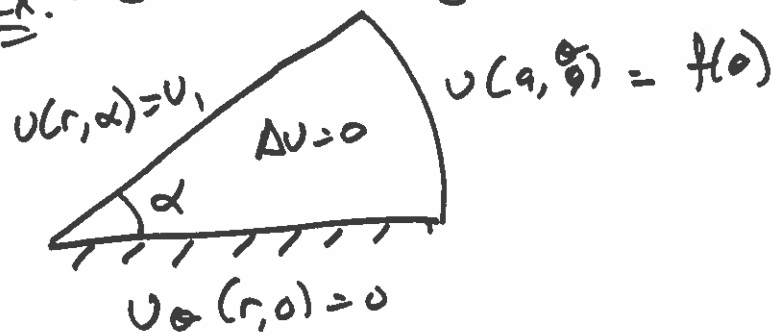
More examples on Laplace's equation on Circular domain

Ex. #2 A wedge with an inhomogeneous B.C.



First remove the inhomogeneous B.C.

Let : $u(r,\theta)=w(\theta)+v(r,\theta)$

We need $w(\theta)$ to satisfy 2 B.C's, so we choose.

$w(\theta) = A\theta + B.$

$w'(0) = A = 0$ & $w(\alpha) = U_1 = B.$

So, $w(\theta) = U_1$

~~Key~~ Now Find the B.V.P for $v(r,\theta)$.

$$U_{rr}+\frac{1}{r}U_r+\frac{1}{r^2}U_{\theta\theta} = \underbrace{\left(w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}\right)}_{=0.}+$$

$$\left(V_{rr}+\frac{1}{r}V_r+\frac{1}{r^2}V_{\theta\theta}\right)=0$$

Hence,

$$\Delta V = 0.$$

B.C's:

#1 $V_\theta(r,0) = W_\theta(0) + V_\theta(r,0) = 0$

$$= 0 + V_\theta(r,0) = 0.$$

#2 $U(r,\alpha) = w(\alpha) + V(r,\alpha) = U_1$

$$= U_1 + V(r,\alpha) = V_1$$

$$V(r,\alpha) = 0.$$

#3. $U(a,\theta) = w(\theta) + V(a,\theta)$

$$= U_1 + V(a,\theta) = f(\theta).$$

$$V(a,\theta) = f(\theta) - U_1$$

Now, we can solve the D.V.D.

$$\begin{cases} \Delta V = 0 \\ V_\theta(r,0) = 0 = V(r,\alpha) \\ V(a,\theta) = f(\theta) - U_1 \end{cases}$$

We know that this is a mixed type 2.

$$\mu_n = \frac{(2n-1)\pi}{2\alpha}, \quad n = 1, 2, 2..$$

$$\Theta_n = \cos(\mu_n \theta).$$

Recall that $u(r,\theta)$ is finite as $r \to 0$, so $a_n = 0$.

$$V(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\mu_n} \cos(\mu_n \theta)$$

To find A_n,

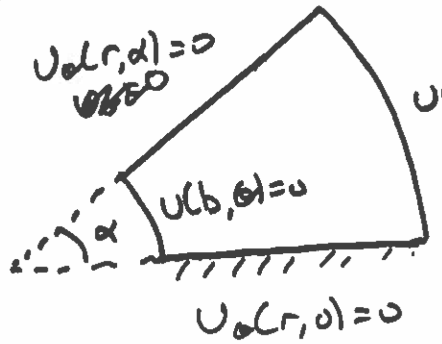
$$f(\theta) - u_1 = v(a, \theta) = \sum_{n=1}^{\infty} \underbrace{A_n a^{\mu_n}}_{a_n^{f-u_1}} \cos(\mu_n \theta).$$

$$A_n a^{\mu_n} = a_n^{f-u_1} = \frac{2}{\alpha} \int_0^{\alpha} \left\{ f(\theta) - u_1 \right\} \cos(\mu_n \theta)\, d\theta.$$

$$u(r,\theta) = w(\theta) + v(r,\theta).$$

$$= u_1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{\mu_n} a_n^{f-u_1} \cos(\mu_n \theta).$$

Ex. #3. A circular wedge with a cut-out.



4 B.C's

$$\hookrightarrow \begin{cases} u_\theta(r,0)=0 \\ u_\theta(r,\alpha)=0 \\ u(b,\theta)=0 \\ u(a,\theta)=f(\theta). \end{cases}$$

Homogeneous problem with a forcing boundary.

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Let: $u(r,\theta) = R(r)\cdot\Theta(\theta)$. ← Separation of Variables

$$R''\cdot\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Multiply by $r^2R^{-1}\Theta^{-1}$ yields.

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0.$$

So,

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \mu^2$$

An eigenvalue problem in Θ.

$\Theta]$ $\quad \Theta'' + \mu^2 \Theta = 0. \rightarrow \Theta = A\cos(\mu\theta) + B\sin(\mu\theta)$

B.C $\rightarrow$ $\Theta'(0) = \Theta'(\alpha) = 0$

$$\Theta' = -A\mu \sin(\mu\theta) + \cancel{B}\, B\mu \cos(\mu\theta).$$

$\Theta'(0) \rightarrow \quad 0 = B$

$\Theta'(\alpha) \rightarrow \quad 0 = -A\mu \sin(\mu\alpha). \longrightarrow \mu_n \in \left\{0, \frac{n\pi}{\alpha}\right\}_{n=1,2\ldots}$

$$\Theta_n = \cos\left(\frac{n\pi\theta}{\alpha}\right)$$

$R]$ $\quad r^2 R'' + rR' - \mu^2 R = 0.$

$\mu = 0: \quad R_0(r) = C_0 + D_0 \ln(r) \leftarrow$ Recall : $R = r^\gamma$

$\mu \neq 0: \quad R_n(r) = C r^{\mu_n} + D r^{-\mu_n} \leftarrow$ Recall : $R = r^\gamma$

The general solution is:

$$u(r,\theta) = R_0(r)\cos(0) + \sum_{n=1}^{\infty} R_n(r)\cos(\mu_n\theta)$$

$$= A_0 + \alpha_0 \ln(r) + \sum_{n=1}^{\infty} \left\{ A_n r^{\mu_n} + \alpha_n r^{-\mu_n} \right\} \cos(\mu_n\theta).$$

To find the unknown coefficients, we apply the B.C.'s.

#1 $u(b,\theta)=0$.

Both ~~separate~~ solutions ($\mu=0$; $\mu\neq 0$) need to satisfy this condition.

$$A_0+\alpha_0\ln(b)=0.\Rightarrow A_0=-\alpha_0\ln(b)$$

$$\left\{A_n b^{\mu_n}+\alpha_n b^{-\mu_n}\right\}\cos(\mu_n\theta)=0 \Rightarrow \alpha_n=-A_n b^{2\mu_n}$$

So now we can write the solution as.

$$u(r,\theta)=\alpha_0\ln\left(\frac{r}{b}\right)+\sum_{n=1}^{\infty}A_n\left\{r^{\mu_n}-b^{2\mu_n}\cdot r^{-\mu_n}\right\}\cos(\mu_n\theta).$$

This should satisfy: $u(a,\theta)=f(\theta)$.

$$u(a,\theta)=f(\theta)=\underbrace{\alpha_0\ln\left(\frac{a}{b}\right)}_{\frac{a_0^f}{2}}+\sum_{n=1}^{\infty}\underbrace{A_n\left\{a^{\mu_n}-b^{2\mu_n}\cdot a^{-\mu_n}\right\}}_{a_n^f}\cos(\mu_n\theta).$$

This is a ~~[illegible]~~ Fourier cosine expansion of $f(\theta)$.

$$\Rightarrow \alpha_0 \ln\left(\frac{a}{b}\right) = \frac{a_0^f}{2} = \left(\frac{1}{2}\right)\frac{2}{\alpha}\int_0^\alpha f(\theta)\,d\theta$$

$$\alpha_0 = \frac{a_0^f}{2\ln(a/b)} = \frac{1}{(\alpha)\ln(a/b)}\int_0^\alpha f(\theta)\,d\theta$$

<u>Expansion term</u>

$$a_n^f = A_n\left\{a^{\mu_n} - b^{2\mu_n}\cdot a^{-\mu_n}\right\} = \frac{2}{\alpha}\int_0^\alpha f(\theta)\cos(\mu_n\theta)\,d\theta.$$

$$A_n = \frac{2}{\alpha\left\{a^{\mu_n} - b^{2\mu_n}\cdot a^{-\mu_n}\right\}}\int_0^\alpha f(\theta)\cos(\mu_n\theta)\,d\theta$$

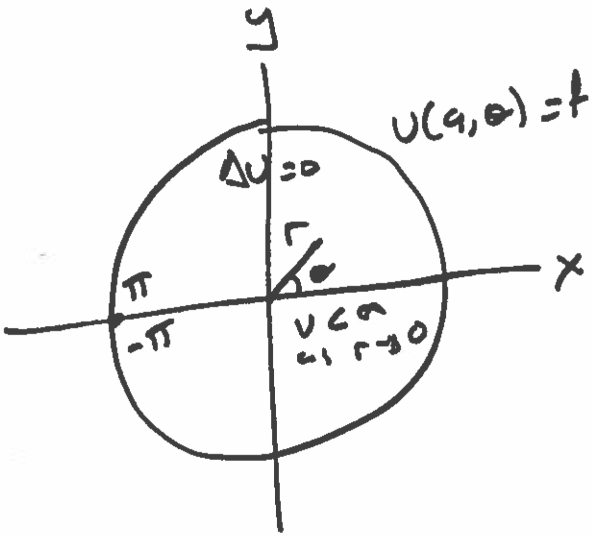
So now we can write,

$$u(r,\theta) = \frac{a_0^f}{2\ln(a/b)}\cdot\ln(r/b) + \sum_{n=1}^{\infty} A_n\left\{\left(\frac{r}{b}\right)^{\mu_n} - \left(\frac{b}{r}\right)^{\mu_n}\right\}\cos(\mu_n\theta)$$

Note: You can also write A_n as

$$A_n = \frac{a_n^f}{\left(\frac{a}{b}\right)^{\mu_n} - \left(\frac{b}{a}\right)^{\mu_n}}$$

Example # 4. Dirichlet problem in the interior of a circle.



$0<r<a$

$-\pi<\theta<\pi$.

Periodic $\to$ $u(r,\pi)=u(r,-\pi)$

$u_\theta(r,\pi)=u_\theta(r,-\pi)$

$$\Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

Separation of variables: $u(r,\theta)=R(r)\cdot\Theta(\theta)$

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\mu^2.$$

$\Theta]$ $\left.\begin{aligned}&\Theta''+\mu^2\Theta=0\\&\Theta(\pi)=\Theta(-\pi)\\&\Theta'(\pi)=\Theta'(-\pi)\end{aligned}\right\}$ $\mu_n\in\left\{0,\frac{n\pi}{\pi}\right\}$

$\Theta_n\in\{1,\cos(n\theta),\sin(n,\theta)\}$

This will lead us to a full range Fourier series.

R] $r^2R'' + rR' - \mu^2 R = 0.$

Again, guess $R = r^{\gamma}$

$\mu = 0$: $R_0(r) = A_0 + B_0 \ln(r)$

$\mu \neq 0$: $R_n(r) = \overset{C_N}{\cancel{A_n}}\, r^{\mu_n} + \overset{D_N}{\cancel{B_n}}\, r^{-\mu_n}$

However, we know that $U < \infty$ as $r \to 0$, ie. finite

So B_0 & $\overset{D_N}{\cancel{B_n}}$ have to be zero.

Therefore, the general solution is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

Note the change in constants.

Now, apply the B.C. $f(\theta) = u(a,\theta)$ to find

$A_0, A_n, B_n.$

$$f(\theta) = U(a,\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

$$= \frac{a_0^f}{2} + \sum_{n=1}^{\infty} a_n^f \cos(n\theta) + b_n^f \sin(n\theta)$$

So, $A_0 = \frac{a_0^f}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$.

$$a_n^f = a^n A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta$$

or $A_n = \dfrac{a_n^f}{a^n}$

$$b_n^f = a^n B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta$$

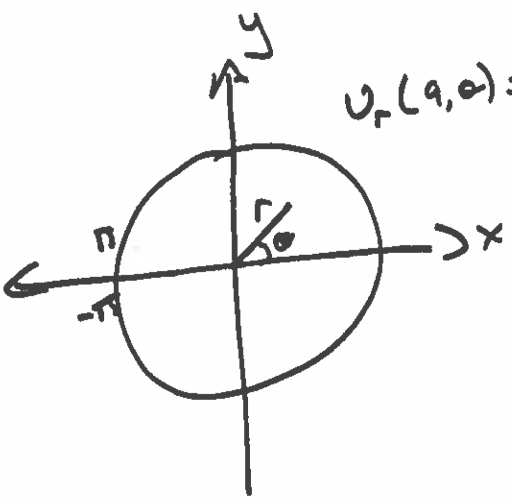
or

$$B_n = \frac{b_n^f}{a^n}$$

$$U(r,\theta) = \frac{a_0^f}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left\{ a_n^f \cos(n\theta) + b_n^f \sin(n\theta) \right\}$$

Show matlab plot for $f(\theta) = \sin(2\theta)$. (membrane deflection)

Ex #5. Neumann problem in the interior of a circle.



$u_r(a,\theta) = f(\theta)$

$0 < r < a$

$-\pi < \theta < \pi.$

Periodic $\rightarrow$ $u(r,\pi) = u(r,-\pi)$

$u_\theta(r,\pi) = u_\theta(r,-\pi).$

Finite u as $r \rightarrow 0.$

We can simply take the general solution from the last case:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

To find the coefficients, we use $u_r(a,\theta) = f(\theta).$

$$u_r(r,\theta) = \sum_{n=1}^{\infty} n r^{n-1} \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

$$u_r(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} n a^{n-1} \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

$$a_n^f = n a^{n-1} A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cdot\cos(n\alpha)\,d\alpha.$$

$\underline{\text{or}}$ $A_n = \dfrac{a_n^f}{n a^{n-1}}$ (Note that $a_0^f = 0$) since $\int_{-\pi}^{\pi} f(\alpha)\,d\alpha = 0$ for steady-state solutions

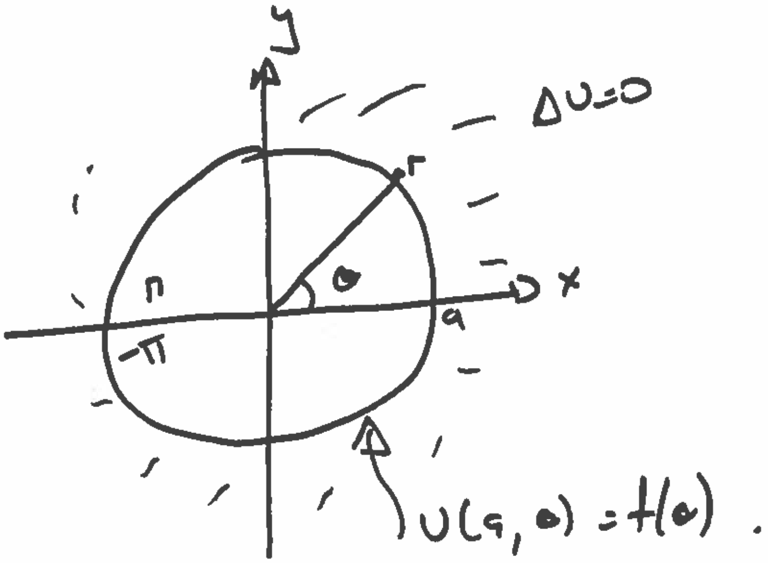
$$b_n^f = n a^{n-1} B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cdot\sin(n\alpha)\,d\alpha.$$

$\underline{\text{or}}$ $B_n = \dfrac{b_n^f}{n a^{n-1}}$

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} \frac{a}{n}\left(\frac{r}{a}\right)^n \left\{ a_n^f \cos(n\theta) + b_n^f \sin(n\theta) \right\}$$

This problem is known up to an arbitrary constant.

Ex. #6. Dirichlet problem on domain exterior to a circle.



u is finite as $r \to \infty$.

$a < r < \infty$

$-\pi < \theta < \pi$.

Periodic $\to$ $u(r, \pi) = u(r, -\pi)$

$u_\theta(r, \pi) = u_\theta(r, -\pi)$.

$$\Delta U = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0.$$

Separation of variables gives:

$$\Theta] \quad \left.\begin{aligned} &\Theta'' + \mu^2 \Theta = 0 \\ &\Theta(\pi) = \Theta(-\pi) \\ &\Theta'(\pi) = \Theta'(-\pi) \end{aligned}\right\} \quad \begin{aligned} &\mu_n \in \{0, n\} \\ &\Theta_n \in \{1, \cos(n\theta), \sin(n\theta)\} \end{aligned}$$

R] $r^2 R'' + rR' - \mu^2 R = 0.$ Guess $R(r) = r^\gamma$

$\mu = 0$: $R_0(r) = A_0 + B_0 \ln(r)$

$\mu \neq 0$: $R_n(r) = C r^{\mu_n} + D r^{-\mu_n}$

we know that as $r \to \infty$, u must be finite

hence, $B_0 = C = 0$.

Solution takes the form.

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^{-n} \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

We ~~so~~ use the B.C. $u(a, \theta) = f(\theta)$ to find the coefficients

$$f(\theta) = u(a, \theta) = A_0 + \sum_{n=1}^{\infty} a^{-n} \left\{ A_n \cos(n\theta) + B_n \sin(n\theta) \right\}$$

$$= \frac{a_0^f}{2} + \sum_{n=1}^{\infty} a_n^f \cos(n\theta) + b_n^f \sin(n\theta)$$

So,

$$A_0 = \frac{a_0^f}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta.$$

$$a_n^f = a^{-n} A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta.$$

or// $a\, A_n = a_n^f a^n$

$$b_n^f = a^{-n} B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta.$$

or// $B_n = b_n^f a^n$

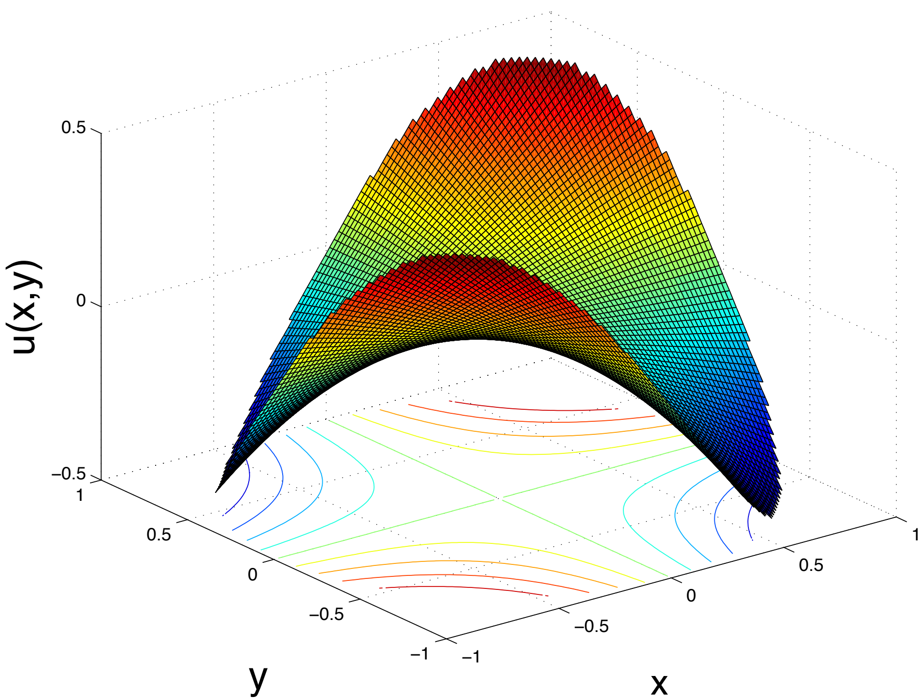
Hence,

$$u(r,\theta) = \frac{a_0^f}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{-n}\left\{a_n^f\cos(n\theta) + b_n^f\sin(n\theta)\right\}$$

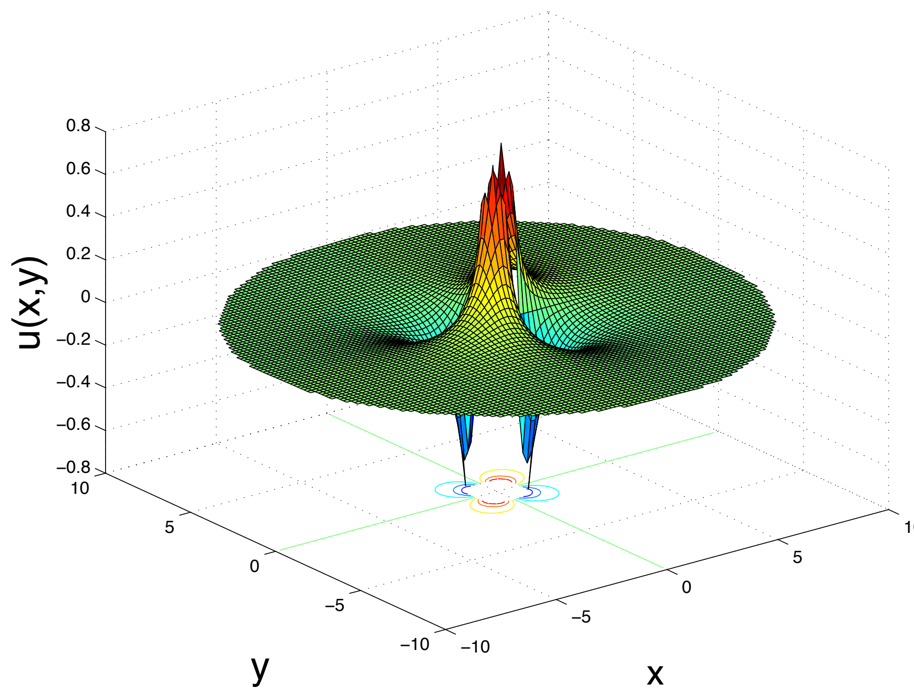
Show plot for

$$f(\theta) = \sin(2\theta).$$

Lecture 29



0.5
0
−0.5
u(x,y)
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
x



0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
u(x,y)
10
5
0
−5
−10
y
−10
−5
0
5
10
x