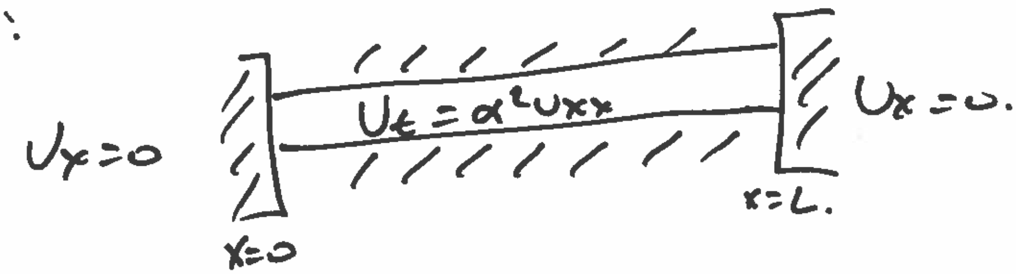
Last time: Neumann B.C's.

Today: Same.

Recall that we found the general solution

for:



to be:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha^2 (n\pi/L)^2 t} \cdot \cos\left(\frac{n\pi x}{L}\right).$$

Now, we need to find the coeff's.

<u>Apply the I.C.</u>

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

- This projects $f(x)$ onto two basis functions
  1) $\cos\left(\frac{n\pi x}{L}\right)$
  2) $1.$

<u>Orthogonality of cosine function.</u>

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n. \\ L, & m = n. \\ 2L, & m = n = 0. \end{cases}$$

- Orthogonal over $[0, 2L]$ when $m \neq n$.

<u>For $m \neq n$.</u>

We note that: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$+ \cos(A-B)$ " $+$ "

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

<u>Solving n.</u>

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{\pi(n+m)x}{L}\right) + \cos\left(\frac{\pi(n-m)x}{L}\right)\right] dx$$

$$= \frac{L}{2\pi(n+m)} \sin\left(\frac{\pi(n+m)x}{L}\right)\Bigg|_{-L}^{L} + \frac{L}{2\pi(n-m)} \sin\left(\frac{\pi(n-m)x}{L}\right)\Bigg|_{-L}^{L}$$

$$= 0 + 0 = 0.$$

$\underline{M = n.}$

$$\cos^2 A = \frac{1}{2}\left(\cos(2A) + 1\right).$$

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{2n\pi x}{L}\right) + 1\right] dx.$$

$$= \frac{1}{2}\left(\frac{L}{2\pi n}\right) \cdot \sin\left(\frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} + \frac{1}{2}x\Bigg|_{-L}^{L}$$

$$= L$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

Multiply first by the Eigenfunction $X_0 = 1$.

$$\int_{-L}^{L} f(x) \cdot 1 \, dx = \int_{-L}^{L} A_0 \, dx + \int_{-L}^{L} \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$= A_0 x \Big|_{-L}^{L} + \sum_{n=1}^{\infty} A_n \left(\frac{L}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}$$

$$= 2A_0 L + 0.$$

$$\boxed{\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x) \cdot dx. \\ &= \frac{1}{L}\int_{0}^{L} f(x) \cdot dx. \end{aligned}}$$

2) Multiply $\cdot$ $\sum = \cos\left(\frac{m\pi x}{L}\right)$.

$\sum_n$: $$\int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} A_0 \cdot \cos\left(\frac{m\pi x}{L}\right) dx +$$

$$\int_{-L}^{L} \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx.$$

$$\int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \frac{A_0 L}{m\pi} \sin\left(\frac{m\pi x}{L}\right)\Big|_{-L}^{L} + A_m \cdot L.$$

So, $$\boxed{A_m = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx.}$$

$$A_m = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = a_m$$

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \cdot dx = \frac{a_0}{2}$$

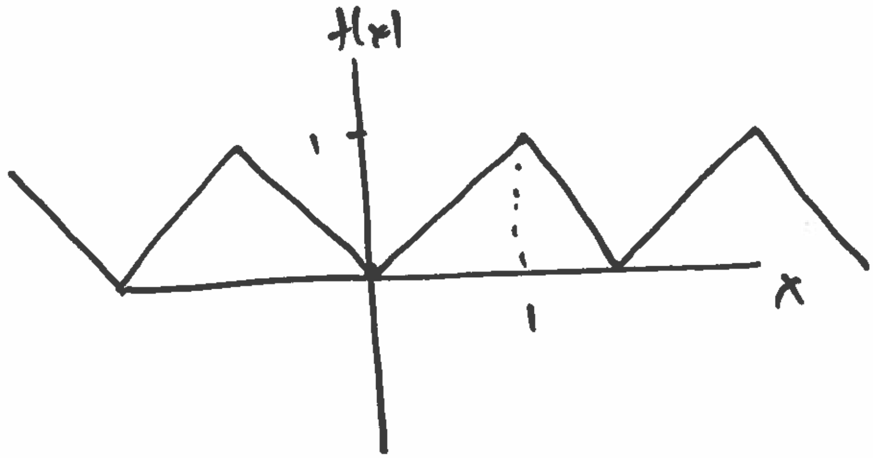
Hence, the general solution is

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \cdot \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \quad ; \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Ex.

$f(x) = x.\quad , 0 < x < L \qquad L = 1.$



An even extension of $f(x)$.

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} |x|\,dx = \frac{1}{1}\int_{-1}^{1} |x|\,dx = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= 2\int_0^L \underset{u}{x}\,\underbrace{\cos\left(\frac{n\pi x}{L}\right) dx}_{dv.}$$

$$= 2\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^1 - \int_0^1 \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx\right] \quad \boxed{L=1}$$

$$= 2\left[0 + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1\right]$$

$$= \frac{2}{(n\pi)^2}\left[\cos(n\pi) - 1\right] = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{-4}{(n\pi)^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

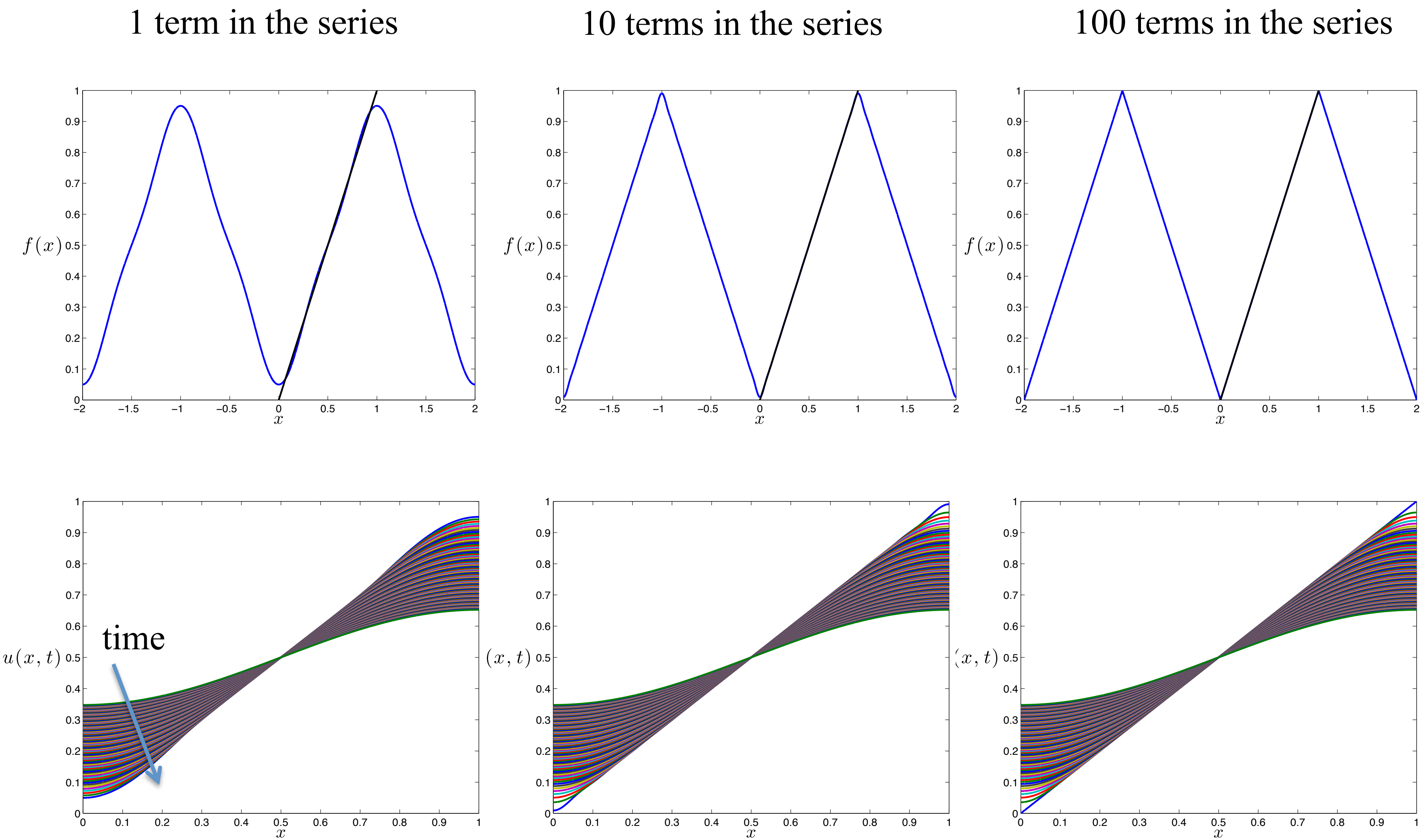
Change of variable: $n = 2K+1$.

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{K=0}^{\infty} \frac{\cos\left[(2K+1)\pi x\right]}{(2K+1)^2}$$

the general solution for $f(x) = x$.

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{K=0}^{\infty} e^{-\alpha^2\left[(2K+1)\pi\right]^2 t} \cdot \frac{\cos\left[(2K+1)\pi x\right]}{(2K+1)^2}$$

Lecture 16


1 term in the series
10 terms in the series
100 terms in the series
$f(x)$
x
$u(x,t)$
time

Example of an eigenvalue problem.

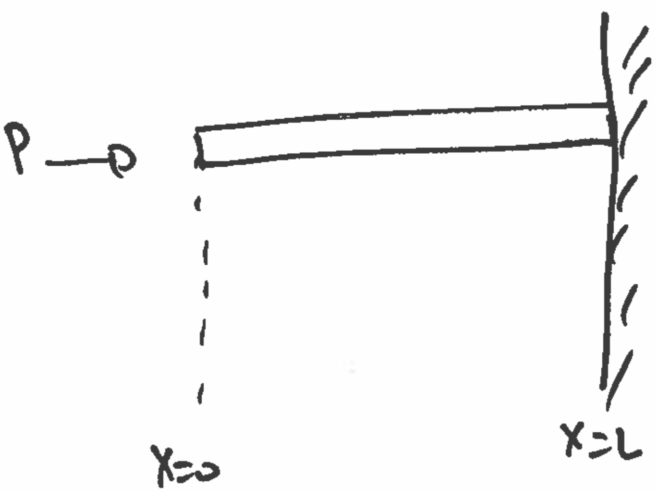
- Buckling of an elastic beam.

→ Euler - Bernoulli's beam equation.

$$y^{(4)} + \lambda y'' = 0 \quad \rightarrow \quad \lambda = \frac{P}{EI}$$

P → load.
E → Young's modulus
I → Moment of inertia.

Boundary Conditions.



free end.

$y''(0) = 0$ (no bending moment)

$y'''(0) + \lambda y'(0) = 0$ (no shear)

$y(L) = 0$ (no deflection)
$y'(L) = 0$ & no slope

$y^{(4)} + \lambda y'' = 0$

guess: $y = e^{rx}$.

Subbing into the ODE.

$r^4 e^{rx} + \lambda r^2 e^{rx} = 0$.

$r^2 (r^2 + \lambda) = 0$

$r_{1,2} = 0$ (repeated) $\qquad r_{3,4} = \pm i\mu$. $\quad (\lambda = \mu^2)$

The general solution

$y(x) = C_1 + C_2 x + C_3 \cos(\mu x) + C_4 \sin(\mu x)$.

$y'(x) = C_2 - C_3 \mu \sin(\mu x) + C_4 \mu \cos(\mu x)$.

$y''(x) = - C_3 \mu^2 \cos(\mu x) - C_4 \mu^2 \sin(\mu x)$

$y'''(x) = C_3 \mu^3 \sin(\mu x) - C_4 \mu^3 \cos(\mu x)$.

$$y''(0) = 0 \rightarrow C_3 = 0.$$

$$y'''(0) + \mu^2 y'(0) = 0 \rightarrow -C_4\mu^3 + \mu^2 C_2 + C_4\mu^3 = 0$$

$$C_2 = 0.$$

$$y(L) = 0 \rightarrow C_1 + C_4 \sin(\mu L) = 0$$

$$C_1 = -C_4 \sin(\mu L) \rightarrow C_4 = \frac{-C_1}{\sin(\mu L)}.$$

$$y'(L) = 0 \rightarrow \underbrace{C_4 \mu \cos(\mu L)}_{} = 0.$$

$$\mu L = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, 3, \ldots$$

The general solution is then, for

$$\mu_n = \frac{(2n+1)\pi}{2L} \quad \text{or} \quad \lambda_n = \mu_n^2.$$

$$y(x) = C_1 - \left(\frac{C_1}{\sin(\mu_n L)}\right) \sin(\mu_n x).$$

Lets assume $C_1 = 1$.

Eigenfunctions $\rightarrow$ $y_n(x) = 1 - \dfrac{\sin(\mu_n x)}{\sin(\mu_n L)}$

$$\mu_n = \frac{(2n+1)\pi}{2L} \quad , \quad n = 0, 1, 2, 3 \ldots$$

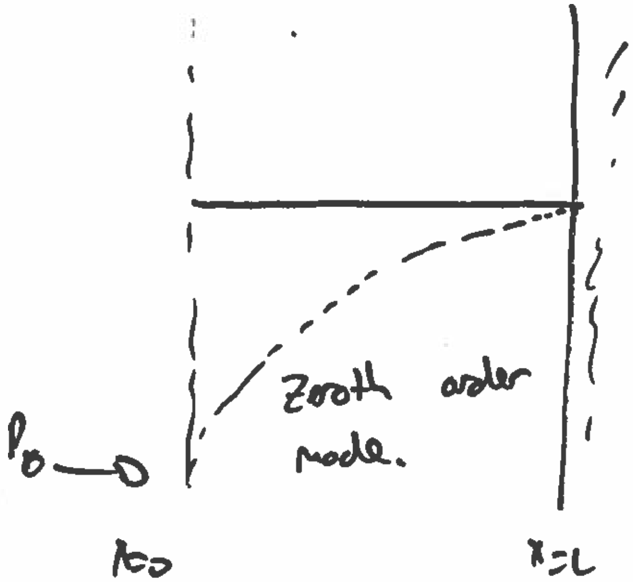
- The first mode corresponds to the smallest load that causes buckling.

$$\mu_0 = \frac{\pi}{2L} = \sqrt{\lambda_0} = \sqrt{\frac{P_0}{EI}}$$

$$\rightarrow \boxed{P_0 = EI\left(\frac{\pi}{2L}\right)^2.}$$

Shape of the buckled column for the lowest load.

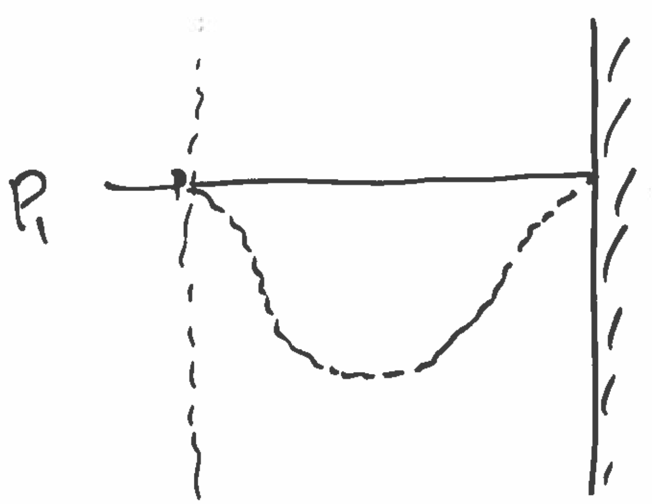
$$y_0(x) = 1 - \frac{\sin\left(\frac{\pi x}{2L}\right)}{\sin\left(\frac{\pi}{2L} \cdot L\right)} = 1 - \sin\left(\frac{\pi x}{2L}\right)$$



<u>n=1</u>

$$\mu_1 = \frac{3\pi}{2L} = \sqrt{P_1/EI} \to P_{cr1} = EI\left(\frac{3\pi}{2L}\right)^2.$$

$$y_1(x) = 1 - \frac{\sin\left(\frac{3\pi x}{2L}\right)}{\sin\left(\frac{3\pi}{2}\right)} = 1 + \sin\left(\frac{3\pi x}{2L}\right).$$

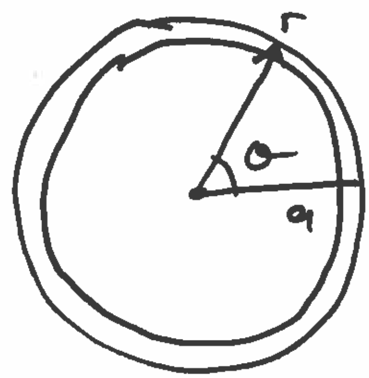


Periodic Boundary Conditions.

Consider the heat-diffusion in a very thin circular ~~circular~~ ring:

$u(r, \theta, t)$ polar coordinates.

$$\frac{du}{dt} = \alpha^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$



$X = a\theta$

Assume the ring is very thin and with no radial dependence

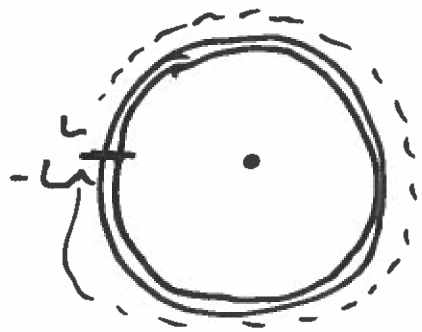
$$u(r, \theta, t) = u(\theta, t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{1}{a^2} \frac{\partial^2 u}{\partial \theta^2} \right) = \alpha^2 \left(\frac{1}{a^2} \frac{\partial^2 u}{\partial (x/a)^2} \right)$$

$$= \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

So, the periodic B.V.P can be written as.

PDE: $U_t = \alpha^2 U_{xx}$ $\quad -L < X < L$, $t > 0$.



B.C #1: $U(-L,t) = U(L,t)$

B.C #2: $U_x(-L,t) = U_x(L,t)$

eg. In relation to our problem

$u(\pi, t) = u(-\pi, t)$

$\frac{\partial u}{\partial \theta}(\pi, t) = \frac{\partial u}{\partial \theta}(-\pi, t)$

I.C $\to$ $U(x,0) = f(x)$.

PDE $\to$ $U_t = \alpha^2 U_{xx}$. $-L < x < L$, $t > 0$.

$U(x,t) = X(x) \cdot T(t)$ (separation of variables)

$U_t = X(x) \cdot \dot{T}(t)$

$U_{xx} = X''(x) \cdot T(t)$.

Subing into our PDE..

$U_t = X(x) \cdot \dot{T}(t) = \alpha^2 X''(x) \cdot T(t)$.

Re-arrange we get.

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

1) $\dot{T}(t) = \alpha^2 \lambda T(t) \to T(t) = Ce^{\alpha^2 \lambda t}$

2) $X'' - \lambda X = 0$.

1) $u(-L,t) = u(L,t) = X(-L)\cdot T(t) = X(L)\cdot T(t)$

$$X(-L) = X(L).$$

2) $u_x(-L,t) = u_x(L,t) = X'(-L)\,T(t) = X'(L)\cdot T(t)$

so, $X'(L) = X'(-L)$.

$$\left.\begin{aligned} X'' - \lambda X &= 0 \\ X(-L) &= X(L) \\ X'(-L) &= X'(L) \end{aligned}\right\}$$ An eigenvalue problem.

Evaluate eigenvalue.