Recap: implicit diff., related rates

This week: critical points, mean value theorem (Ref book: 3.1, 3.2)
(1^{st} derivative test)

Next week: 1^{st} derivative test (Ref. book 3.3)

Reminder: Critical points → derivative does not exist
→ derivative is zero
→ end points

Example: How many critical points does the function $y = 3x^{\frac{2}{3}} - x^2$ have?

a) 0 b) 1 c) 2 d) 3 e) none of the above!

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^{\frac{2}{3}}\right) - \frac{d}{dx}\left(x^2\right) = 3 * \frac{2}{3}\, x^{\frac{2}{3}-1} - 2x = 2x^{-\frac{1}{3}} - 2x$$

$$= \frac{2}{x^{\frac{1}{3}}} - 2x = \frac{2 - 2 * x * x^{1/3}}{\sqrt[3]{x}} = \frac{2(1 - x^{4/3})}{\sqrt[3]{x}}$$

$\frac{dy}{dx}$ DNE at $\boxed{x = 0}$

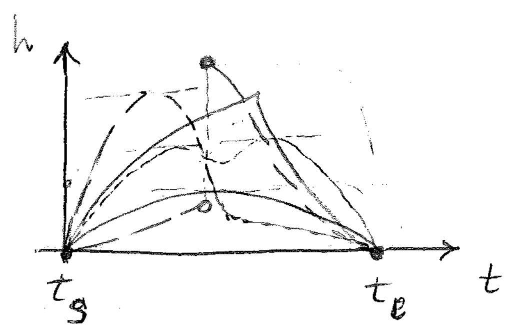
$$\frac{dy}{dx} = 0 \rightarrow 2 - 2x^{4/3} = 0 \Rightarrow 1 = x^{4/3} \Rightarrow 1^3 = x^4 \Rightarrow x^4 = 1$$

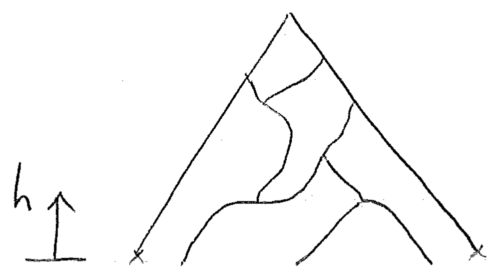
$$\Rightarrow \boxed{x = \mp 1}$$

Assume that you live in Gage tower and run to the class every morning. The distance is about 700m and it takes you 3 minutes to get here. What is average runnig speed? $V_{ave} = \frac{700\text{m}}{3\text{s}} = \frac{700}{3}\frac{\text{m}}{\text{s}}$

Is it possible that your instantaneous velocity has been less/more than $\frac{700}{3}\frac{\text{m}}{\text{s}}$ throughout the trip?

So at some instant during the trip you have been runnig at the instantaneous velocity of $\frac{700}{3}\frac{\text{m}}{\text{s}}$





Average vertical velocity: $\frac{h(t_e) - h(t_s)}{t_e - t_s} = 0$

So there's some instant when the instantaneous velocity of the trainee is $0\ \frac{\text{m}}{\text{s}}$

$\boxed{\text{Theorem}}$ Rollé's theorem: If $f(a) = f(b)$ and $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, then there's at least one number c, between a and b, so that $f'(c) = 0$.

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Example:

If I throw a ball up with an initial velocity of $10\frac{m}{s}$ ~~from~~ with an initial height of 1 m, the height of the ball as a function of time is given by $h(t) = -5t^2 + 10t + 1$, where t is in seconds and h in meters. Show that the height of the ball satisfies the hypotheses of Rolle's theorem on the interval $[0.5s, 1.5s]$, and find the value of c which the theorem says exists.

$$h(0.5) = -5 * \left(\frac{1}{2}\right)^2 + 10 * \frac{1}{2} + 1 = \frac{-5}{2*2} + \frac{10}{2} + 1 = -1.25 + 5 + 1$$

$$= 6 - 1.25 = 4.75 \text{ m}$$

$$h(1.5) = -5 * \left(\frac{3}{2}\right)^2 + 10 * \frac{3}{2} + 1 = \frac{-5*3*3}{2*2} + 5*3 + 1 = \frac{-45}{4} + 15$$

$$= \frac{-40}{4} - \frac{5}{4} + 16 = -10 - 1.25 + 16 = 6 - 1.25 = 4.75 \text{ m}$$

$h(0.5) = h(1.5) = 4.75$ So there's at least one ~~point~~ number c between 0.5s and 1.5s so that $h'(c) = 0$

$$h' = \frac{dh}{dt} = \frac{d}{dt}\left(-5t^2 + 10t + 1\right) = -10t + 10$$

$$\frac{dh}{dt} = 0 \Rightarrow -10t + 10 = 0 \Rightarrow 10t = 10 \Rightarrow t = 1$$

(4.5)

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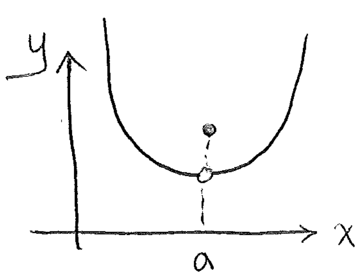
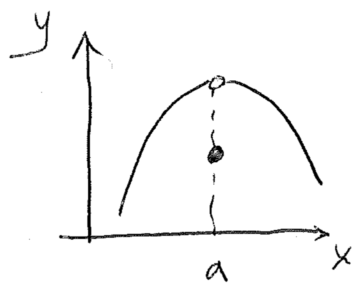
Recap: ~~the~~ Rolles theorem, MVT: f continuous on $a \le x \le b$, differentiable on $a < x < b$ $\Rightarrow$ at least one number $a < c < b$ exists so that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Today: MVT

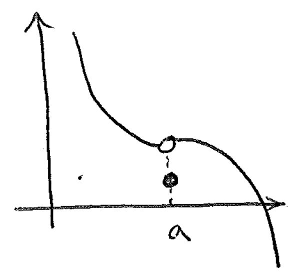
Quick Q: Indicate if y has a local min, max or neither at $x = a$.



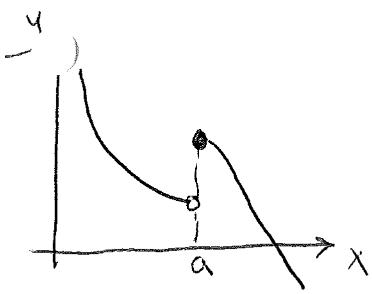
Max



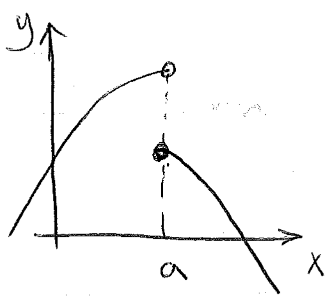
min



min



max



neither

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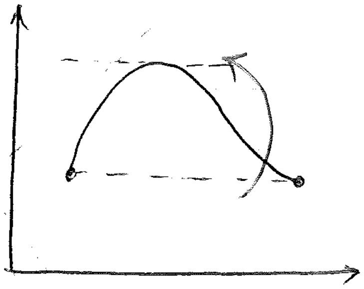
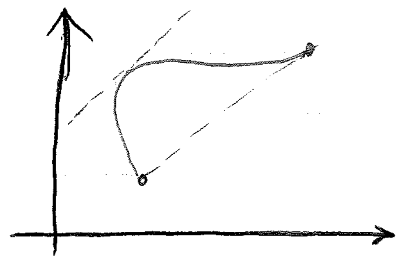
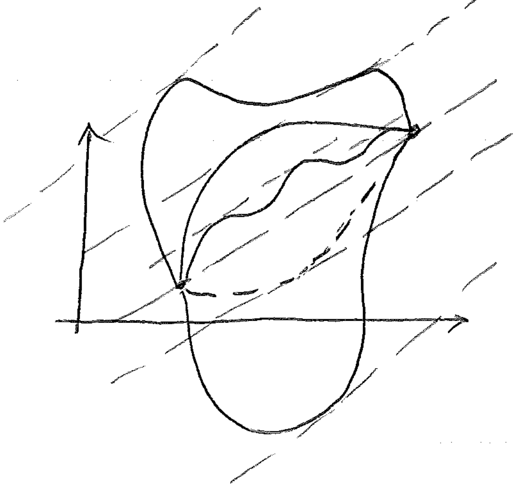




$\boxed{\text{Theorem}}$ Mean Value Theorem

If $f(x)$ is contiuous for $a \leq x \leq b$ and differentiable for $a < x < b$, then there's at least one number c, between a and b, so the tangent line at c is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



Reminder: running example
(Gage to class)

Mon. Class ended here

Q: If $f(x) = \dfrac{1}{(\sin(x))^2}$, then $f(\frac{\pi}{4}) = f(-\frac{\pi}{4}) = 2$. Can you find the value of the number(s) "c" exists?

a) yes b) no Why? $f(x)$ is not

Q: If $f(x) = \sqrt[3]{x^2}$, then $f(8) = f(-8) = 4$. Can you find the value of the number(s) "c", $-8 < c < 8$, such that $f'(c) = 0$?

a) yes b) no

Why? $f(x)$ is not differentiable on $(-8, 8)$

$f'(0)$ DNE

$$\frac{df}{dx} = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Q: If $f(x) = \frac{1}{x^2 - 4}$, then $f(3) = f(-3) = \frac{1}{5}$. Can you find the value of the number "c" such that $f'(c) = 0$?

a) yes b) No

Does Rolle's theorem apply here? a) yes b) no

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{1}{x^2-4}\right) = \frac{d}{du}\left(\frac{1}{u}\right) * \frac{du}{dx} \qquad u = x^2 - 4$$

$$\frac{df}{dx} = \frac{-1}{u^2} * \frac{d(x^2-4)}{dx} = \frac{-1}{(x^2-4)^2} * 2x = \frac{-2x}{(x^2-4)^2}$$

$$\frac{df}{dx} = 0 \Rightarrow \frac{-2x}{(x^2-4)^2} = 0 \Rightarrow x = 0$$

Example: Show that the only root of $f(x) = 2x^3 + 2x - 4$ is $x = 1$.

Contradiction proof

Well... let's 1st check: $f(1) = 2 + 2 - 4 = 0$

Now let's assume that f has another root a such that $f(a) = 0$

Then, since f is continuous and differentiable for all x between 1 and a, ~~then~~ there's at least one number c between a and 1 so that $f'(c) = \frac{f(a) - f(1)}{a - 1} = 0$

However,

$f'(x) = \frac{df}{dx} = \frac{d}{dx}(2x^3 + 2x - 4) = 6x^2 + 2 > 0$ for all x

So such a number c does not exist. So this contradicts the statement that a number c must exist such that $f'(c) = 0$.

We reached this contradiction by assuming that f has at least 2 roots. Since this assumption leads to a contradiction, it must be false and f can only have 1 root

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Fri. Jan 29

Q: ~~You are~~ The toll taker says "Based on the elapsed time from when the car entered the toll road until the car stopped at my booth, I know the average speed of the car was $110 \frac{km}{h}$. I didn't actully see the car speeding, but I know it was and I gave the driver a speeding ticket." Assuming that the speed limit is $100 \frac{km}{h}$ and that you are a traffic court judge, do you think this appropriate? a) yes b) no

Why? if p is the position of the car and t_e and t_b are when the car entered the toll road and when it got to the booth, we know that

$$\text{average velocity}: \frac{P(t_b) - P(t_e)}{t_b - t_e} = 110 \frac{km}{h}$$

Since velocity is the derivative of position, $V(t) = \frac{dP}{dt}$, MVT states that there is a time c

$$p'(c) = 110 \frac{km}{h}, \quad t_e < c < t_b$$

i.e. the car must have been driving at $110 \frac{km}{h}$ at some point.

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True or false: If a, b (with $a<b$) are two roots of $f(x)$ and $f(x)$ is differentiable for $a<x<b$ and continuous for $a \le x \le b$, then $f'(x)$ has at least one root between a and b.

a) True   b) false.

Use Rolle's theorem

$f(a) = f(b) = 0$
continuity ✓
differentiability ✓

? there's at least one number c, $a<c<b$ so that $f'(c) = 0$

Note that c is a critical number of f
How can we tell if it is a max or min?

$\boxed{\text{Definition}}$: The function f is increasing on (a,b) if $a<x_1<x_2<b$ implies $f(x_1) < f(x_2)$

The function f is decreasing on (a,b) if $a<x_1<x_2<b$ implies $f(x_1) > f(x_2)$

f is monotonic on (a,b) if f is increasing on (a,b) or if f is decreasing on (a,b).

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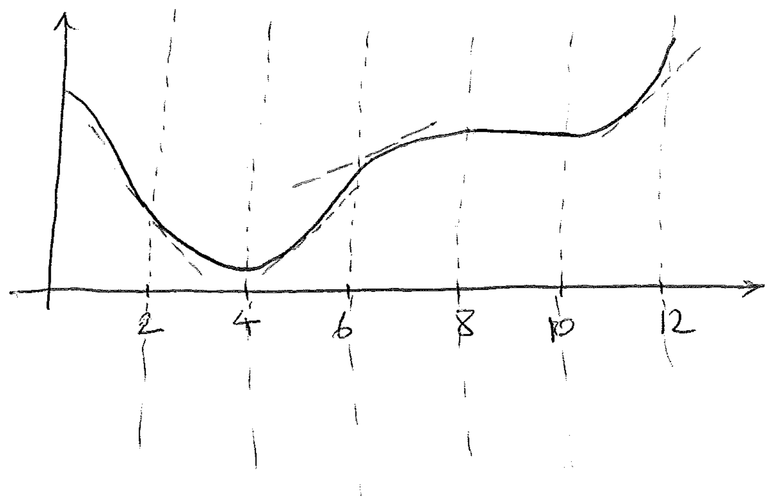


decreasing on $[0,4]$,

increasing on $[4,8]$, $[10,12]$

constant on $[8,10]$

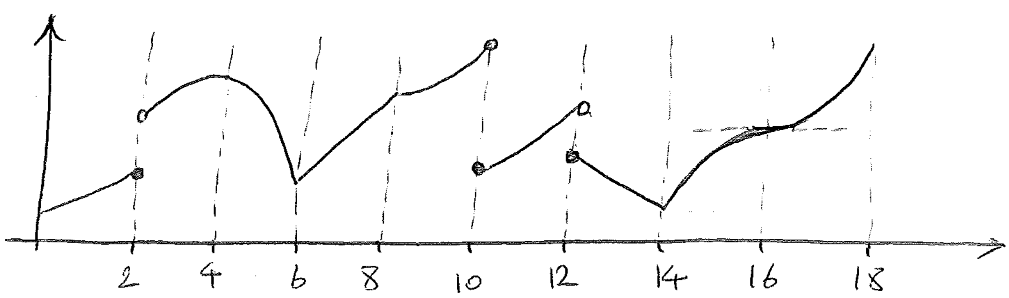
Theorem 1^{st} shape theorem

For a function f which is differentiable on an interval (a,b)

i) if f is increasing on (a,b), then $f'(x) \geq 0$ for all x in $(a$

ii) if f is dereasing on (a,b), then $f'(x) \leq 0$ for all x in $(a,b$

iii) if f is constant on (a,b), then $f'(x)=0$ for all x in $(a,b$



decreasing on $(4,6)$, $(12,14)$

increasing on $(0,4)$, $(6,10)$, $(10,12)$, $(14,18)$

Note that its ok that ① f is not diff. at 8, ② f is not continuous at 2, ③ $f'(16)=0$

Fri. Class Ended here

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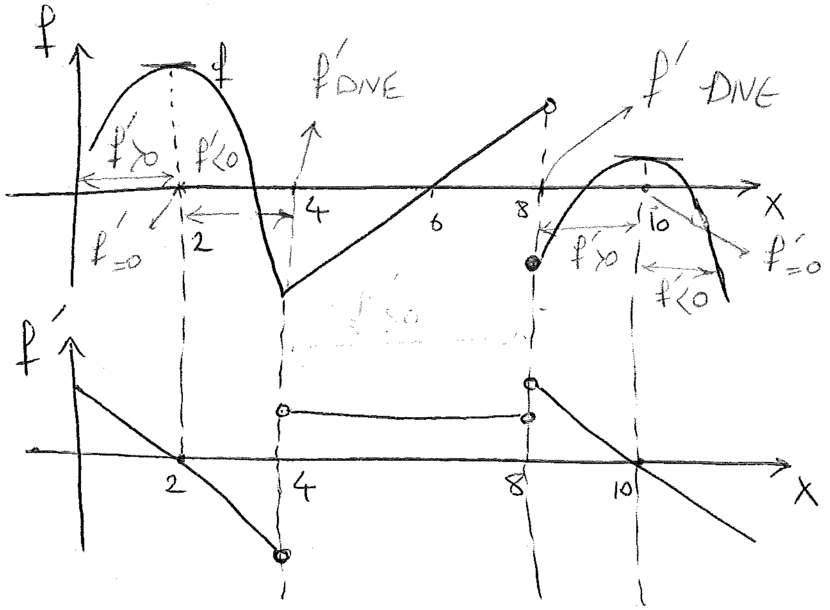
Now can you sketch the graph of f' given the graph of f



Critical numbers:
2, 4, 8, 10

How about sketching f given the graph of f'?

$\boxed{\text{Theorem}}$ 2^{nd} shape theorem

For a function f which is differentiable on an interval I

i) if $f'(x)>0$ for all x in the interval I, then f is increasing on I

ii) if $f'(x)<0$ for all x in the interval I, then f is decreasing on I

iii) if $f'(x)=0$ for all x in the interval I, then f is constant on I.

→ Proof on page 12

Example: Use information about f' to help graph $f(x) = \frac{x^3}{3} - 3x^2 + 5x - 1$

$$\frac{df}{dx} = x^2 - 6x + 5 \qquad \frac{df}{dx} = 0 \Rightarrow x = \frac{6 \mp \sqrt{36-20}}{2} = \frac{6 \mp \sqrt{16}}{2}$$

1 − 4 , 5

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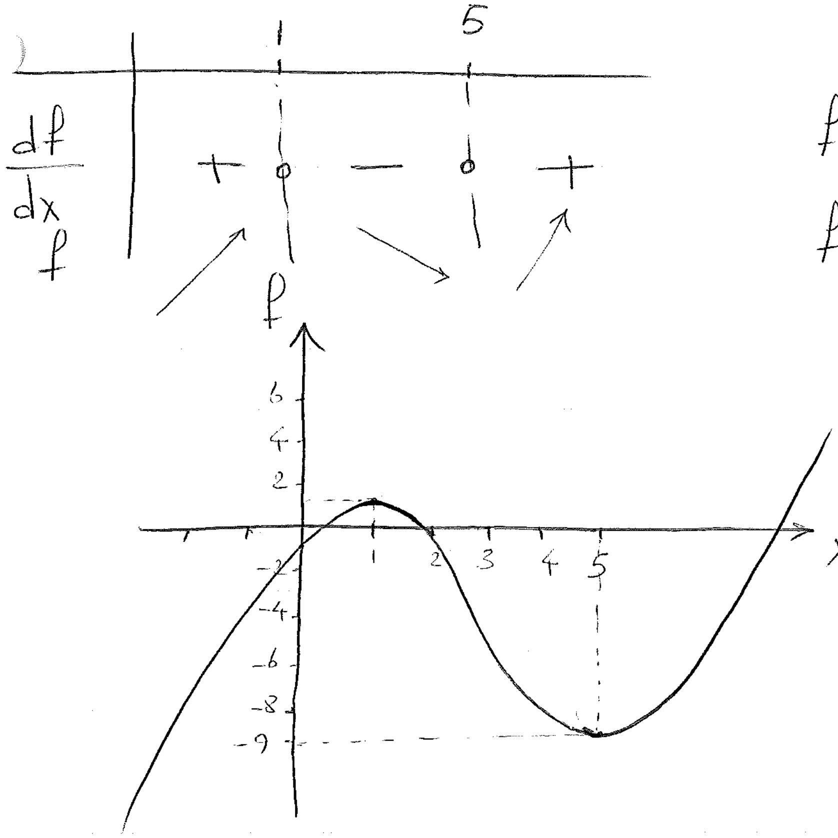


$f(1) = \frac{1}{3} - 3 + 5 - 1 = \frac{4}{3}$

$f(5) = \frac{5^3}{3} - 3 * 25 + 5 * 5 - 1$

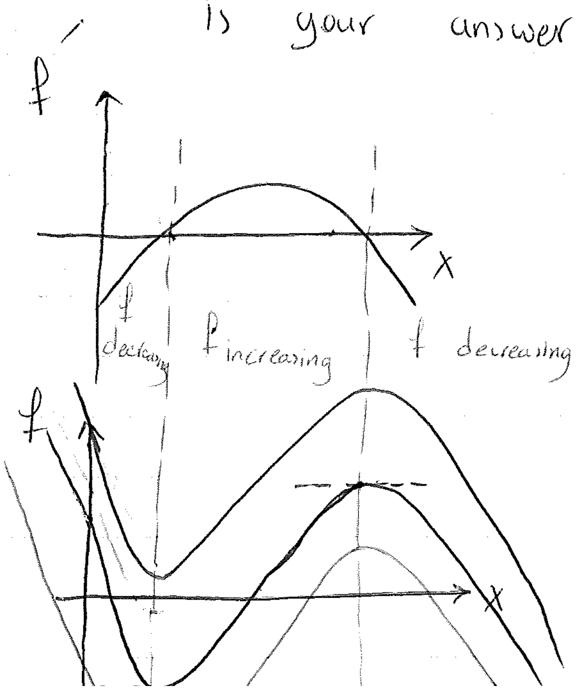
$= 5^2\left(\frac{5}{3} - 3 + 1\right) - 1$

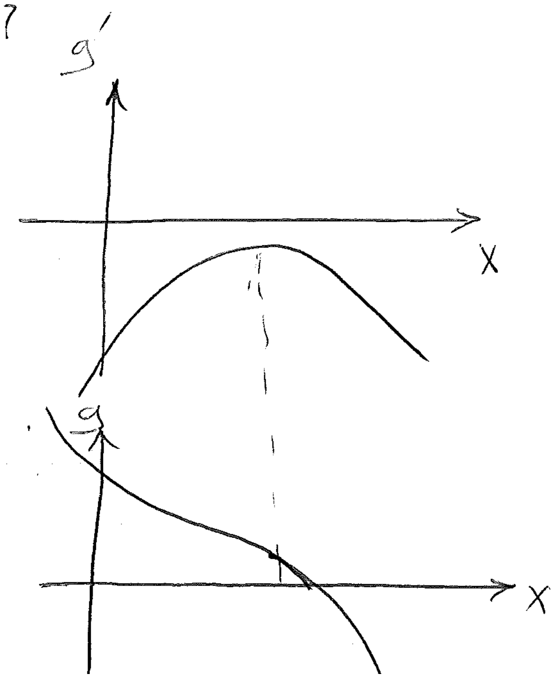
$= 5^2\left(\frac{5 - }{3}\right) - 1 = \frac{-25}{3} \quad 1$

$= -\frac{28}{3} \simeq -9.$

Example: Use the graph of $f'(x)$ to make a graph of $f(x)$. Does a graph of f' completely determine the graph of f?

is your answer unique?





Proof of i: (Contradiction proof)

Let's assume that f is not increasing on I, then there exist $a < b$ in I so that $f(b) < f(a)$.

Since f' is defined for all x in I, f must be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$.
So MTV states that there is at least one number c so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad a < c < b$$

However, since $a < b$, we have $b - a > 0$. Also, $f(b) < f(a)$; therefore $f(b) - f(a) < 0$. So we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} < 0$$

This contradicts the statement that $f' > 0$ for all x in I.

So the assumption that f is not increasing on I must be false.