) = r\sin(0) = 0 \end{cases}$$

$$\Rightarrow A = 0$$

$$\theta = \frac{\pi}{3} \Rightarrow \begin{cases} w = 2r\cos(\pi - 2 * \frac{\pi}{3}) = 2r\cos(\frac{\pi}{3}) = 2r * \frac{1}{2} = r \\ h = r\sin(\pi - 2 * \frac{\pi}{3}) = r\sin(\frac{\pi}{3}) = r * \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} r \end{cases}$$

$$A = \frac{1}{2} * \frac{\sqrt{3}}{2} r * (r + 2r) = \frac{3\sqrt{3}}{4} r^2$$

Since $0 < r^2 < \frac{3\sqrt{3}}{4} r^2$, A has its abs. max

at $\theta = \frac{\pi}{3}$. $A = \frac{3\sqrt{3}}{4} r^2$