Clarifying points

- For ordinary points, it is possible to find a solution using the power series expansion:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad x_0 - \text{ordinary point.}$$

- The radius of convergence of a solution is at least as large as the distance from the expansion point to the nearest singular point.

Ex.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

$x_{sp} = \pm 1.$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$ will definitely work $|x| < 1.$

- What you find is that two solutions arise, i.e. 2 recurrence relations.

If $\alpha > 0$, <u>one</u> solution terminates after a finite number of terms, and hence converges for all $\underline{x}$.

e.g. $\alpha = 1$, $y_2(x) = x$.

- See Anthony Peirce's notes (Lecture 3) for more examples.

Ex.

Legendre equation.

$$(1-x^2)y'' \underbrace{-2x}_{Q(x)}y' + \underbrace{\alpha(\alpha+1)}_{R(x)}y = 0.$$

$P(x) = 1-x^2 = 0 \quad x_0 = \pm 1$

$$\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)}$$

$$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}$$

$x_0 = 1$

$$\lim_{x \to +1} (x-1)\left(\frac{-2x}{1-x^2}\right) = \lim_{x \to +1}\left(\frac{2x}{1+x}\right) = 1$$

$$\lim_{x \to +1} (x-1)^2\left(\frac{\alpha(\alpha+1)}{1-x^2}\right) = \lim_{x \to +1}\left(\frac{x-1}{1+x}\right)\alpha(\alpha+1) = 0.$$

Series solutions near singular points

$$P(x)y'' + Q(x)y' + R(x)y = 0. \quad (*)$$

If x_0 is a regular singular point,

ie. $\lim_{x \to x_0} (x-x_0)\dfrac{Q(x)}{P(x)}$ & $\lim_{x \to x_0} (x-x_0)^2 \dfrac{R(x)}{P(x)}$

are finite, and analytic, they have a convergent Taylor series.

$$\bar{p}(x) = (x-x_0)\frac{Q(x)}{P(x)} = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \dots$$

$$\bar{q}(x) = (x-x_0)^2 \frac{R(x)}{P(x)} = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \dots$$

The convergent Taylor / power series expansions take the form

$$\bar{p} = \sum_{n=0}^{\infty} p_n x^n \quad \& \quad \bar{q} = \sum_{n=0}^{\infty} q_n x^n.$$

As before, divide (*) by $P(x)$ and multiply by $(x-x_0)^2$.

$$(x-x_0)^2 y'' + \frac{Q(x)}{P(x)}(x-x_0)^2 y' + \frac{R(x)}{P(x)}(x-x_0)^2 y = 0.$$

$$(x-x_0)^2 y'' + (x-x_0)\bar{p}(x) y' + \bar{q}(x) y = 0.$$

- This is similar to the C-E. equation

i.e. $\alpha \to \bar{p}(x)$, $\beta \to \bar{q}(x)$.

$$(x-x_0)^2 y'' + (x-x_0)\left[p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \dots\right]y'$$

$$+ \left[q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \dots\right]y = 0.$$

Now, we evaluate the BIG vs. small terms.

→ Since we're interested in solutions near our R.S.P x_0,

$$P_1(x-x_0) < P_0 \quad ; \quad q_1(x-x_0) < q_0$$

- Group the terms.

$$\overbrace{(x-x_0)^2 y'' + (x-x_0)p_0 y' + q_0 y}^{\text{BIG}} +$$

$$p_1(x-x_0)^2 y' + p_2(x-x_0)^3 y' + q_1(x-x_0)y + q_2(x-x_0)^2 y + \dots = 0$$

small.

Now, we solve to leading order.

$$(x-x_0)^2 y_0'' + (x-x_0) p_0 y_0' + q_0 y_0 = 0.$$

- This is the C-E. equation and has a solution of the form $y_0(x) = (x-x_0)^r$

- The values of 'r' are called the exponents at the singularity and determine the qualitative behaviour of the solution near the regular singular point.

- Because of the ~~presence~~ presence of the small terms, we modify our solution y_0 by adding a particular solution.

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

correction.

cateher the singularity.

This is a Frobenius series.

Consider the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

- If x_0 is a R.S.P, the Frobenius solution near this point has the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

For a solution to be complete, we need.

1) Find the values of 'r'

2) Find the recurssion relation for 'n'.

3) Find the radius of convergence of

$$\sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$$

Ex.

$$2x^2y'' - xy' + (1-x)y = 0.$$

R.S.P $\Rightarrow x_0 = 0$

$P(x) = 2x^2$

$Q(x) = -x$

$R(x) = 1-x.$

$$p_0 = \lim_{x\to 0} (x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} \frac{-x^2}{2x^2} = -1/2.$$

$$q_0 = \lim_{x\to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x\to 0} \frac{x^2(1-x)}{2x^2} = \frac{1}{2}$$

The corresponding C-E equation is then

$$x^2y'' + p_0 x y' + q_0 y = 0$$

$$x^2y'' - \frac{x}{2}y' + \frac{y}{2} = 0.$$

we know that the roots are

$r = 1, \frac{1}{2}$.

The ~~the~~ Frobenius solution is:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$ starts at $n=0$ since $r \neq 0$.

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

ODE is $\rightarrow$ $2x^2 y'' - x y' + (1-x) y = 0$

1) sub into ODE.

$$2x^2 y'' - x y' + (1-x) y = 0$$

$$2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

Need to shift the index.

$$n+r+1 = m+r$$

$$n = m-1.$$

Start the series now at $m=1$

(pull out first term from each remaining series).

$$2a_0 r(r-1)x^r - a_0 r x^r + a_0 x^r$$

$$+ \sum_{n=1}^{\infty} \left[2a_n(n+r)(n+r-1) - a_n(n+r) \overset{+a_n}{\cancel{-}} a_{n-1} \right] x^{n+r} = 0$$

$$x^r : \quad a_0 \left[2r(r-1) - r + 1 \right] x^r = 0.$$

$$2r(r-1) - r + 1 = 0$$

$$r = 1, 1/2.$$

$$\underset{n \geq 1}{x^{n+r}} : \quad a_n = \frac{a_{n-1}}{(n+r)(2(n+r)-3)+1}$$

Find the recursion relationship for $r_{1,2}$.

$$r_1 = \tfrac{1}{2} \; : \; a_n = \frac{a_{n-1}}{(n+\tfrac{1}{2})(2n+1-3)+1} = \frac{a_{n-1}}{n(2n-1)}$$

$$y_1(x) = a_0 x^{1/2}\left[1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \cdots\right].$$

Second solution, $r_2 = 1$.

$$a_n = \frac{a_{n-1}}{(n+1)\left[2(n+1)-3\right]+1} = \frac{a_{n-1}}{n(2n+1)}.$$

$$y_2(x) = a_0 x\left[1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \cdots\right].$$

Check radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_n x^n}{a_{n-1}x^{n-1}}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{(n+r)((2n+r)+3)+1}\right|$$

$$= 0 \qquad \rho = \infty.$$

The general solution.

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 x^{1/2}\left[1 + x + \frac{x^2}{60} + \frac{x^3}{90} + \cdots\right]$$

$$+ c_2 x\left[1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \cdots\right]$$

$$= c_1 x^{r_1}\sum_{n=0}^{\infty} a_n x^n + c_2 x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$

Another method.

$$Ly = 2x^2y'' - xy' + (1-x)y = 0.$$

$P(x_0) = 0$ for $x_0 = 0$.

As $x \to 0$ we see that $(1-x) \to 1$

and $y \gg xy$

So,

$$2x^2y'' - xy' + y - \underbrace{xy'}_{\to 0} = 0$$

This is an approximate equation.

Choosing : $y = (x-x_0)^r = x^r$

$$y' = r x^{r-1}$$

$$y'' = r(r-1)\, x^{r-2}.$$

Subing in we get.

$$\left[2r(r-1) - r + 1\right]x^r = 0$$

$$r = 1/2, 1.$$

$$y_0 = c_1 x^{1/2} + c_2 x^1$$

Now, we need to correct for the forcing term (i.e. xy) that we removed.

$$2x^2y'' - xy' + y = xy_0 \quad (*)$$

$$= c_1 x^{3/2} + c_2 x^2$$

lets guess a particular solution.

$$y_p = A_0 x^{3/2} + B_0 x^2$$

$$y_p' = \frac{3}{2} A_0 x^{1/2} + 2B_0 x$$

$$y_p'' = \frac{3}{4} A_0 x^{-1/2} + 2B_0.$$

Subing into the RHS we get.

$$\frac{3}{2}A_0x^{3/2} + 4B_0x^2 - \frac{3}{2}A_0x^{3/2} - 2B_0x^2 + A_0x^{3/2} + B_0x^2 =$$

$$C_1x^{3/2} + C_2x^2$$

Collecting the terms we get.

$$A_0 = C_1$$

$$B_0 = \frac{C_2}{3}.$$

Hence, $y_1(x) = C_1x^{1/2}(1+x) + C_2x\left(1+\frac{x}{3}\right)$

We can repeat this process by subing in $y_1(x)$ for $y_0(x)$ in (*) to get the higher order terms.