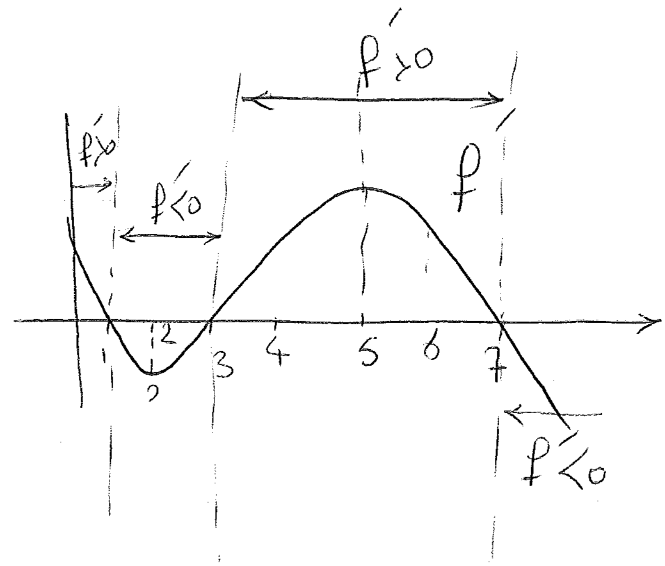
Feb 10, 2016
Review session

Example 1:



Note: f' exists everywhere $\Rightarrow$ f is continuous on the domain presented here

at $\left\{\begin{array}{l} x=1 \\ x=7 \end{array}\right\}$ f' changes sign from $+$ to $-$ $\Rightarrow$ $x=1$, $x=7$ are the locations of two local max. of f.

at $x=3$, f' changes sign from $-$ to $+$ $\Rightarrow$ f has a local min at $x=3$

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Example 2: Given the equation $x^3y^2 + y^3 \sin(xy) = 0$ find $\frac{dy}{dx}$?

$$\frac{d}{dx}\left(x^3y^2 + y^3 \sin(xy)\right) = \frac{d}{dx}(0) = 0$$

$$\frac{d}{dx}(x^3y^2) + \frac{d}{dx}(y^3 \sin(xy)) = 0$$

product rule

$$\frac{d(x^3)}{dx} * y^2 + x^3 \frac{d(y^2)}{dx} + \frac{d(y^3)}{dx} * \sin(xy) + y^3 \frac{d(\sin(xy))}{dx} = 0$$

$u = xy$

$$3x^2y^2 + x^3 \frac{d(y^2)}{dy} * \frac{dy}{dx} + \frac{d(y^3)}{dy} * \frac{dy}{dx} * \sin(xy) + y^3 \frac{d\sin(u)}{du} * \frac{du}{dx} =$$

$$3x^2y^2 + x^3 * 2y * \frac{dy}{dx} + 3y^2 * \frac{dy}{dx} * \sin(xy) + y^3 \cos(u) \frac{d(xy)}{dx} = 0$$

product rule

$$3x^2y^2 + 2x^3y \frac{dy}{dx} + 3y^2 \sin(xy) \frac{dy}{dx} + y^3 \cos(xy)\left[\frac{dx}{dx} * y + x \frac{dy}{dx}\right] = 0$$

$$\downarrow$$

$$3x^2y^2 + \left[2x^3y + 3y^2 \sin(xy)\right] \frac{dy}{dx} + y^3 \cos(xy)\left(y + x\frac{dy}{dx}\right) = 0$$

$$3x^2y^2 + y^4 \cos(xy) = -\left[+2x^3y + 3y^2 \sin(xy) + xy^3 \cos(xy)\right] \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-\left(3x^2y^2 + y^4 \cos(xy)\right)}{2x^3y + 3y^2 \sin(xy) + xy^3 \cos(xy)}$$

Example 3: Find all the critical points of the function $f(x) = x^{1/3} e^{-3x}$

Identify all the local min and max.

$$f(x) = x^{+1/3} e^{-3x}$$

$$\frac{df}{dx} = \frac{d}{dx}\left(x^{1/3} e^{-3x}\right) = \frac{d}{dx}(x^{1/3}) * e^{-3x} + x^{1/3} * \frac{d}{dx}\left(e^{-3x}\right)$$

$$\frac{df}{dx} = \frac{1}{3} x^{\frac{1}{3}-1} * e^{-3x} + x^{1/3} * (-3) * e^{-3x}$$

$$= e^{-3x}\left(\frac{1}{3} x^{-\frac{2}{3}} \cancel{*} - 3 x^{\frac{1}{3}}\right)$$

$$= e^{-3x}\left(\frac{1}{3 x^{\frac{2}{3}}} - 3 x^{\frac{1}{3}}\right)$$

$$= e^{-3x}\left(\frac{1 - 9x}{3 x^{2/3}}\right)$$

$f'(x) = 0 \Rightarrow \begin{cases} e^{-3x} = 0 & \text{does not have a solution. i.e } e^{-3x} > 0 \text{ for all } x \in \mathbb{R} \\ \text{OR} \\ 1 - 9x = 0 \Rightarrow x = \frac{1}{9} \end{cases}$

$f'(x)$ DNE $\Rightarrow x = 0$

		0		$1/9$	
$3x^{2/3}$	$+$		$+$		$+$
$1-9x$	$+$		$+$	ϕ	$--$
f'	$+$		$+$		$-$

Critical numbers: $x = 0, 1/9$

f does not have any local min

f is increasing on $(-\infty, \frac{1}{9})$

f is decreasing on $(\frac{1}{9}, \infty)$

Hypotheses

Example 4: MVT: if $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, then there is at least one number, c, between a and b so that the tangent line at c is parallel to the secant line through the points $(a, f(a))$, $(b, f(b))$:

Conclusion $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

f and f' are ~~continuous~~ & differentiable for all x $\Rightarrow$ f and f' are continuous for all x

Define $g(x) = f'(x)$

$g(2) = f'(2) = 3$

$g(5) = f'(5) = 0$

MVT states that at least one number $2 < c < 5$ exists such that

$$g'(c) = \frac{g(5) - g(2)}{5 - 2} = \frac{0 - 3}{5 - 2} = -1$$

Now notice that $g'(x) = \frac{dg}{dx} = \frac{d}{dx}(f'(x)) = f''(x)$

So $f''(c) = -1$ has at least one real solution with $2 < c < 5$.

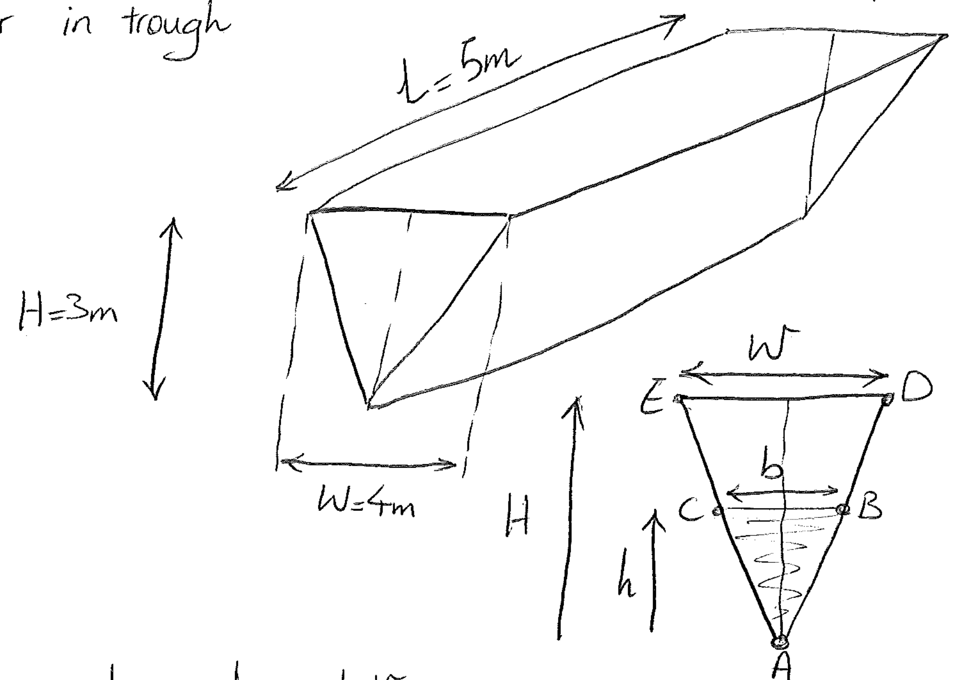
Example 5:

V: volume of water in trough

Steps 0, 1

$$\frac{dV}{dt} = 2\ \frac{m^3}{min}$$

$h = 1m$



Step 2: $\frac{dh}{dt}$

Step 3: equation that relates h and V

$$V = \left(\frac{1}{2} h b\right) * L$$

Do not plug any numbers unless you are 100% sure that it does not change w/ time. If it changes w/ time or you are unsure, USE A SYMBOL

Step 4: V, b and h change w/ time $\Rightarrow V(t)$, $b(t)$, $h(t)$

L is constant, $\frac{dL}{dt} = 0$

$\Rightarrow$ Can plug in the value of L

Step 5: diff with respect to time:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2} b h L\right) = \frac{L}{2}\frac{d}{dt}(bh) = \frac{L}{2}\left(\frac{db}{dt} * h + b\frac{dh}{dt}\right)$$

We are looking for $\frac{dh}{dt}$. We need to relate $\frac{db}{dt}$ to

$b, h \ldots ?$ Similar triangles: $\hat{ABC}$ and $\hat{ADE}$ $\Rightarrow \frac{h}{H} = \frac{b}{W}$

$\Rightarrow b = \frac{W}{H} h$ W, H do not change w/ time

$\Rightarrow \frac{db}{dt} = \frac{d}{dt}\left(\frac{W}{H} h\right) = \frac{W}{H} \frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{L}{2} * \left(\underbrace{\left(\frac{W}{H} * \frac{dh}{dt}\right)}_{\frac{db}{dt}} * h + \frac{dh}{dt} * \overbrace{\left(\frac{W}{H} h\right)}^{b} \right)$$

Solve for $\frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{L}{2} * \left[\frac{Wh}{H} + \frac{Wh}{H}\right] \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2}{L * \left[\frac{2Wh}{H}\right]} * \frac{dV}{dt} = \frac{2}{\frac{2LWh}{H}} \frac{dV}{dt} = \frac{2H}{2LWh} \frac{dV}{dt}$$

Step 6:

Text

$$\frac{dh}{dt} = \frac{3m}{5m * 4m * 1m} * 2 \frac{m^3}{min} = \frac{6}{20} \frac{m}{min} = \frac{3}{10} \frac{m}{min}$$

for example 6, see the solution of quiz 2

Example: Find the local min and max of $f(x) = x\sqrt{x - x^2}$.

product rule

$$\frac{df}{dx} = \frac{d}{dx}\left(x\sqrt{x-x^2}\right) = \frac{dx}{dx} * \sqrt{x-x^2} + x\,\frac{d}{dx}\left(\sqrt{x-x^2}\right)$$

chain rule

$$\frac{df}{dx} = 1 * \sqrt{x-x^2} + x\,\frac{d\sqrt{u}}{du} * \frac{du}{dx} \qquad u = x - x^2$$

$$\frac{df}{dx} = \sqrt{x-x^2} + x * \frac{1}{2\sqrt{u}} * \frac{d}{dx}\left(x - x^2\right)$$

$$\frac{df}{dx} = \sqrt{x-x^2} + \frac{x}{2\sqrt{x-x^2}} * (1-2x)$$

$$= \frac{\sqrt{x-x^2} * 2\sqrt{x-x^2}}{2\sqrt{x-x^2}} + \frac{x - 2x^2}{2\sqrt{x-x^2}}$$

$$= \frac{2(x-x^2) + x - 2x^2}{2\sqrt{x-x^2}} = \frac{3x - 4x^2}{2\sqrt{x-x^2}}$$

$$\frac{df}{dx} = 0 \Rightarrow 3x - 4x^2 = 0 \Rightarrow x(3-4x) = 0 \Rightarrow \begin{cases} x = 0 \\ x = \frac{3}{4} \end{cases}$$

$$\frac{df}{dx}\ \text{DNE} \Rightarrow \sqrt{x-x^2} = 0 \Rightarrow x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 1 \end{cases}$$

for $x=0$ both the numerator and the denominator become zero. We need to check $\lim_{x\to 0} \frac{df}{dx}$

$$\lim_{x\to 0}\frac{df}{dx} = \lim_{x\to 0}\frac{3x - 4x^2}{2\sqrt{x-x^2}} = \lim_{x\to 0}\frac{x(3-4x)}{2\sqrt{x(1-x)}} = \lim_{x\to 0}\frac{x(3-4x)}{2\sqrt{x}\sqrt{1-x}} = \lim_{x\to 0}\frac{\sqrt{x}(3-4x)}{2\sqrt{1-x}}$$

$\frac{df}{dx} = 0 \implies$ $x = 0$, $x = \frac{3}{4}$

$\frac{df}{dx}$ DNE $\implies$ $x = 1$

a) $x = 0, 1, \frac{3}{4}$

c) $x = \frac{3}{4}$

e) $x = 1$

b) $x = 0, \frac{3}{4}$

d) $x = 0, 1$

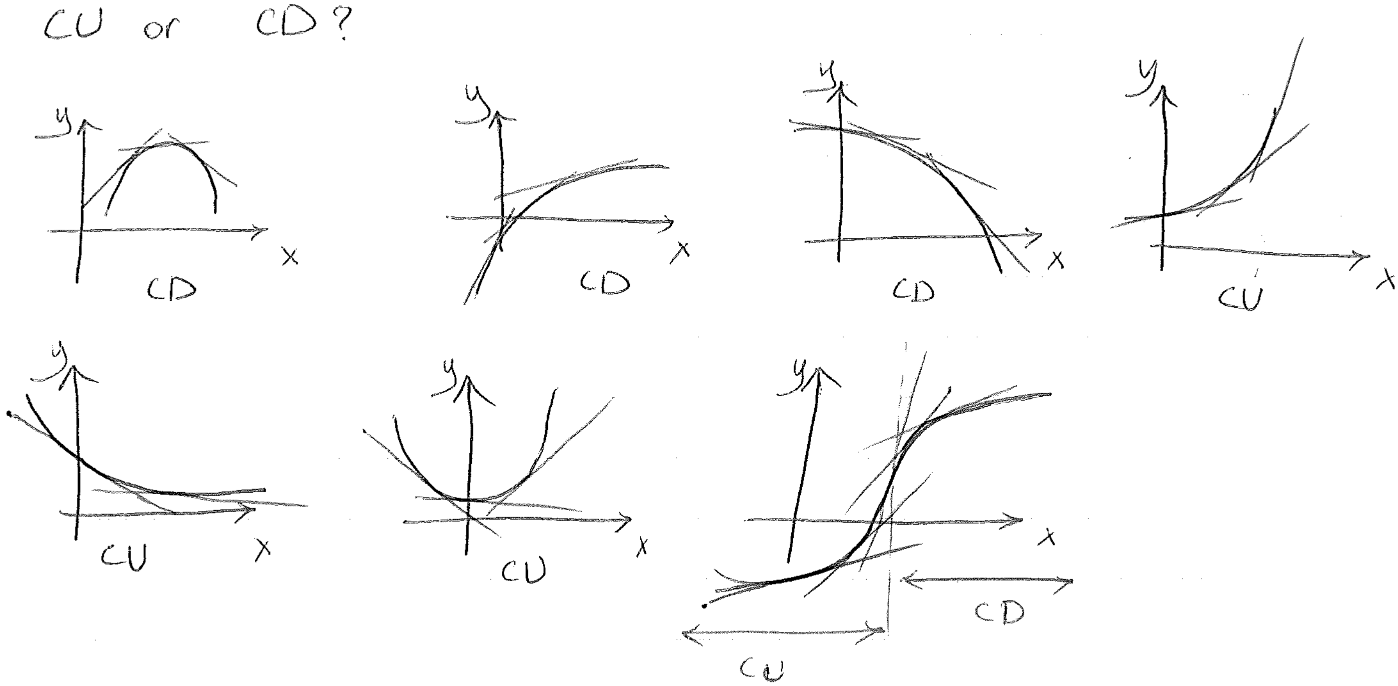
$x = 0, 1$ are end points. So $(0, f(0))$ and $(1, f(1))$ are not local extrema

Ida Karimfazli
No class or office hours next week

Today: concavity, 2^{nd} derivative test

Graphically a function is co cave up (CU) if its graph is curved w/ the opening upward. Similarly a function is concave down (CD if its graph opens downward.

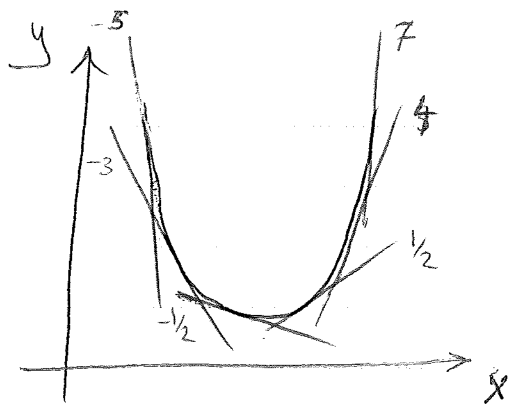
CU or CD?

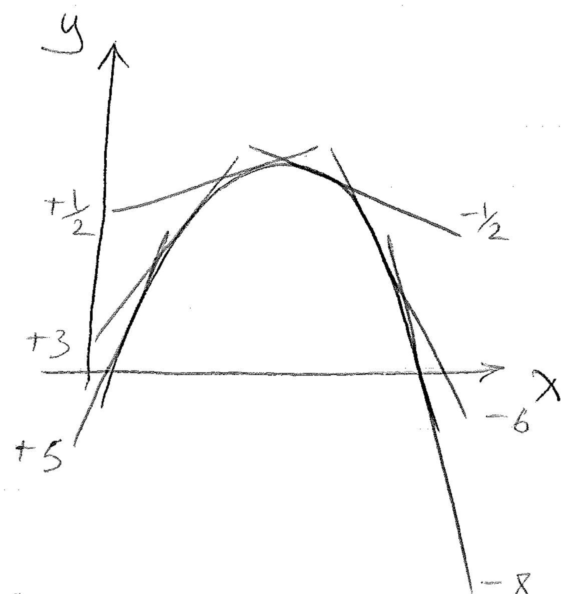


Definition Let f be a differentiable.

* f is concave up at a if the graph of f is above the tangent line L to f for all x close to a (but not equal to a): $f(x) > L(x) = f(a) + f'(a)(x-a)$

* f is concave down at a if the graph of f is below the tangent line L to f for all x close to a (but not equal to a): $f(x) < L(x)$





The 2nd derivative condition for concavity

a) if $f''(x) > 0$ on an interval I, then $f'(x)$ is increasing on I and f is concave up on I.

b) if $f''(x) > 0$ on an interval I, the $f'(x)$ is decreasing on I and f is concave down on I.

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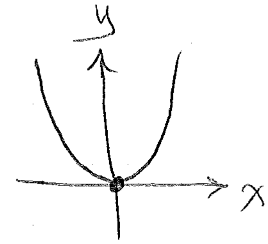
Example: Discuss the following curves with respect to concavity and local min and max.

a) $y = x^2$

$y' = 2x$

$y'' = 2$

		0	
y''	CU +		+ CU
y'	−		+
y	↘	min	↗

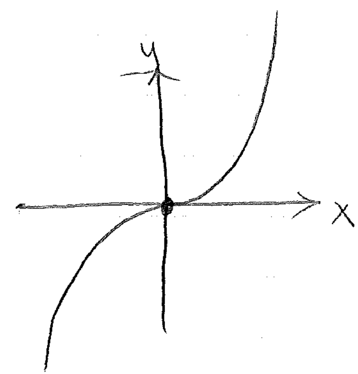


b) $y = x^3$

$y' = 3x^2$

$y'' = 6x$

		0	
y''	CD −	0	+ CU
y'	+	0	+
y	↗	not a local extremum	↗

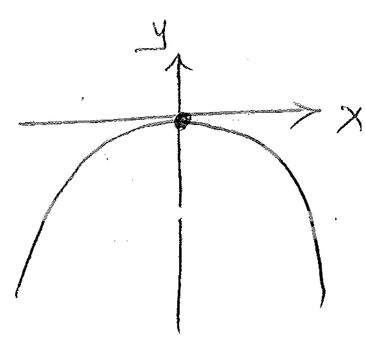


c) $y = -x^4$

$y' = -4x^3$

$y'' = -12x^2$

		0	
y''	CD −	0	− CD
y'	+	0	−
y	↗	max	↘



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The Second derivative test for extremes:

a) if $f'(c)=0$ and $f''(c)<0$, then f is concave down and has a local maximum at $x=c$ (e.g $y=-x^2$)

b) if $f'(c)=0$ and $f''(c)>0$, the f is concave up and has a local minimum at $x=c$ (e.g $y=x^2$)

c) if $f'(c)=0$ and $f''(c)=0$, then f may have a local maximum, a minimum or neither (e.g $y=x^4$, $y=-x^4$, $y=x^3$)

Definition An inflection point is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

e.g. Example on page 3 $\Rightarrow$
- a) no inflection point
- b) $x=0$ is an inflection point
- c) no inflection point

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find y''

Example: Find where the function $y = x^4 - 4x^3$ is concave up and down. Find all the local min and max. and find y'

$$y = x^4 - 4x^3$$

$$y' = \frac{dy}{dx} = 4x^3 - 12x^2 = (x^2)(4x - 12)$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(4x^3 - 12x^2\right) = 12x^2 - 24x = (12x)(x-2)$$

		0		3	
x^2	+	0	+	|	+
$4x-12$	−	|	−	0	+
y'	−	0	−	0	+

		0		2	
$12x$	−	0	+	|	+
$x-2$	−	|	−	0	+
y''	+	0	−	0	+

		0		2		3	
y'	−	0	−	|	−	0	+
y''	CU +	0	CD −	0	CU +	|	CU +
y	↘	ip	↘	ip	↘	min	↗

ip : inflection point