Assignment 5 - Solution

Problem 1.

This problem is a variant of the "silly walks" problem from the workshop. Suppose you are in a room that is a meters wide and 10 meters long. You can "wall walk" along the a meter long wall at s meters per second and "floor walk" in any direction at 1 meter per second. You start at one corner of the room, wall walk along the wall of length a meters for some distance, and then floor walk straight to the corner opposite from where you started.

Suppose that the fastest route to the opposite corner of the room is to directly floor walk from one corner of the room to the other (i.e. you wall walk for 0 distance). What can be said about the relationship between a and s?

Solution:

We will begin by solving the optimization problem, and then we will determine the conditions on a and s.

We need to compute the total time t. Write this time $t = t_1 + t_2$, where t_1 is the wall-walking time and t_2 is the floor-walking time.

Let x be distance walked across the wall. Then x is between 0 and a inclusive. We have that $t_1 = \dfrac{x}{s}$.

The floor-walking distance is $\sqrt{100 + (a - x)^2}$. Because my floor-walking speed is 1 meter per second, we have that $t_2 = \sqrt{100 + (a - x)^2}$.

We therefore have

$$t = t_1 + t_2$$

so that

$$t = \frac{x}{s} + \sqrt{100 + (a - x)^2}.$$

Taking the derivative gives

$$t'(x) = \frac{1}{s} + 1/2(100 + (a - x)^2)^{-1/2}(2(a - x))(-1)$$

which is

$$t'(x) = \frac{1}{s} + (x - a)(100 + (a - x)^2)^{-1/2}.$$

Now we set $t'(x) = 0$ to get

$$(a - x) = \frac{1}{s}(100 + (a - x)^2)^{1/2}$$

and square both sides:

$$(a - x)^2 = \frac{1}{s^2}(100 + (a - x)^2)$$

so

$$\left(1 - \frac{1}{s^2}\right)(a - x)^2 = \frac{100}{s^2}$$

and taking a square root gives:

$$\left(1 - \frac{1}{s^2}\right)^{1/2}(a - x) = \frac{10}{s}$$

so

$$x = a - \frac{10}{s}\left(1 - \frac{1}{s^2}\right)^{-1/2}$$

The square root makes sense as long as $s > 1$. Clearly this value of x is always smaller than a. We can simplify this expression a little bit:

$$x = a - \frac{10}{s}\left(\frac{s^2 - 1}{s^2}\right)^{-1/2}$$

and rewrite this as

$$x = a - \frac{10}{s}\frac{s}{\sqrt{s^2 - 1}}$$

and cancel out the s's:

$$x = a - \frac{10}{\sqrt{s^2 - 1}}$$

So this critical point is in $[0, a]$ provided that a is larger than $\dfrac{10}{\sqrt{s^2 - 1}}$. As stated before, the expression is not even defined unless $s > 1$.

Let's compute $t''(x)$. We get

$$(100 + (a - x)^2)^{-1/2} - (x - a)^2(100 + (a - x)^2)^{-3/2}$$

which is

$$(100 + (a - x)^2)^{-3/2}((100 + (a - x)^2 - (x - a)^2))$$

this comes out to

$$100(100 + (a - x)^2)^{-3/2},$$

a positive number. So if there is a critical point in $[0, a]$ it must be a local minimum, and as the unique local minimum it must be the global minimum.

It's clear that t' is positive at $x = a$; therefore, t will *never* have a global minimum there, as we can always reduce t by reducing x. Therefore, if there is no critical point on $[0, a]$ then the *global minimum occurs at $x = 0$*. If there is a critical point on $(0, a)$, then we have from the work above that the global minimum occurs strictly inside the interval $(0, a)$. Therefore, the global minimum will only occur at $x = a$ if either $s < 1$ or if $a \le \dfrac{10}{\sqrt{s^2 - 1}}$.

Problem 2.

You can solve multivariable optimization problems by first treating one of the variables as a parameter. This problem will illustrate how this is done.

We want to solve the following optimization problem: suppose x, y, and z are positive numbers that sum to 10. What is the largest possible value of $xy + xz + yz$?

To solve this optimization problem, use the following steps:

a) First, suppose that z is a fixed parameter. Then we have to find nonnegative numbers x and y (depending on the fixed value z such that $x + y = 10 - z$), such that $xy + xz + yz$ is maximized.

b) To find the global maximum, plug in the values for x and y obtained in (a) in order to rewrite $xy + xz + yz$ as a function of a single variable. Then maximize this function.

Solution: First, we'll do step 1. Let's fix the parameter z. Then $y = 10 - z - x$, so we get that our objective function is

$$f(x) = x(10 - z - x) + xz + (10 - z - x)z$$

This is
$$f(x) = 10x - zx - x^2 + xz + 10z - z^2 - xz$$

so differentiating gives
$$f'(x) = 10 - z - 2x + z - z$$

which is
$$f'(x) = 10 - 2x - z$$

thus $z = 10 - 2x$ so $x = \dfrac{10 - z}{2}$

Notice that the second derivative is -2, which is negative; therefore, this is a *maximum*.

Now, let's do step 2. z is allowed to vary now. We have that $z = 10 - 2x$, so our quantity is
$$f(x) = xy + x(10 - 2x) + (10 - 2x)y$$

and where
$$10 = x + 10 - 2x + y$$

so
$$x = y$$

Therefore, our expression reduces to
$$f(x) = x^2 + x(10 - 2x) + (10 - 2x)x$$

which is
$$f(x) = x^2 + 10x - 2x^2 + 10x - 2x^2$$

which comes out to
$$f(x) = -3x^2 + 20x$$

and taking the first derivative gives
$$f'(x) = -6x + 20$$

which has a critical point at $x = \dfrac{10}{3}$. $f''(x) = -6$, so this is a *maximum*.

Therefore, we get the maximum at $x = \dfrac{10}{3}$, $y = \dfrac{10}{3}$, and $z = \dfrac{10}{3}$.