The 1D wave equation.

$$U_{tt} = c^2 U_{xx}$$

- Recall that we need two initial conditions and two B.C's.

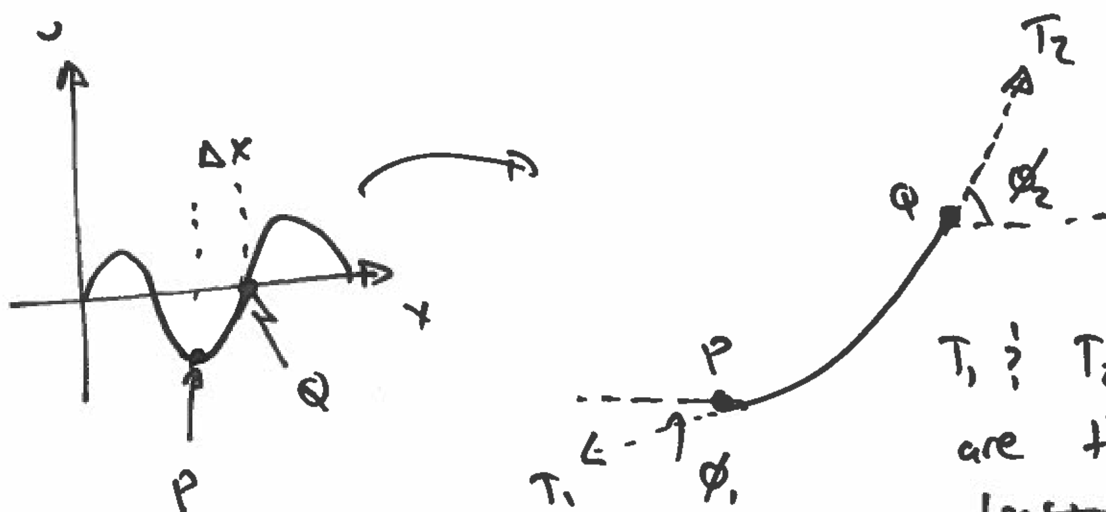
Eg.

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{ I.C.}$$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{ B.C.}$$

Physical model.

1) Vibrating string.



T_1 & T_2 are the ~~tensions~~ tensions at point P & Q.

No motion in the horizontal direction.

$$T_1 \cos\phi_1 = T_2 \cos\phi_2 = T$$

<u>Newton's second law.</u>

$$T_2 \sin\phi_2 - T_1 \sin\phi_1 = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$\rho = \frac{kg}{m}$ (ignoring in plane length / height).

Re-arranging, we get.

$$\tan\phi_2 - \tan\phi_1 = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

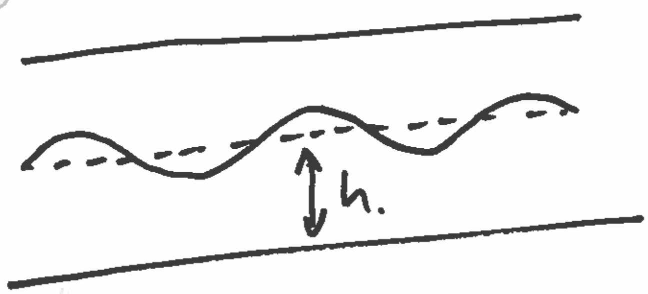
$$\tan\phi_2 = \left.\frac{\partial u}{\partial x}\right|_{x}$$

$$\tan\phi_1 = \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}$$

$$\frac{1}{\Delta x}\left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x} \right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

as $\Delta x \to 0$ $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \qquad c = \sqrt{T/\rho}$$

Shallow water waves



$h \sim 10^3 m.$

$g \sim 10 m/s^2$

Here $\quad c \approx \sqrt{10^4} = 100 m/s.$

The 1D wave equation can be decomposed into a right and left ~~trav~~ moving wave:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right] u = 0$$

Remember that we can look for an exponential solution:

$$u(x,t) = e^{ikx + \sigma t}$$

Differentiate and sub into wave equation

$$\sigma^2 e^{ikx+\sigma t} = -k^2c^2 e^{ikx+\sigma t}.$$

$$\sigma^2 = -k^2c^2. \quad \underline{\text{or}} \quad \sigma = \pm ikc$$

← dispersion relation

Our guess solution will have the form.

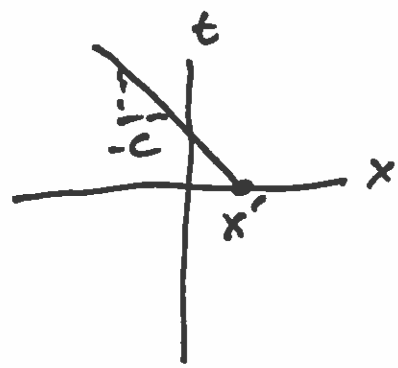
$$u(x,t) = e^{ik(x\pm ct)}.$$

Galilean transform.

- By transforming $x' = x \pm ct$, this transforms to the left, the frame of reference moving to the left, or to the right at speed c.

- If the wave is traveling to the left

$x' = x + ct.$



• $u(x,t) = G(x+ct)$. ← left moving wave.

$U_t = cG'$ $\quad U_x = G'$

$U_{tt} = c^2 G''$ $\quad U_{xx} = G''$

Hence, we get

$$U_{tt} - c^2 U_{xx} = c^2 G'' - c^2 G'' = 0.$$

D'Alembert solution.

$$U_{tt} = c^2 U_{xx}.$$

Subject to: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

• Because the wave equation is linear, the general solution is a linear combination of the left and right moving ~~waves~~ waves

$$u(x,t) = F(x-ct) + G(x+ct).$$

Apply the IC's.

$\forall$ $f(x) = u(x,0) = F(x) + G(x)$.

$g(x) = u_t(x,0) = -cF'(x) + cG'(x)$.

$$\rightarrow \int_0^x -cF'(x) + cG'(x)\, \partial x = \int_0^x g(s)\,ds + A.$$

$$\hookrightarrow \quad -cF(x) + cG(x) = \int_0^x g(s)\,ds + A. \quad ①$$

$\forall \rightarrow \quad cF(x) + cG(x) = cf(x). \quad ②$

$$① + ② \rightarrow \quad G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{A}{2c}$$

$$② - ① \rightarrow \quad F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{A}{2c}.$$

$\therefore$ The & time dependent solution is.

$$u(x,t) = F(x-ct) + G(x+ct).$$

$$u(x,t) = \frac{1}{2}\left\{ f(x-ct) + f(x+ct) \right\} + \frac{1}{2c}\left\{ \int_{x-ct}^{0} g(s)\, ds + \int_{0}^{x+ct} g(s)\, ds \right\} + \frac{A}{2c} - \frac{A}{2c}.$$

or

$$\boxed{u(x,t) = \frac{1}{2}\left\{ f(x-ct) + f(x+ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) \cdot ds.}$$

Left / right waves from initial displacement.

depends on initial velocity.

Solving the wave equation using seperation of variables.

- Consider the vibrating ~~string~~ string. problem

$$U_{tt} = c^2 U_{xx} \qquad 0 < x < L.$$

B.C: $U(0,t) = U(L,t) = 0.$

I.C: $U(x,0) = f(x)$; $U_t(x,0) = g(x).$

$$U(x,t) = X(x) \cdot T(t).$$

Subing in, we get

$$X(x) \cdot \ddot{T}(t) = c^2 X''(x) \cdot T(t)$$

$$\frac{\ddot{T}(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\mu^2 \neq 0.$$

X] $X'' + \mu^2 X = 0. \rightarrow X(x) = A\cos(\mu x) + B\sin(\mu x).$

$X(0) = X(L) = 0.$

$X(0) = 0 = A.$

$X(L) = 0 \rightarrow B\sin(\mu L) = 0$

$\mu_n = \frac{n\pi}{L} \quad X_n = \sin\left(\frac{n\pi x}{L}\right)$

T] $\ddot{T} + \mu^2 c^2 T = 0.$

$T(t) = A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t)$

Our solution is then:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) \right\} \sin(\mu_n x).$$

To apply the I.C, we take the time derivative

$$u_t(x,t) = \sum_{n=1}^{\infty} \left\{ -A_n \mu_n c \sin(\mu_n c t) + B_n \mu_n c \cos(\mu_n c t) \right\} \cdot$$

$\sin(\mu_n x).$

I.C #1

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\mu_n x).$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin(\mu_n x)\, dx = b_n^f$$

IC #2.

$$U_t(x,0) = \sum_{n=1}^{\infty} B_n \mu_n c \cdot \sin(\mu_n x) = g(x).$$

$$B_n = \frac{2}{\mu_n c L} \int_0^L g(x) \cdot \sin(\mu_n x)\, dx.$$

$$= \frac{b_n^g}{\mu_n c}$$

So, the solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ b_n^f \cos(\mu_n c t) + \frac{b_n^g}{\mu_n c} \sin(\mu_n c t) \right\} \sin(\mu_n x).$$

- How does this compare to the D'Alembert's solution.

① $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A.$

② $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

③ $\sin A \cos B = \frac{1}{2}\left\{ \sin(A+B) + \sin(A-B) \right\}$

④ $\sin A \sin B = \frac{1}{2}\left\{ \cos(A-B) - \cos(A+B) \right\}.$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{b_n^f}{2} \left\{ \sin\left[\mu_n(x+ct)\right] + \sin\left[\mu_n(x-ct)\right] \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^g}{2\mu_n c} \left\{ \cos\left[\mu_n(x-ct)\right] - \cos\left[\mu_n(x+ct)\right] \right\}$$

Calling the two sums u^f and u^g.

$$u(x,t) = u^f + u^g$$

Also, recall that at $t=0$, we found that the IC's are.

$$f^o(x) = \sum_{n=1}^{\infty} b_n^f \sin(\mu_n x). \leftarrow$$ odd extension of $f(x)$.

$$g^o(x) = \sum_{n=1}^{\infty} b_n^g \sin(\mu_n x). \leftarrow$$ odd extension of $g(x)$.

So,

$$u^f = \sum_{n=1}^{\infty} \frac{b_n^f}{2} \left[\sin\left(\mu_n (x+ct)\right) + \sin\left(\mu_n (x-ct)\right)\right]$$

$$= \frac{1}{2}\left\{ f^o(x+ct) + f^o(x-ct) \right\}$$

From (**),

$$g^o(x) = \sum_{n=1}^{\infty} b_n^g \sin(\mu_n x).$$

$$\int_0^x g^o(s)\,ds = \sum_{n=1}^{\infty} \int b_n^g \sin(\mu_n x)\,dx.$$

$$\int_0^x g(s)\,ds = \sum_{n=1}^{\infty} \frac{-b_n^g}{\mu_n} \cos(\mu_n x) + A.$$

$$u^g = \sum_{n=1}^{\infty} \frac{b_n^g}{2\mu_n c} \cos(\mu_n(x-ct)) - \sum_{n=1}^{\infty} \frac{b_n^g}{2\mu_n c} \cos(\mu_n(x+ct)).$$

$$= \frac{1}{2c}\int_{x-ct}^{0} g^o(s)\,ds + \cancel{A} + \frac{1}{2c}\int_0^{x+ct} g^o(s)\,ds - \cancel{A}$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g^o(s)\,ds.$$

$$u(x,t) = u^f + u^g$$

$$= \sum_{n=1}^{\infty} \left\{ b_n^f \cos(\mu_n ct) + \frac{b_n^g}{\mu_n c} \sin(\mu_n ct) \right\} \sin(\mu_n x)$$

$$= \frac{1}{2}\left\{ f^o(x+ct) + f^o(x-ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^o(s)\,ds.$$

$f^o(x)$ & $g^o(x)$ are odd extensions.

Modes of vibration.

$$p_n = \frac{n\pi}{L}.$$

frequency & period.

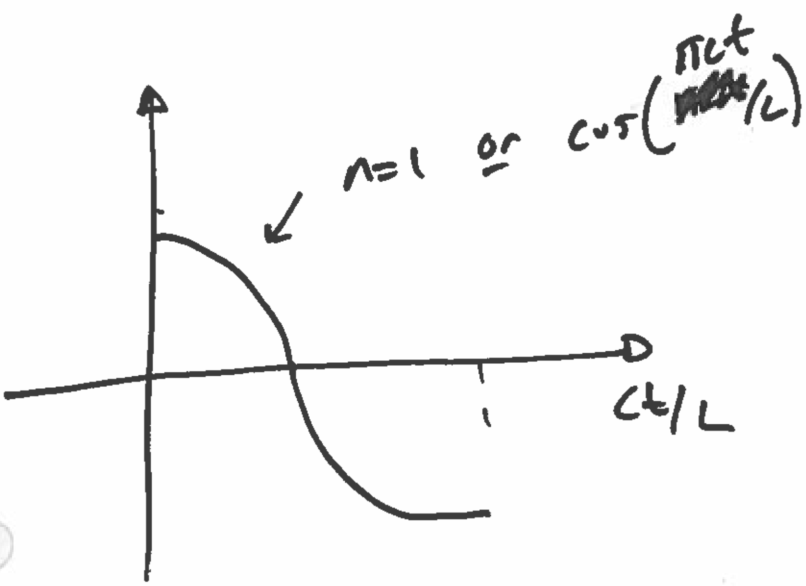
$$\cos\left[p_n c(t + T_n)\right] = \cos\left[p_n c t\right]$$

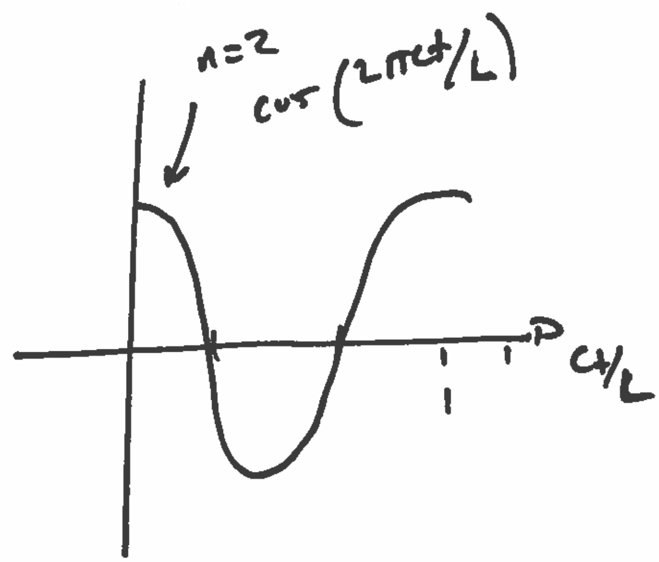
$$T_n p_n c = 2\pi$$

$$T_n = \frac{2L}{nc} \quad \text{(period)}$$

The frequency is:

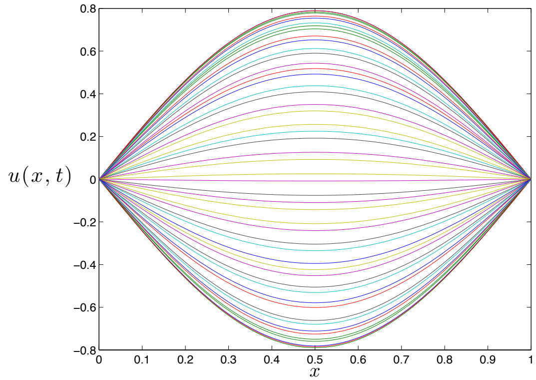
$$f_n = \frac{1}{T_n} = \frac{nc}{2L}.$$

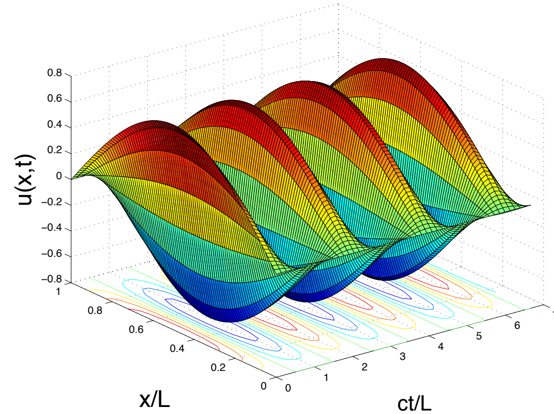




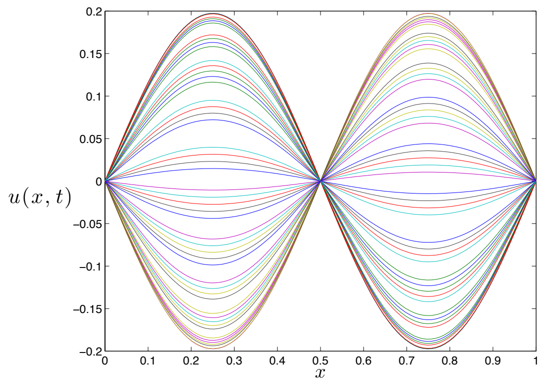
Lecture 26

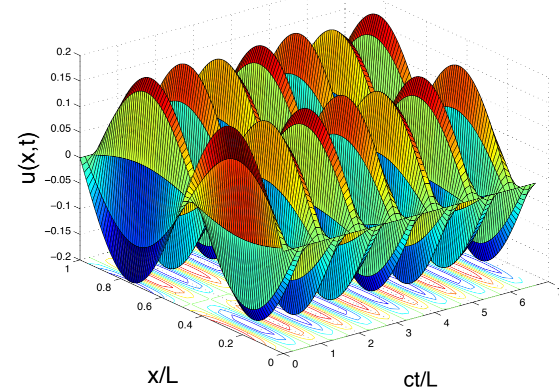
Fundamental





First overtone





Second overtone