Clarifying points

- For ordinary points, it is possible to find a solution using the power series expansion:
  \[ y(x_0) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x_0 - \text{ordinary point.} \]

- The radius of convergence of the solution is at least as large as the distance from the expansion point to the nearest singular point.

Ex.

\[(1-x^2)y'' - 2xy' + \alpha (\alpha + 1)y = 0.\]

\[X_{sp} = \pm 1.\]

\[y(\pm 1) = \sum_{n=0}^{\infty} a_n x^n \quad \text{will definitely work} \quad |x| < 1.\]
What you find is that two solutions arise, i.e., a recurrence relation.

If \( \alpha > 0 \), one solution terminates after a finite number of terms, and hence converges for all \( x \).

E.g., \( \alpha = 1 \), \( y_2(x) = x \).

See Anthony Peirce's notes (Lecture 3) for more examples.
Ex.  

Legendre equation: \( Q(x) = P(x) \).

\[
(1-x^2) y'' - 2x y' + \alpha(\alpha+1)y = 0.
\]

\( P(x) = 1-x^2 \geq 0 \quad x_0 = \pm 1 \)

\[
\lim_{x \to x_0} \frac{Q(x)}{P(x)}
\]

\[
\lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)}
\]

\[
\frac{1}{x_0+1} \lim_{x \to x_0} (x-1) \frac{-2x}{1-x^2} = \frac{1}{x_0+1} \lim_{x \to x_0} \frac{2x}{1+x} = 1
\]

\[
\lim_{x \to x_0+1} (x-1)^2 \frac{\alpha(\alpha+1)}{1-x^2} = \lim_{x \to x_0+1} \frac{X-1}{1+x} \alpha(\alpha+1) = 0.
\]
Series solutions near singular points

\[ P(x) y'' + Q(x) y' + R(x) y = 0. \]  \hspace{1cm} (\star)

If \( x_0 \) is a regular singular point,

ie. \( \lim_{x \to x_0} \frac{Q(x)}{P(x)} \) \( \lim_{x \to x_0} \frac{R(x)}{P(x)} \)

are finite and analytic, they have a convergent Taylor series.

\[ p(x) = (x-x_0) \frac{Q(x)}{P(x)} = p_0 + p_1 (x-x_0) + p_2 (x-x_0)^2 + \ldots \]

\[ q(x) = (x-x_0)^2 \frac{R(x)}{P(x)} = q_0 + q_1 (x-x_0) + q_2 (x-x_0)^2 + \ldots \]
The convergent Taylor/power series expansion take the form

\[
p = \sum_{n=0}^{\infty} p_n x^n \quad \bar{z} = \sum_{n=0}^{\infty} q_n x^n.
\]

As before, divide \((\ast)\) by \(P(x)\) and multiply by \((x-x_0)^2\).

\[
(x-x_0)^2 y'' + \frac{Q(x)}{P(x)} (x-x_0)^2 y' + \frac{R(x)}{P(x)} (x-x_0)^2 y = 0.
\]

\[
(x-x_0)^2 y'' + (x-x_0) \bar{p}(x) y' + \bar{q}(x_0) y = 0.
\]

This is similar to the C-E. equation

i.e. \(\alpha \rightarrow \bar{p}(x)\) , \(\bar{p} \rightarrow \bar{q}(x)\).
\[(x-x_0)^2 y'' + (x-x_0) \left[ p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \ldots \right] y' + \left[ q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \ldots \right] y = 0.\]

Now, we evaluate the BIG- vs. small terms.

Since we're interested in near our solution, let's consider the terms:

- \( p_1(x-x_0) \ll p_0 \quad \text{and} \quad q_1(x-x_0) \ll q_0 \)

**Group the terms.**

\[
\overbrace{(x-x_0)^2 y'' + (x-x_0)p_0 y' + q_0 y}^\text{BIG-} + \]

\[
\overbrace{p_1(x-x_0)^2 y' + p_2(x-x_0)^3 y' + q_1(x-x_0)y + q_2(x-x_0)^2 y}^\text{small.} = 0
\]
Now, we solve to leading order.

\[(x-x_0)^2 y'' + (x-x_0) p_0 y' + q_0 y = 0.\]

- This is the C-E. equation and has a solution of the form \( y_0(x) = (x-x_0)^n \).

- The values of \( n \) are called the exponent at the singularity and determine the qualitative behaviour of the solution near the regular singular point.

- Because of the presence of the small terms, we modify our solution \( y_0 \) by adding a particular solution.

\[ y(x) = (x-x_0)^n \sum_{n=0}^{\infty} a_n (x-x_0)^n. \]

Catch the singularity. This is a Frobenius series.
Consider the ODE

\[ P(x)y'' + Q(x)y' + R(x)y = 0. \]

If \( x_0 \) is a R.S.P., the Frobenius solution near this point has the form

\[ y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n \]

For a solution to be complete, we need

1) Find the value of \( r \).
2) Find the recursion relation for \( a_n \).
3) Find the radius of convergence of

\[ \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r} \]
Ex.

\[ 2x^2y'' - x y' + (1-x)y = 0. \]

\[ P(x) = 2x^2 \]
\[ Q(x) = -x \]
\[ R(x) = 1 - x. \]

\[ p_0 = \lim_{x \to 0} \frac{(x-x_0)Q(x)}{P(x)} = \lim_{x \to 0} \frac{-x^2}{2x^2} = -\frac{1}{2}. \]

\[ q_0 = \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2(1-x)}{2x^2} = \frac{1}{2}. \]

The corresponding C-E equation is then

\[ 2x^2y'' + p_0 x y' + q_0 y = 0 \]
\[ x^2y'' - \frac{x}{2} y' + \frac{1}{2} y = 0. \]
We know that the roots are
\[ r = 1, -\frac{1}{2}. \]

The Frobenius solution is:
\[
y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}
\]

\[
y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}
\]

\[
y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}
\]

ODE is:
\[ 2x^2y'' - xy' + (1-x)y = 0 \]

1) Sub into ODE:
\[ 2x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \]
\[ + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=2}^{\infty} a_n x^{n+r+1} = 0. \]
Need to shift the index.

\[ n+r+1 = m+r \]

\[ n = m-1. \]

Start the series now at \( m=1 \).

(pull out first term from each remaining series).

\[ 2a_0 r(r-1)x^r - a_0 r^2x^r + a_0 x^r + a_n \]

\[ + \sum_{n=1}^{\infty} \left[ 2a_n (n+r)(n+r-1) - a_n(n+r) \right] x^{n+r} = 0 \]

for:

\[ x^r : a_0 \left[ 2r(r-1) - r + 1 \right] x^r = 0. \]

\[ 2r(r-1) - r + 1 = 0 \]

\[ r = 1, \frac{1}{2}. \]

\[ x^{n+r} : a_n = \frac{a_{n-1}}{(n+r)(2(n+r) - 3) + 1} \]
Find the recursion relationship for $\gamma_{1,2}$.

$$\gamma_1 = \frac{1}{2} : \ a_n = \frac{a_{n-1}}{(n+\frac{1}{2})(2n+1-3)+1} = \frac{a_{n-1}}{n(2n-1)}$$

$$y_1(x) = a_0 x^{\frac{1}{2}} \left[ 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \ldots \right] .$$

Second solution, $\gamma_2 = 1$.

$$a_n = \frac{a_{n-1}}{(n+1) \left[ 2(n+1)-3 \right] + 1} = \frac{a_{n-1}}{n(2n+1)} .$$

$$y_2(x) = a_0 x \left[ 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \ldots \right] .$$
Check radius of convergence:

\[
\lim_{n \to \infty} \left| \frac{a_n x^n}{a_{n-1} x^{n-1}} \right| = |x| \lim_{n \to \infty} \left| \frac{1}{(n+r)(2n+r)+3} \right| = 0 \quad \therefore \quad r = \infty.
\]

The general solution:

\[y(x) = c_1 y_1(x) + c_2 y_2(x)\]

\[= c_1 x^{\frac{1}{2}} \left[ 1 + x + \frac{x^2}{60} + \frac{x^3}{90} + \ldots \right] + c_2 x \left[ 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \ldots \right]\]

\[= c_1 x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n + c_2 x^{\frac{3}{2}} \sum_{n=0}^{\infty} b_n x^n.\]
Another method.

$ly = 2x^2 y'' - xy' + (1 - x)y = 0.$

$P(x_0) = 0$ for $x_0 = 0$.

As $x \to 0$, we see that $(1 - x) \to 0$ and $y \to xy$.

So,

$2x^2 y'' - xy' + y - xy' = 0$.

This is an approximate equation.

Choose: $y = (x - x_0)^r = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$.
Subbing in we get:

\[ 2r(r-1) - r + 1 \] \[ x^r = 0 \]

\[ r = \frac{1}{2}, 1. \]

\[ y_0 = c_1 x^{\frac{1}{2}} + c_2 x \]

Now, we need to correct for the forcing term (i.e. \( xy \)) that we removed.

\[ 2x^2 y'' - xy' + y = x y_0 \quad (\#) \]

\[ = c_1 x^{3/2} + c_2 x^2 \]

Let's guess a particular solution:

\[ y_p = A_0 x^{3/2} + B_0 x^2 \]

\[ y'_p = \frac{3}{2} A_0 x^{1/2} + 2 B_0 x \]

\[ y''_p = \frac{3}{4} A_0 x^{-1/2} + 2 B_0. \]
Subbing into the RHS we get.

\[
\frac{3}{2} A_0 x^{3/2} + 3 B_0 x^2 - \frac{3}{2} A_0 x^{3/2} - 2 B_0 x^2 + A_0 x^{3/2} + B_0 x^2 = C x^{3/2} + C_2 x^2
\]

Collecting the terms we get.

\[
A_0 = C_1 \\
B_0 = \frac{C_2}{3}.
\]

Hence, \( y_1(x) = C_1 x^{3/2} (1 + x) + C_2 x (1 + \frac{x}{3}) \)

We can repeat this process by subbing in \( y_1(x) \) for \( y_0(x) \) in \((2)\) to get the higher order terms.