# Hedging Agricultural Commodities: A Structural Analysis

James Vercammen

University of British Columbia*

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ABSTRACT

Despite the predominant view that firms hedge to reduce price risk, it has long been known that firms also time their hedges with the goal of increasing profits [Working, 1953, 1962]. In this paper a structural model of commodity pricing is used to examine the determinants of hedging profits. The model builds on the well-known theory of the supply and demand for storage [Working, 1948, Brennan, 1958, Telser, 1958] by incorporating seasonal production and futures contracts with different times to maturity. The structural relationship between stocks and the size of the spot - futures price gap (i.e., the basis) results in predictable seasonal variation in hedging profits. Seasonality in the price discounts across spatially separated markets result in a second source of seasonal variation in hedging profits. The pattern of seasonality in hedging profits is shown to be very different if the market stocks out versus carries stocks across years. A secondary contribution of this paper is that it establishes the role of the forward curve as a tool for analyzing hedging profits, and it clarifies the dual interpretation of backwardation and contango.

Key words: hedging profits, convenience yield, carrying-charge, backwardation, contango

JEL codes: G11, G13, Q11, Q14

*James Vercammen has a joint appointment with the Food and Resource Economics group and The Sauder School of Business at the University of British Columbia. He can be contacted by mail at 2357 Main Mall, Vancouver, British Columbia, V6T 1Z4, by phone at (604) 822-5667, or by e-mail at james.vercammen@ubc.ca.

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# 1 Introduction

Working [1953, 1962] made convincing arguments that agri-business firms such as flour mills hedge in commodity markets for reasons that go well beyond that of reducing price risk. Of particular interest is Working's concept of *carrying-charge* hedging, which occurs when firms hold stocks in order to directly profit from storage.[1] Working [1953] noted that predicting a commodity's price as part of the storage decision is highly unreliable. In contrast, predicting the relationship between a commodity's spot price and futures price is highly reliable, and these predictions can be used to earn positive profits from carrying-charge hedging. Working [1953] demonstrated that if the commodity in question is a crop with an annual harvest such as wheat, corn or soybeans, then hedging profits will vary considerably over the marketing year, and the optimal timing of the hedge becomes an important decision. Working's arguments concerning the reasons why agribusiness firms hedge are consistent with the extensive work on this topic by Williams [1986].

Working's concept of carrying-charge hedging is highly relevant for current agricultural commodity markets despite the continued emphasis on risk reduction as the primary motivation for hedging. Hedging professionals such as consultants and agricultural extension agents seldom use the carrying-charge terminology but they do describe the role of the local market basis and the forward curve in the futures market as important sources of information when hedging with a goal of increasing profits. The basis is defined as the spot price minus the price of the next-to-expire futures contract, and it is how the basis changes over time that determines the profitability of a hedge. The forward curve is the set of futures prices for contracts with different expiry dates, all quoted at the same point in time. Kluis [2000] and Hofstrand [2009] describe how U.S. corn and soybean farmers can use information from the seasonal basis to identify profitable

[1] See page 145 of Leuthold, Junkus, and Cordier [1989] for a detailed description of a carrying-charge hedge.

hedging opportunities. Mitchell [2010] describes how soybean crushers can increase profits by optimally choosing when during the marketing year to lock in a crush margin via hedging. Whalen [2018] describes how U.S. hog producers can use the CBOT forward curve to anticipate profitable hedging opportunities. Bektemirova [2014] explains how elevator companies attempt to profitably trade the basis when buying grain from farmers and later selling it to overseas customers. Finally, Yoon and Brorsen [2002] examine the extent that price inversions are an effective decision tool for hedging and other stock management decisions.

The purpose of this paper is to examine seasonal hedging profits within an equilibrium pricing model that is approximately calibrated to the U.S. corn market. While accounting for storage costs and convenience yield which vary with the level of seasonal stocks, the pricing equilibrium consists of a sequence of eight quarterly spot prices at the U.S. Gulf Coast over a two year period. A location adjustment function, which depends on the cost of transportation, the current level of stocks and the season, determines the size of the gap between the Gulf Coast export price and the Chicago spot price. The Chicago spot price is required because the futures contract specifies Chicago as the delivery point. A second location adjustment factor determines the size of the gap between the Gulf Coast export price and a U.S. midwest local market spot price (in this case, Kansas City). Variations in the size of the expected year 2 harvest generates variations in the equilibrium sequence of storage and seasonal price patterns at the U.S. Gulf Coast and Kansas City. These price patterns are examined as part of the analysis of hedging.

The structural hedging model presented below is unique in the way it ties together the various theoretical concepts from the commodity futures literature. These concepts include the demand and supply of storage, convenience yield, convergence of the spot and futures price, a market stock out, a price inversion, transportation costs, a basis which can take on a positive or negative value, the forward curve, normal backwardation and contango. Standard models of hedging consider only a subset of these features within a partial equilibrium framework. Particularly important in the current analysis is the endogenous sequence of carrying costs because this sequence, which depends on both the season and level of stocks, is the primary determinant of hedging profits. For example, the sequence of carrying costs and thus the pattern of seasonality is very different if the year 1 market stocks out versus carries over stock to year 2.

As noted above, the basis and the forward curve inform the hedging decision when the goal is to increase profits. It is well known that a change in the basis across the hedging time period is a measure of the expected profitability of the short hedge. What is less well known for the case of commodities with seasonal production is the relationship between the price spreads in the forward curve at the point in time when the hedge is placed and the expected profitability of the hedge.[2] The results from the current analysis show that seasonality in the price discounts across spatially separated markets limit the usefulness of the forward curve price spreads as an indicator of hedging profits. This finding implies that for agricultural commodities with seasonal production it is important to use the change in the basis rather than forward curve price spreads when analyzing hedging profits.

The theory of convenience yield and the supply of storage [Working, 1948, Brennan, 1958, Telser, 1958], which explains systematic variations in commodity prices, is widely accepted. Much more controversial is the existence of a risk premium in commodity price data, which can be attributed to the hedging pressure theory of commodity prices. The hedging pressure theory began with Keynes' theory of normal backwardation within which short hedgers implicitly pay a risk premium to long hedgers [Johnson, 1960, Stoll, 1979]. The observation that hedging pressure may bias the futures price upward and thus give rise to contango [Hirshleifer, 1988, 1990] gained traction in recent years when hedge funds and other institutional investors were accused of being responsible for the commodity price boom and bust in the 2005 - 2008 period. Fama and French [1987] and Dewally, Ederington, and Fernando [2013] test and compare the hedging pressure theory and the supply of storage theory, and comment on the implications of these two theories for expected hedging profits. Gorton, Hayashi, and Rouwenhorst [2012] use a two period model to show how the risk premium from hedging pressure varies over time and depends on the level of inventories. After extensive empirical analysis Hambur and Stenner [2016] conclude that the risk premium varies widely across commodities and across the maturity date of the futures contract.

[2] [Jovanovic, 2014] and [Geman, 2015] analyze the relationship between the forward curve and hedging profits. However, their models do not allow for seasonal production of the commodity and price inversions due to a market stockout.

Outside of the agricultural economics discipline where the focus is on commodities such as oil that do not have seasonal production, hedging pressure as a reason for the slope of the forward curve is seldom discussed. The emphasis instead is on temporary shortages or surpluses which are responsible for tilting the forward curve down or up. In this literature the term backwardation (contango) is used to describe a situation where a temporary shortage (surplus) causes the forward curve to slope down (up). With this interpretation of backwardation (contango), the temporary shortage (surplus) causes the spot and futures prices to rise above (fall below) their long run equilibrium levels, in which case it is rational for traders to believe that spot and futures prices will eventually decrease (increase) in order to regain the long run equilibrium.[3] This paper uses simulation results to provide a more detailed comparison of the hedging pressure interpretation and the temporary shortage/surplus interpretation of backwardation and contango.

In the next section of this paper, the basic assumptions are laid out and the equilibrium conditions are specified. Simulated pricing outcomes are then presented with parameter values which reflect the general features of the U.S. corn market. Section 2 concludes with a description of the methods and results from the estimation of the Gulf Coast - Chicago and Gulf Coast - Kansas City pricing discount functions. In Section 3 the results from the estimation of the pricing discount functions are incorporated into the simulation model, and the focus then shifts to the pricing of futures contracts and the local market basis. It is here that the main results concerning hedging profits are set forth. Section 4 presents a short graphical analysis of how the traditional view of backwardation and contango affect hedging profits. The role of convenience yield in the temporary shortage and surplus interpretation of backwardation and contango is also discussed. Concluding remarks are provided in Section 5.

[3] For example, when oil stocks were unusually large in 2015 the price of a futures contract with a nearby expiry was trading well below the price of a contract with a more distant expiry. The strong upward slope to the forward curve and the belief by traders that prices will rebound was referred to as a state of "super contango" [Brusstar and Norland, 2015].

# 2 Spot Market

## 2.1 Basic Assumptions

The spot market consists of a set of competitive farmers producing a homogeneous commodity and selling this commodity to a set of competitive merchants. There are two marketing years, and each year is divided into four quarters/seasons: $n \in \{\text{fall}, \text{winter}, \text{spring}, \text{summer}\}$. Harvesting of the commodity takes place in fall, which is the first quarter/season of the marketing year. Year 1 harvest in quarter 1 (Q1) occurs with certainty at level $H$, and year 2 harvest in Q5 is uncertain, with a 50 percent chance that it will be the same size as the year 1 harvest, and a 50 percent chance that it will be lower at level $H - L$.[4] The size of $L$ relative to a pre-determined threshold value, $L^*$, determines whether the year 1 market will stock out in Q4 (i.e., $S_5 = 0$) or whether inventory will be carried over from year 1 to year 2 (i.e., $S_5 > 0$). Note that $S_t$ denotes the level of stocks which leave quarter $t - 1$ and arrive at quarter $t$. By assumption, stocks are zero prior to year 1 harvest, and are also zero when the market ceases to operate at the end of Q8. Thus, $S_0 = S_9 = 0$.

The analysis is simplified by assuming that demand is stable across the eight seasons. Specifically, inverse demand in quarter $t$ is given by $P_t = a - bX_t$ where $P_t$ is the market price and $X_t$ is the level of consumption. The merchants' cost of storing the marginal unit of the commodity from one quarter to the next consists of a physical storage cost and an opportunity cost of the capital that is tied up in the inventory. The capital cost should depend on the commodity's price but this linkage is ignored in order to simply the analysis. Instead, assume the marginal overall cost of storage is given by the increasing function $k_t = k_0 + k_1 S_t$ . This specification ensures that marginal storage costs are highest in the fall quarter when stocks are at a maximum and gradually decline as the marketing year progresses. Merchants also receive a convenience yield

[4]The analysis is simplified considerably by assuming that the size of the harvest is exogenous and thus not dependent on the price of the commodity.

from owning the stocks rather than having to purchase on short notice. [5] Let $c_t = c_0 - c_1 S_t$ denote the marginal convenience yield for quarter $t$. This function is a decreasing function of stocks because the transaction cost associated with external procurement is assumed to be highest (lowest) when stocks are lowest (highest). Combined storage cost and convenience yield is referred to as the carrying cost. Following [Brennan, 1958] let $m_t = k_t - c_t$ denote the marginal carrying cost for period $t$.

Competition amongst merchants ensures that the expected compensation for supplying storage, $E\{P_{t+1}\} - P_t$, is equal to the net cost of carry, $m_t$, provided that stocks are positive (i.e., no stock out). Substituting in the expressions for $k_t$ and $c_t$, the supply of storage equation can be written as

$$E\{P_{t+1}\} - P_t = m_0 + m_s S_t \tag{1}$$

where $m_0 = k_0 - c_0$ and $m_1 = k_1 + c_1$. If the market has stocked out because merchants are moving from a high-priced pre-harvest quarter to a low-priced post-harvest quarter then equation 1 holds as an inequality rather than an equality. Brennan [1958] defines the demand for storage by first noting that period $t$ consumption, $X_t$, can be written as $X_t = S_{t-1} + H_t - S_t$. Inverse demand in quarter $t$ can therefore be expressed as $P_t = a - b(S_{t-1} + H_t - S_t)$, and the demand for storage function is

$$P_{t+1} - P_t = [a - b(S_t + H_{t+1} - S_{t+1})] - [a - b(S_{t-1} + H_t - S_t)] \tag{2}$$

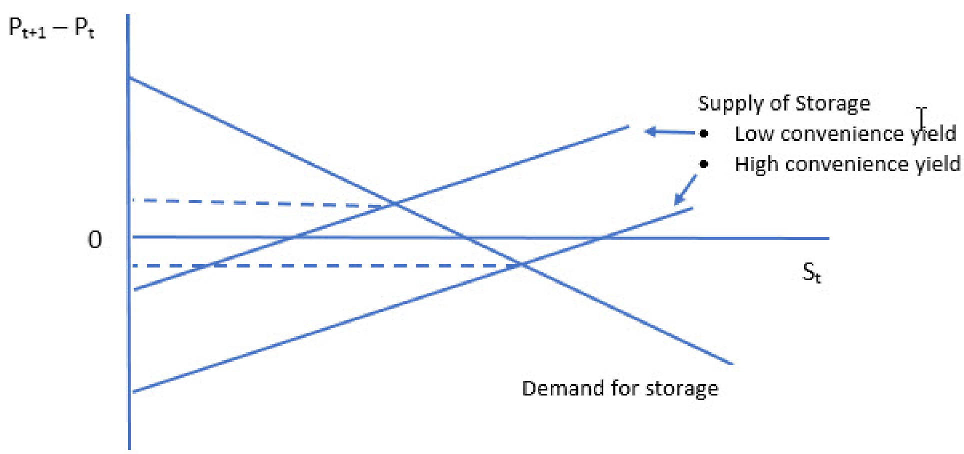
Equation (2) shows that $P_{t+1} - P_t$ is a decreasing function of $S_t$ and thus represents a demand for storage. Brennan [1958]) explains that higher stocks carried out of period $t$ as measured by $S_t$ is associated with an increase in $P_t$ (since less is available for consumption in period $t$) and a decrease in $P_{t+1}$ (since more is available for consumption in period $t + 1$).

Figure 1 shows the supply and demand for storage as given by equations (1) and (2) for the case where there is no production uncertainty and no stockout. Assume that $c_0 > k_0$ to ensure that the marginal convenience yield exceeds the marginal cost of storage at low levels

[5] A standard explanation of convenience yield is that by having stocks on hand a firm can fill unexpected sales orders or create sales opportunities that would otherwise not be possible due to the high transaction costs associated with short-notice spot market transactions.

of $S_t$. The intersection of the supply and demand for storage implies equilibrium values for the stocks carried out of period $t$ and the change in price between period $t$ and $t+1$. The top (bottom) supply of storage schedule corresponds to a relatively low (high) convenience yield and a positive (negative) price change. The fact that stocks can be carried forward despite a negative change in the price is an important part of the theory. Equation (2) shows that the demand for storage function shifts out with a higher beginning inventory, as measured by $S_{t-1} + H_t$, and shifts in with a higher future harvest, as measured by $H_t$. These outcomes are expected because a larger initial inventory will depress both $P_t$ and $P_{t+1}$ but the former will decrease by more than the latter.

Figure 1: Brennan's (1958) Theory of the Supply of Storage with Linear Schedules




The set of equations which describe the market equilibrium can be written as

$$E\{P_{t+1}\} - P_t \begin{cases} = m_0 + m_1 S_t & \text{if } S_t > 0; \\ < m_0 + m_1 S_t & \text{if } S_t = 0. \end{cases} \tag{3}$$

$$P_t = a - bX_t \tag{4}$$

$$S_t = S_{t-1} - X_t \tag{5}$$

Equation (3) is the supply of storage, equation (4) is quarterly demand for the commodity and equation (5) is the equation of motion, which ensures that for those quarters without a harvest ending stocks must equal beginning stocks minus consumption.

The model is solved in the Appendix. For the first four quarters uncertainty has yet to be resolved and thus storage and consumption decisions for these quarters are based on the expected Q5 (post-harvest) price. For this reason there is one set of values for stocks, consumption and price in Q1 through Q4. For the last four quarters each endogenous variable has two values corresponding to whether the year 2 harvest outcome is normal ($H$) or low ($H - L$). The linearity of the various functions imply that merchants' expectation of the Q5 price when making storage decisions in Q1 through Q4 is the straight average of the Q5 price with a normal year 2 harvest and the Q5 price with a low year 2 harvest.

## 2.2 Simulation Results

The previous set of equations are too complicated to analyze using formal comparative statics. As an alternative the model is calibrated to a set of values which are approximately representative for the U.S. corn market. The calibrated model is then used to generate simulation results. In 2018 the U.S. produced 14.63 billion bushels of corn and the average price was $3.70/bu. If U.S. production was evenly consumed over the four quarters then 14.63/4 = 3.66 is an estimate of the average value of $X_t$ in the simulation model. Invert equation (2) to obtain $X_t = a/b - (1/b)P_t$. Assuming the demand elasticity evaluated at $X = 3.66$ and $P = 3.70$ is equal to -2.0, it follows that $a = 5.550$ and $b = 0.506$.[6]

Due to lack of data, it is more challenging to specify values for the $m_0$ and $m_1$ parameters which define the supply of storage function. If harvest is 14.63 billion bushels and this amount is consumed evenly throughout the year then it is reasonable to use one half of this value (7.315) for the purpose of calculating the cost of storage and convenience yield. Suppose the marginal carrying cost (i.e., marginal storage cost minus marginal convenience yield) is equal to $0.0925 per bushel per quarter, which is equivalent to 2.5 percent of the $3.70 average price. Thus, on average $P_{t+1} - P_t = 0.0925 = m_0 + m_1 S_t$. Using $S_t = 7.315$ and assuming the elasticity for

[6]A demand elasticity of -2.0 is much larger in absolute value than the estimates from the literature. This value was chosen because an unrealistically large movement in price from the beginning to the end of the marketing year would otherwise emerge.

the supply of storage schedule is 0.5 it can be shown that $m_0 = 0.0462$ and $m_1 = 0.0063$. These imprecise estimates are rounded to $m_0 = 0.05$ and $m_1 = 0.005$.

Two alternative values of the production loss parameter, $L$, are specified in order to create two different storage scenarios. Specifically, assume $L = 1.750$ (12 percent of production) in the low-loss scenario and $L = 7.315$ (50 percent of production) in the high-loss scenario. This pair of values imply $L^* = 4.670$ for the loss threshold variable. Consequently, in the low-loss scenario the market stocks out in year 1 because $1.75 < 4.570$, and in the high-loss scenario there is positive carry over from year 1 to year 2 because $7.327 > 4.670$.[7] The two alternative values of $L$ implies that there are four distinct scenarios to consider: (1) the market stocks out and year 2 harvest is normal ($L \leq L^*$, $H_2 = H$); (2) the market stocks out and year 2 harvest is low ($L \leq L^*$, $H_2 = H - L$); (3) the market does not stock out and year 2 harvest is normal ($L > L^*$, $H_2 = H$); and (4) the market does not stock out and year 2 harvest is low ($L > L^*$, $H_2 = H - L$). The full set of parameters for the four scenarios are summarized in Table 1.

Table 1: Parameter Values for Spot Market Simulation

| | **Scenario 1-N** Year 1 Stockout, $H_2$ is Normal | **Scenario 1-L** Year 1 Stockout, $H_2$ is Low | **Scenario 2-N** Year 1 Carryover, $H_2$ is Normal | **Scenario 2-L** Year 1 Carryover, $H_2$ is Low |
|---|---|---|---|---|
| Yr2 Loss Size | $L = 1.75$ | $L = 1.75$ | $L = 7.315$ | $L = 7.315$ |
| Yr2 Harvest | $H_2 = 14.63$ | $H_2 = 12.88$ | $H_2 = 14.63$ | $H_2 = 7.315$ |

Demand: $a = 5.55$, $b = 0.50$ Supply of Storage: $m_0 = 0.05$, $m_1 = 0.005$

Figure 2 shows the simulation results for the stockout case (Scenarios 1-N and 1-L). In the top chart, for each of the eight quarters, the total height of the column represents stocks at the end of the quarter assuming a normal harvest in year 2. For the last four quarters the height of the solid-shaded column represents ending stocks assuming that the year 2 harvest incurred a loss of size $L$. For the first four quarters the stocks variable takes on a single value because there

[7]In real world commodity markets some commodity is always carried over from one marketing year to the next. These so-called pipeline stocks are treated as exogenous in this analysis and set equal to zero in order to simplify the notation.

is no production uncertainty in the first four quarters. The central column labelled "Harvest", which corresponds to an instantaneous harvest period, shows the cumulative post-harvest stocks (i.e., carry-in stocks plus harvest).

The bottom chart in Figure 2 shows the level of consumption (columns) and price (lines) for each of the eight quarters. For the last four quarters consumption and price are shown for the normal and low year 2 harvest levels. A third line graph shows the expected price, which is a straight average of the pair of prices for the two harvest scenarios (recall that $H_2 = \{H, H - L\}$ with equal probability). Notice that for Q1 to Q4, and for the two $H_2$ scenarios in Q5 to Q8, price rises at a rate equal to the marginal carrying charge (i.e., $P_{t+1} - P_t = m_0 + m_1 S_t$). The slight concavity of the pricing schedule reflects the negative theoretical relationship between the size of the carrying cost and the level of stocks. The market stocks out in year 1 because $E\{P_5\} - P_4 = m_0 + m_1 S_4$ fails to hold for all positive values of $S_t$ due to a relatively small value of the year 2 loss parameter, $L$. Instead, the stockout implies $E\{P_5\} - P_4 < m_0 + m_1 S_4$, and typically $E\{P_5\} - P_4 < 0$, which is equivalent to a rapid drop in the expected price with the arrival of the year 2 harvest.[8]

[8]This expected pricing pattern of a gradual intra-year increase and then a sharp decrease with the arrival of a new harvest is referred to as the "saw tooth" pattern of pricing for a storable agricultural commodity.

Figure 2: Commodity Stocks, Consumption and Price in the Stockout Scenario

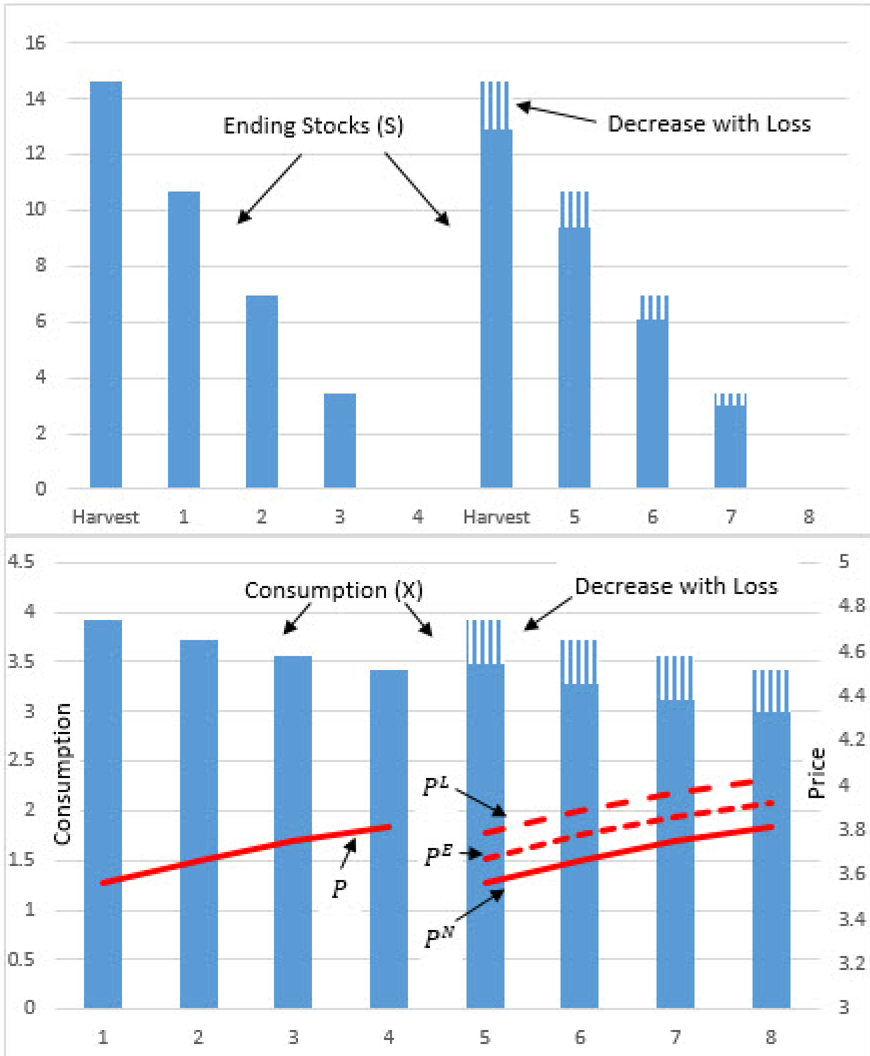
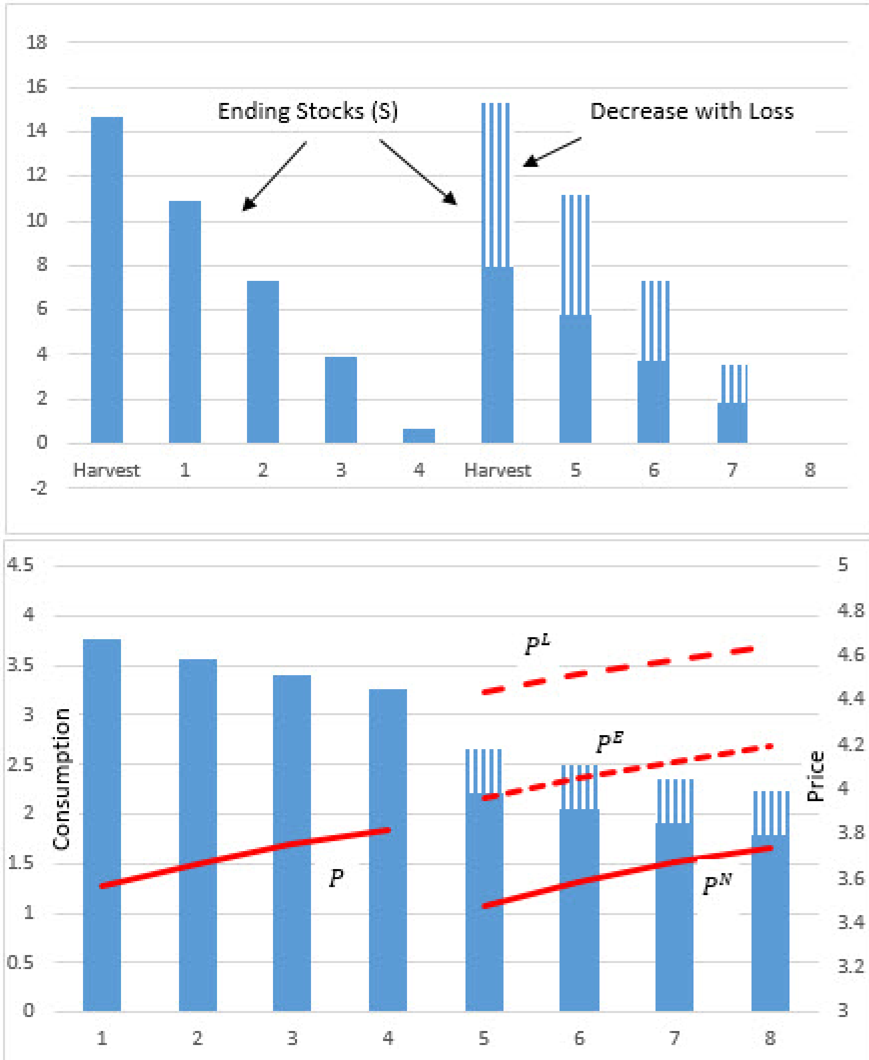



Figure 3 shows the same stock, consumption and price information for the carryover case, which corresponds to scenarios 2-N and 2-L in Table 1. The larger reduction in year 2 production results in a year-to-year carry over of approximate 8 percent of year 1 production. The positive carryover from year 1 to year 2 ensures that the transition pricing equation, $E\{P_5\} - P_4 = m_0 + m_1 S_4$, holds as an equality for all eight quarters, and price is higher in all eight quarters, as compared to the previous stock out case. Notice in Figure 3 that the rise in the commodity's actual price in Q1 to Q4 continues in Q5 to Q8 when price is viewed as an expected value (middle of the three pricing schedules) rather than a conditional value.

Figure 3: Commodity Stocks, Consumption and Price in the Carryover Scenario




## 2.3 Estimation of Location Adjustment Functions

In the previous section the implicit assumption is that prices are discovered within a generic spot market. The analysis in this section assembles data from three locations in the U.S. spot market for corn. The assembled data is used to estimate the new parameters of an enhanced futures/spot simulation model, which is presented in the next section. The three locations consist of: the local spot market in Kansas City, Missouri; the large export spot market on the U.S. Gulf Coast, which consists of the set of terminal elevators at the mouth of the Mississippi River on the Louisiana coast; and a small export spot market on the Great Lakes near Chicago. According to Denicoff, Prater, and Bahizi [2014], between 2007 and 2011 about 15 percent of U.S. corn production was exported from the U.S. with the majority of exports moving through the Gulf Coast. In 2004, for example, of total corn exports 65.1 percent went through the Gulf Coast, 13.1 percent went through the Pacific Northwest and less than 1 percent went through the Great Lakes. Despite the small export share in the U.S. corn market, the analysis is simplified considerably by assuming that the price of corn is fully discovered in the Gulf Coast export market. In the simulation model to follow the Chicago export price is calculated by subtracting from the equilibrium Gulf Coast price a discount which depends on the season and the level of stocks. A similar procedure is used to calculate the Kansas City price. The difference between the Kansas City and Chicago prices can fluctuate significantly over time without drawing arbitrage activity because corn is not shipped from Kansas City to Chicago.[9]

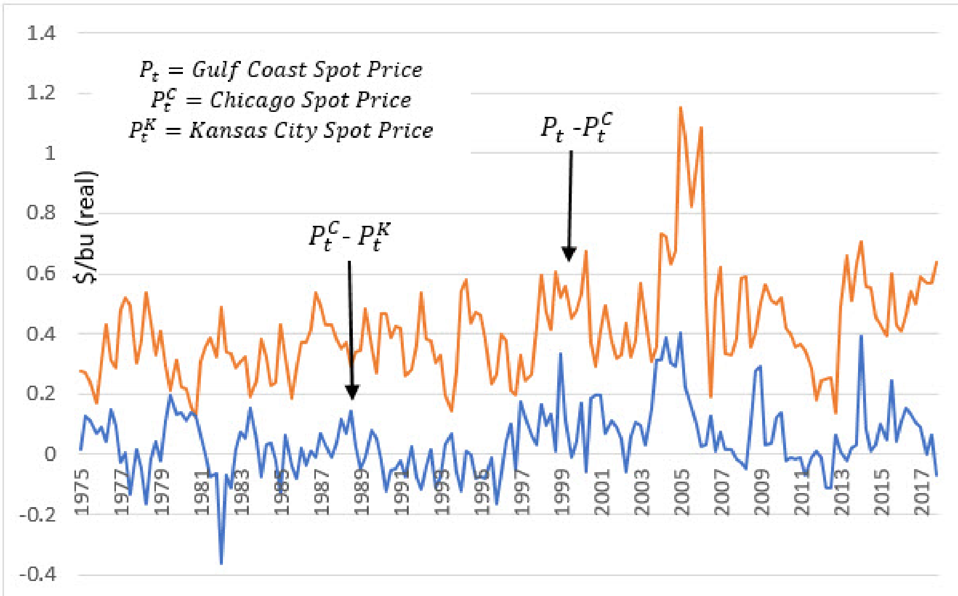
The following procedure is used to estimate the Gulf Coast - Chicago and Gulf Coast - Kansas City price discounts that will be used in the enhanced simulation model. Step A is to create a series of quarterly deflated spot price differences for the three locations. First, monthly USDA-reported spot prices for no. 2 corn are collected for the Gulf Coast, Chicago and Kansas City markets over the September, 1975 - December, 2018 period. Second, the spot prices are deflated by the the Bureau of Labour Statistics (BLS) producer price index (PPI) for all grains (WPU012).[10] Third, two new data series are created by calculating the Gulf Coast - Chicago and Gulf Coast - Kansas City price differences. Fourth, the values for these new series are averaged over three month periods to convert the monthly data into quarterly data. Let $P_t - P_t^C$ and $P_t - P_t^K$ denote these quarterly values for the Chicago and Kansas City discounts, respectively. Figure 4 shows the deflated series for $P_t - P_t^C$ and $P_t^C - P_t^K$. The Chicago - Kansas City price differences are much smaller on average than the Gulf Coast - Kansas City price differences

[9] The Gulf Coast and Chicago export markets serve different overseas customers, and these two markets are not directly linked in the U.S. transportation network.

[10] The spot prices and quarterly stocks for corn (stocks are discussed below) was downloaded from the USDA ERS feedgrain database (`https://www.ers.usda.gov/data-products/feed-grains-database/`) on May 23, 2019. The PPI was downloaded from the BLS website (`https://data.bls.gov/pdq/SurveyOutputServlet`) on May 23, 2019.

because due to their inland locations, the Chicago and Kansas City markets trade corn at a discount relative to the Gulf Coast market.

Figure 4: USDA Deflated Spot Prices and Price Discounts at Locations of Interest: 1975 - 2017.




Step B in the procedure is to construct forecasted values for $P_t - P_t^C$ and $P_t - P_t^K$, using a time trend, a set of seasonal dummies (fall omitted) and a measure of the stock residuals as the set of explanatory variables. Stock residuals is defined as actual stocks minus expected stocks, where expected stocks are the forecasted stocks from a regression of quarterly stocks on a set of seasonal dummies (fall omitted) and a time trend. The $P_t - P_t^C$ and $P_t - P_t^K$ least square regression results are reported in Table 2. The estimated coefficients for the seasonal adjustments are all negative in value and all but one are statistically significant. Thus, the price discounts for both Chicago and Kansas City are largest during the harvest period, and they steadily decrease over the next three seasons. The estimated coefficient on the residual stocks variable is statistically significant for the Chicago regression but not for the Kansas City regression. The positive value for Chicago implies that higher stocks increase the Gulf Coast - Chicago price discount.

Let $\delta(n, S_t^R)$ and $\tau(n, S_t^R)$ denote the conditional forecasts for $P_t - P_t^C$ and $P_t - P_t^K$, respectively, where $n \in \{\text{fall, winter, spring, summer}\}$ is the season indicator (the coefficient for fall is zero) and $S_t^R$ is a measure of residual stocks in quarter $t$. These conditional forecasts are linear functions based on the quarterly data and the regression results from Table 2. The time trend variable is excluded because the impact of a linear time trend on the forecasted values over a two year period is negligible.[11] In scenario 1 of the simulation model (see Table 1), $S_t^R$ is zero for all eight quarters because there is no residual due to a repeating pattern of stocks each marketing year. In scenario 2, the expected value of $S_t^R$ takes on a positive value for Q1 through Q4 and a negative value for Q5 through Q8 because of the positive carryover from year 1 to year 2.

# 3 Futures Price and Basis

In this section the simulation model from the previous section is generalized by assuming there are three spatially distinct spot markets and a common futures market. The focus on no. 2 corn is maintained in this section and for the remainder of the paper. The crucial assumption is that the set of prices which appear in Figures 2 and 3 are the spot prices which are discovered in the Gulf Coast export market. The Chicago and Kansas City spot prices are assumed to equal the Gulf Coast spot price minus a location-specific price discount, which is given by $\delta(n, S_t^R)$ and $\tau(n, S_t^R)$, respectively. The $\tau(n, S_t^R)$ discount reflects the cost of transporting the commodity between Kansas City and the Gulf Coast. The $\delta(n, S_t^R)$ discount reflects the season-specific and stock-dependent difference in the export prices at the Gulf Coast and Chicago export locations.

The assumption of normal price expectations, which is maintained throughout this section, ensures that the price of a futures contract is equal to the price that traders expect the contract to be priced at as the contract approaches its expiry date. The additional assumption that the Chicago spot price and the futures price are equal (i.e., converge) when the futures contract expires implies that the futures price is equal to traders' expectation of the Chicago spot price in the quarter that the futures contract expires.[12] The basis, which is unique for each futures

[11] The mean value of the time trend, $\bar{t} = 86$, over the 1975 - 2018 period was used to absorb the time trend variable into the constant term.

[12] Suppose instead that the expiring futures price remained above the prevailing spot price. In this case a trader could profitably take a short position in the futures market, purchase the corn in the spot market and immediately deliver the corn to fulfill the conditions of the short futures contract. In the opposite situation the trader would take

Table 2: Regression Model for Forecasting $P_t - P_t^C$ and $P_t - P_t^K$

| Specification: | OLS | |
|---|---|---|
| | (1) | (2) |
| Dependent Variable | $P_t - P_t^C$ | $P_t - P_t^K$ |
| Constant | $0.365^a$ | $0.382^a$ |
| | (0.029) | (0.040) |
| Q2 Dummy | -0.037 | $-0.075^c$ |
| | (0.031) | (0.042) |
| Q3 Dummy | $-0.118^a$ | $-0.124^a$ |
| | (0.031) | (0.043) |
| Q4 Dummy | $-0.103^b$ | $-0.122^a$ |
| | (0.031) | (0.043) |
| Time Trend | $0.0013^a$ | $0.0018^a$ |
| | (0.0002) | (0.0003) |
| Stock Residual | $0.0257^a$ | 0.0200 |
| | (0.0090) | (0.0124) |
| N | 172 | 172 |
| $R^2$ | 0.257 | 0.210 |

$^a p < 0.01$, $^b p < 0.05$, $^c p < 0.1$

*Notes*: Q1 = Sept-Nov, Q2 = Dec - Feb, Q3 = Mar - May, Q4 = Jun - Aug.
The Stock Residual variable is the residuals from a regression of stocks on seasonal dummies and a time trend.

contract expiry date and each spot market location combination, is equal to the spot price minus the futures price. As is shown below, the behavior of the basis over time is a key determinant of hedging profitability.

Assume there are two traded futures contracts, both of which have Chicago as the delivery location. The F4 contract trades in Q1 through Q4 and expires in Q4, which is the summer of year 1. The F7 contract trades in Q1 through Q7 and expires in Q7, which is the spring of year 2. Regarding notation, let $F_t^4$ and $F_t^7$ denote the price of the F4 and F7 futures contract in period $t \in \{1, 2, 3, 4\}$. Both of these prices are constant in Q1 through Q4 because it is only in Q5 when the size of the year 2 harvest becomes known that this pair of prices adjust. The F7 basis for Q1 through Q4 can be expressed as $B_t^G = P_t - F_t^7$ in the Gulf Coast market, and $B_t^K = P_t^K - F_t^7$ in the Kansas City market. Substitute $P_t^C + \delta(n, S_t^R)$ for $P_t$ in the first expression to obtain $B_t^G = P_t^C - F_t^7 + \delta(n, S_t^R)$. Similarly, substitute $P_t - \tau(n, S_t^R)$ for $P_t^K$ in the second expression to obtain $B_t^K = P_t - \tau(n, S_t^R) - F_t^7$. Now substitute $P_t^C + \delta(n, S_t^R)$ for $P_t$ in this latter expression to obtain $B_t^K = P_t^C - F_t^7 + \delta(n, S_t^R) - \tau(n, S_t^R)$. The two basis equations can be summarized as follows:

$$B_t^G = P_t^C - F_t^7 + \delta(n, S_t^R) \qquad\qquad B_t^K = P_t^C - F_t^7 + \delta(n, S_t^R) - \tau(n, S_t^R) \tag{6}$$

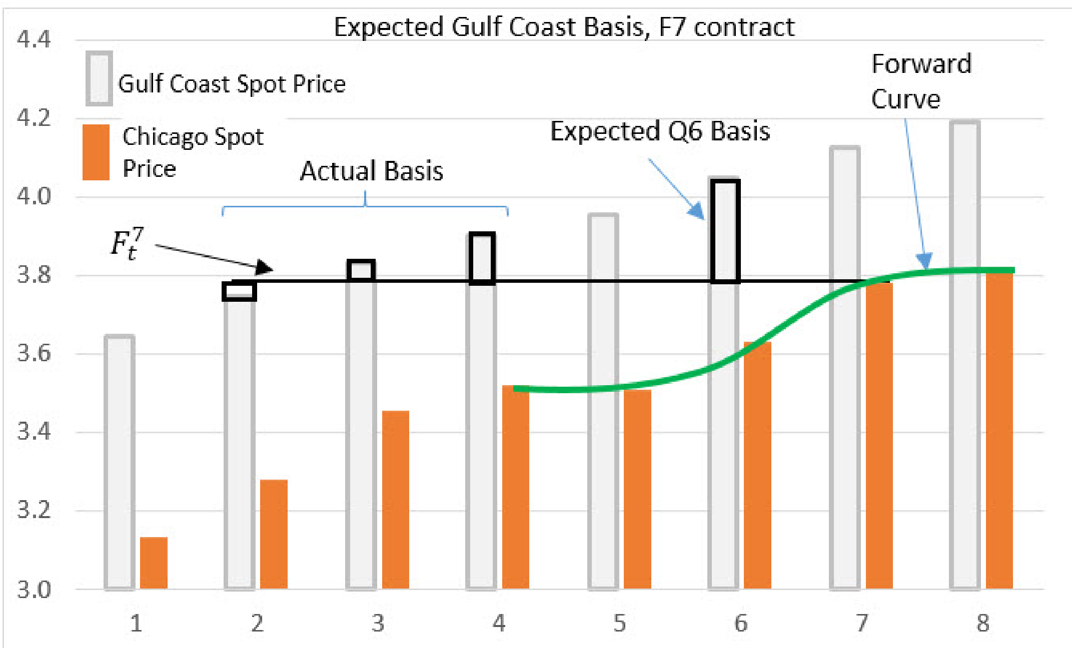
Equation (6) decomposes the basis into a location adjustment component, which is $\delta(n, S_t^R)$ for Chicago and $\tau(n, S_t^R)$ for Kansas City, and a cost of carry component, $P_t^C - F_t^7$. The location adjustment component takes on a relatively large positive value for Chicago and a relatively small negative value for Kansas City (see Figure 4). If other Midwest U.S. locations that ship to the Gulf Coast were considered, the basis would take on positive value if the location was relatively close to the Gulf Coast (e.g., Lubbock, TX) and a larger negative value if the location was relatively far from the Gulf Coast (e.g., Minneapolis, MN).

Figure 5 shows the simulated basis from the perspective of a merchant who operates in the Gulf Coast export market when the market carries over stock due to a large potential loss in the year 2 harvest (i.e., Scenario 2 in Table 1). Figure 6 shows the simulation results from the

a long position in the expiring contract, accept delivery of the corn to fulfill the contract requirements and then immediately sell the corn on the spot market.

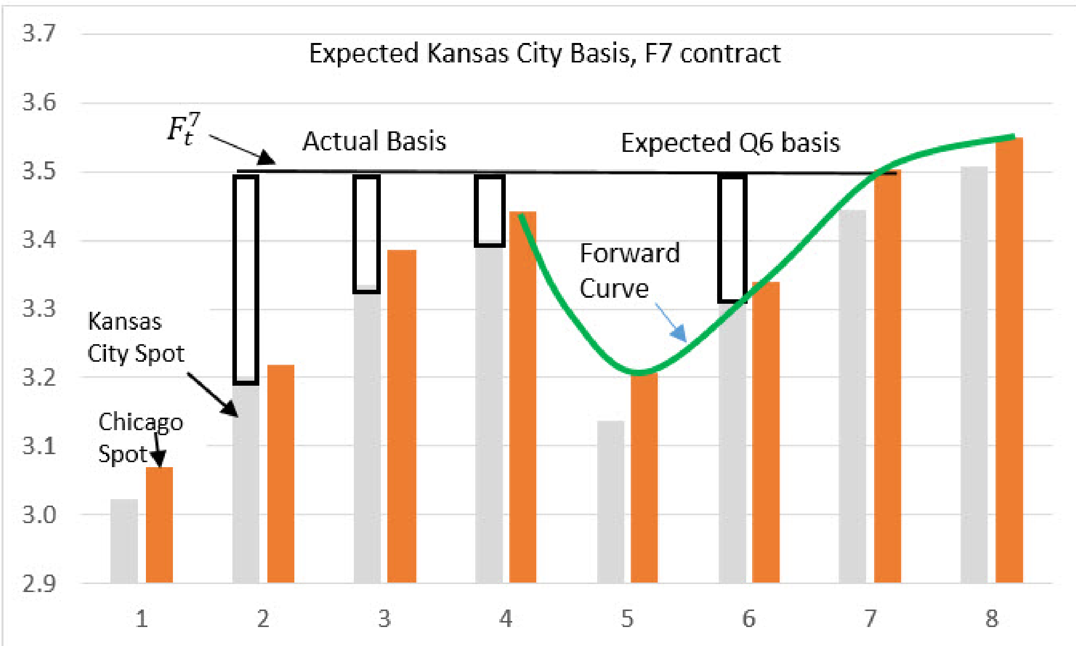
perspective of a merchant who operates in the Kansas City local market when the market stocks out due to a small potential loss in the year 2 harvest (i.e., Scenario 1 in Table 1). In each case the merchant is assumed to be in Q2, Q3 or Q4 (i.e., pre harvest) and is looking forward to Q6 (i.e., post harvest). The futures price that is used to calculate the basis ($F_t^7$) is equal to the expected Q7 Chicago spot price. The basis is the vertical distance (positive or negative) between the horizontal $F_t^7$ line and the column which measures the spot price. The basis for Q2, Q3 and Q4 is the actual basis because prior to Q5 there is no uncertainty. The basis for Q6 is the basis which the merchant expects to observe when Q6 is reached.

Figure 5: Gulf Coast Basis and Forward Curve With Carry Over




There are several important differences in the basis levels which are displayed in Figures 5 and 6. First, the basis values are negative in the Kansas City market and mostly positive in the Gulf Coast market. This is because the spot price trades above the futures price in the Gulf Coast market and below the futures price in the Kansas City market due to the differences in location. Second, in both markets the basis increases in value over Q2 through Q4, becoming more positive in the Gulf Coast market and less negative in the Kansas City market. An increasing basis (either more positive or less negative) is referred to as a strengthening basis. A scenario where the basis is becoming less positive or more negative is referred to as a weakening basis. As will be shown in the section to follow, the expected profitability of the hedge is directly determined by whether the basis strengthens or weakens over the hedging period. The third important difference in the basis for the two markets is that in Gulf Coast market the expected Q6 basis is greater than the Q4 basis whereas, due to the stockout, the opposite is true in the Kansas City market.

Figure 6: Kansas City Basis and Forward Curve With Stock Out




The forward curves in Figures 5 and 6 reflect the spread in the forward prices, including the F7 - F4 spread in the pair of futures prices, once again from the perspective of a trader in Q1 through Q4. As will be shown in the next section, under certain conditions the F7 - F4 price spread is a measure of expected gross profits for the short hedger. In Figure 6 notice that the stock out results in a negative price spread, which is equivalent to a downward sloping portion of the forward curve. A negative spread in futures prices is commonly referred to as an inverted market. In the Kansas City market (see Figure 6) a positive basis is a sufficient condition for an inverted market. In general, it is not possible to identify if a market is inverted by comparing the spot price to the futures price without accounting for the price discount due to differences in location. Indeed, Figure 5 shows why a positive basis is not always a sufficient condition for an inverted market.

# 4 Hedging with Normal Price Expectations

In this section the enhanced simulation model from Section 3 is used to examine how the expected profits of a short hedge depends on the season and level of residual stocks. There is no formal analysis of the long hedge because the theoretical properties of the long hedge are a mirror image of the properties of the short hedge. The standard short hedge involves a merchant with a long position in the spot market (i.e., ownership of the commodity) taking a short position in the futures market, and offsetting this position when the commodity is eventually sold. If the futures and spot prices change in the same direction in response to an information shock then the gains in one market will "largely" offset the losses in the other market, and vice versa. The standard long hedge involves a merchant with a short position in the spot market (i.e., the commodity was implicitly borrowed through a forward sale) taking a long position in the futures market, and offsetting this position when the commodity is eventually purchased in order to implicitly return the borrowed commodity.[13]. For the long hedger, the gains and losses in the two markets are "largely" offsetting but the net profits are theoretically opposite in sign of the expected profits of the short hedger.

Hedging is commonly described as a risk reducing activity because the hedge results in basis risk substituting for price risk. While this is true, the lack of emphasis on expected profits tends to imply that the actual profitability of a hedge is random rather than systematic. In this section it is shown that expected profits from short hedging systematically varies over the marketing year due to seasonal changes in storage costs, convenience yield and location adjustment factors. The analysis below implicitly assumes that the short hedger has physical possession of the commodity when the hedge is placed. An alternative assumption is that a commodity buyer

[13]For example, an ethanol producer is short in the spot market for corn if ethanol that has not yet been produced is forward sold. This short position is eliminated when the corn which is used to produce the ethanol is eventually purchased.

takes a short position in the futures market when a forward price for the spot commodity is contractually offered to a farmer, and the short position is later offset when the farmer delivers the commodity and the merchant sells the commodity. The difference between this scenario and the one considered in this paper is that the forward price offered to the farmer when the hedge is initiated may be different than the prevailing spot price, and the hedger's net cost of carry is zero due to the nature of the forward transaction.

The expected (net) profits from hedging is equal to expected gross profits minus the carrying cost that is associated with the hedge.[14] For the case of the short hedge, the carrying cost consists of the hedger's individual cost of storing the commodity less convenience yield over the hedging period. These individual carrying costs which may be greater than, equal to or less than the carrying cost for the market as a whole. Let $C(n, T)$ denote the hedger's carrying cost when the hedge is placed in season $n$ and lifted $T$ quarters later.[15] It is reasonable to assume that $C(n, T)$ is an increasing function of $T$ but additional properties of this cost function are not specified because the individual carrying cost is likely to vary widely across hedgers.

Short hedges are commonly initiated prior to harvest, which in the current analysis could be the preceding late winter (Q2), spring (Q3) or summer (Q4). Moreover, the hedge is commonly lifted in the post-harvest season, which in the current analysis is early winter (Q6). Prices are non-stochastic prior to Q5, which means that the expected profits from the hedge can be directly compared for the three alternative dates when the hedge is initiated. The analysis proceeds by first calculating profits conditioned on the outcome of the year 2 harvest, and then using the conditional results to derive an expression for the profits the hedger should expect at the time the hedge is initiated. Expected profits from hedging are separately calculated for the Gulf Coast and Kansas City spot markets, and for the two different assumptions about the size of the potential loss in the year 2 harvest. Throughout the analysis the implicit assumption is that one unit of the commodity is hedged.

[14]Hedging costs should also include futures trading transaction costs such as commission fees and the short term cost of financing margin calls. These costs are not included in the discussion below.

[15]The carrying cost function should also depend on residual stocks but this linkage is ignored in order to simplify the analysis.

Although the focus of the analysis is hedging profits in the Gulf Coast and Kansas City spot markets, it is nevertheless useful to derive an expression for hedging profits in the Chicago market for reasons that will become obvious as the analysis proceeds. Let ${}^{N}\mathbf{P}_6^C$ and ${}^{L}\mathbf{P}_6^C$ denote the Chicago spot prices when the hedge is lifted in Q6 and the year 2 harvest in Q5 was normal and low, respectively. Let ${}^{N}\mathbf{F}_6^7$ and ${}^{L}\mathbf{F}_6^7$ denote the corresponding prices in Q6 of a futures contract that expires in Q7 assuming a normal and low Q5 harvest, respectively. When the hedge is initiated in quarter $t \in \{2,3,4\}$, let $P_t^C$ denote the prevailing Chicago spot price, and let $F_t^7$ denote the prevailing futures price of a F7 contract. Conditional gross hedging profits for a period $t$ hedge can be expressed as ${}^{N}\pi_t^C = {}^{N}\mathbf{P}_6^C - P_t^C + F_t^7 - {}^{N}\mathbf{F}_6^7$ when the year 2 harvest is normal, and ${}^{L}\pi_t^C = {}^{L}\mathbf{P}_6^C - P_t^C + F_t^7 - {}^{L}\mathbf{F}_6^7$ when the year 2 harvest is low.

To make the connection between basis and hedging profits more explicit, rearrange the previous pair of equations as follows: ${}^{N}\pi_t^C = {}^{N}\mathbf{B}_6^C - B_t^C$ and ${}^{L}\pi_t^C = {}^{L}\mathbf{B}_6^C - B_t^C$. In these expressions ${}^{j}\mathbf{B}_6^C = {}^{j}\mathbf{P}_6^C - {}^{j}\mathbf{F}_6^7$ is the basis when the hedge is lifted in Q6 and the year 2 harvest had state $j \in \{Normal, Low\}$, and $B_t^C = P_t^C - F_t^7$ is the basis when the hedge is placed in quarter $t \in \{2,3,4\}$. Recall that the normal and low outcomes for the year 2 harvest each occur with 0.5 probability. Expected gross hedging profits in the Chicago market when the hedge is placed in quarter $t \in \{2,3,4\}$ and lifted in quarter 6 can therefore be expressed as

$$\bar{\pi}_t^C = \bar{B}_6^C - B_t^C \qquad \text{where} \qquad \bar{B}_6^C = 0.5({}^{N}\mathbf{B}_6^C + {}^{L}\mathbf{B}_6^C) \tag{7}$$

Equation (7) reveals the well-known result that expected profits for the short hedger is the expected change in the basis over the hedging period.[16] It is important to note that in the Chicago market the change in the basis over the hedging period is equivalent to the change in the carrying cost over the hedging period. Isolating the change in the carrying cost is useful because this measure can be used to decompose hedging profits in the Gulf Coast and Kansas City markets into a cost of carry component and a change in the location-specific price discount component.

[16] Basis risk, which is the difference in how the basis changes across the hedging period when the year 2 harvest is low versus high, results in profit uncertainty for the short hedger. Examining basis risk in the simulation results goes beyond the scope of this analysis. Nevertheless, it should now be obvious why hedging is often described as substituting basis risk for price risk.

Expected profits for a short hedger in the Gulf Coast market can be expressed as $\bar{\pi}_t = \bar{B}_6 - B_t$. It is straight forward to show that $\bar{B}_6 = \bar{B}_6^C + \bar{\delta}(2, S_6^R)$ where $\bar{\delta}(2, S_6^R)$ is the value of $\delta(2, S_6^R)$ averaged over the normal and low year 2 harvest outcomes. Similarly, $B_t = B_t^C + \delta(t, S_t^R)$. This allows the expression for expected profits for the short hedger in the Gulf Coast market to be written as

$$\bar{\pi}_t = \bar{\pi}_t^C + \bar{\delta}(2, S_6^R) - \delta(t, S_t^R) \quad (8)$$

Noting that $\bar{\pi}_t^C$ is a measure of the change in the carrying cost over the hedging period, equation (8) decomposes the profits expected by a short hedger in the Gulf Coast market into a carrying cost component and a change in the Gulf Coast - Chicago price discount component. This is an important to result because it shows that seasonality in the spot prices at the two export markets directly affects the profits of the short hedger.

The derivation of expected profits for a short hedger in the Kansas City market is similar to the derivation in the Gulf Coast market. The only difference is that the Kansas City spot price can be expressed as $P_t^K = P_t^C + \delta(n, S_t^R) - \tau(n, S_t^R)$. Thus, expected profits for a short hedger in the Kansas City market can be expressed as

$$\bar{\pi}_t = \bar{\pi}_t^C + \bar{\delta}(2, S_6^R) - \delta(t, S_t^R) - [\bar{\tau}(2, S_6^R) - \tau(t, S_t^R)] \quad (9)$$

Equation (9) shows that the difference in the change in the Gulf Cost - Chicago price discount and the Gulf Coast - Kansas City price discount is a direct determinant of hedging profits in the Kansas City market. As will be shown in the simulation results in the next section, a moderately strong positive correlation of the seasonality within $\delta(n, S_t^R)$ and $\tau(n, S_t^R)$ results in a comparatively small overall impact of location on expected hedging profits in the Kansas City market.

To conclude this section is it useful to examine the extent that the F7 - F4 price spread in the forward curve is a reasonable estimate of expected profits for the short hedger. Suppose the hedge was initiated in the Chicago spot market in Q4 with a F7 futures contract and lifted in Q7. Noting that price convergence ensures $F_4^4 = P_4^C$ and $F_7^7 = P_7^C$, it follows that the basis in Q7 is zero and hedging profits are equal to the negative of the Q4 basis, which can be expressed as $F_4^7 - F_4^4$. This result demonstrates that if the short hedge takes place in the Chicago market and if the timing of the hedge exactly coincides with the expiration dates of the futures contracts used in the hedge, then the price spread when the hedge is initiated is an exact measure of expected hedging profits. In the current analysis where the hedge is lifted in Q6 rather than Q7, and initiated possibly earlier than Q4, the F7 - F4 price spread is not an accurate measure of hedging profits in the Chicago market. The accuracy further declines when using the F7 - F4 price spread to measure expected hedging profits in the Gulf Coast and Kansas City markets because this measure ignores the changes in the $\delta(n, S_t^R)$ and $\tau(n, S_t^R)$ price discount functions.

## 4.1 Hedging Simulation Results

In this section the results from the enhanced simulation model are used to identify the key determinants of expected profits for the short hedger. As shown in the preceding section, expected gross profits for the short hedge is equal to the change in the basis over the hedging period. Moreover, the change in the basis is due to seasonality in the carrying cost (i.e., the marginal cost of storage minus the marginal convenience yield) and seasonality in the Gulf Coast - Chicago and Gulf Coast - Kansas City price discounts.

In Figure 5 expected gross profits for a short hedger in the Gulf Coast can be identified for the case where stocks are carried over from year 1 to year 2. The situation is the same in Figure 6 except now the location is Kansas City and the year 1 market stocks out. In both cases expected gross profits from short hedging can be identified by comparing the size of the expected Q6 basis with the size of the basis when the hedge was initiated in Q2, Q3 or Q4. Figure 5 shows that for each of the three hedge initiation dates the continually rising spot price which results from the carry over causes the expected basis over the hedging period to strengthen. In contrast, Figure 6 shows that because of the price inversion due to the market stock out in year 1, the expected basis in the Kansas City market strengthens over the hedging period if the hedge was initiated in Q2 and weakens if the hedge was initiated in either Q3 or Q4. Keep in mind that a strengthening basis in the Gulf Coast market implies more positive values whereas a strengthening basis in the Kansas City market implies less negative values.

In Figures 5 and 6 expected gross profits from the short hedge are positive with a strengthening basis and negative with a weakening basis. The cost of hedging, as measured by the $C(n, T)$ cost function, must be subtracted from expected gross profits in order to obtain a measure of net expected profits from the short hedge. Without specifying more structure on this cost function it is not possible to determine if net expected hedging profits are positive or negative when the basis strengthens because in this case the rate of increase in the cost of hedging may be greater than or less than the rate of increase in the size of the basis. When the basis weakens it is possible to conclude that the net expected profits from short hedging are negative because in this case the rate of increase in hedging costs must necessarily exceed the rate of increase in the size of the basis.

Table 3 shows the hedging simulation results for the Kansas City local market. The top (bottom) half of the table corresponds to the scenario where the year 2 harvest loss is small (large), in which case the year 1 market stocks out (has positive carry over). The top row in each half of the table is the set of spot prices in the Gulf Coast export market, as illustrated in Figure 6. The second row is the carrying cost over the hedging period, and the third row is the Gulf Coast - Chicago location adjustment for Q7. The sum of the first three rows is the expected Q7 spot price in Chicago, which can be interpreted as the price of a F7 futures contract given the assumption of normal price expectations. Notice that the futures price remains constant for Q2 through Q6 at level \$3.502/bu in the top half of the table, and at level \$3.781/bu in the bottom half. The values in the row labelled $\tau(n, S_t^R)$ are subtracted from the Gulf Coast spot prices to obtain the set of Kansas City spot prices. The difference between the Kansas City spot prices and the previously-calculated futures price is the basis. The change in the calculated variables between when the hedge is lifted in Q6 and when it was initiated in Q2, Q3 or Q4 is reported in the three right-hand columns of Table 3. The negative of the basis values in the second row from the bottom is a measure of the expected gross profits for the short hedger. These calculated profits are reported in the bottom row of each half of Table 3. .

For the stockout scenario, the results in top half of Table 3 reveal positive gross profits of \$0.111/bu if the short hedge is initiated in the late winter (Q2) and negative gross profits equal to -\$0.023/bu and -\$0.088/bu if the short hedge is initiated in the spring (Q3) or summer (Q4),

Table 3: Simulated Expected Hedging Profits in the Kansas City Market

Kansas City Local Market

| Low L (Scenario 1): Year 1 Market Stocks Out | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Q2 | Q3 | Q4 | Q6 | Q2 - Q6 | Q3 - Q6 | Q4 - Q6 |
| Gulf Coast Spot | 3.666 | 3.751 | 3.818 | 3.777 | **-0.111** | **-0.026** | **0.041** |
| Carry Cost to Q7 | 0.194 | 0.109 | 0.042 | 0.083 | **0.111** | **0.026** | **-0.041** |
| $\delta(3, S_7^R)$ | 0.358 | 0.358 | 0.358 | 0.358 | **0** | **0** | **0** |
| $F_t^7$ (Futures) | 3.502 | 3.502 | 3.502 | 3.502 | **0** | **0** | **0** |
| $\tau(n, S_t^R)$ | 0.465 | 0.416 | 0.418 | 0.465 | **0** | **-0.049** | **-0.047** |
| Kansas City Spot | 3.201 | 3.335 | 3.400 | 3.312 | **-0.111** | **0.023** | **0.088** |
| Kansas City Basis | -0.301 | -0.167 | -0.102 | -0.190 | **-0.111** | **0.023** | **0.088** |
| Expected Gross $\pi$ | | | | | **0.111** | **-0.023** | **-0.088** |

| High L (Scenario 2): Year 1 Carry Over Stocks | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Q2 | Q3 | Q4 | Q6 | Q2 - Q6 | Q3 - Q6 | Q4 - Q6 |
| Gulf Coast Spot | 3.747 | 3.833 | 3.903 | 4.048 | **-0.302** | **-0.215** | **-0.146** |
| Carry Cost to Q7 | 0.379 | 0.293 | 0.223 | 0.078 | **0.302** | **0.215** | **0.146** |
| $\delta(3, S_7^R)$ | 0.345 | 0.345 | 0.345 | 0.345 | **0** | **0** | **0** |
| $F_t^7$ (Futures) | 3.781 | 3.781 | 3.781 | 3.781 | **0** | **0** | **0** |
| $\tau(n, S_t^R)$ | 0.465 | 0.416 | 0.418 | 0.465 | **0** | **-0.049** | **-0.047** |
| Kansas City Spot | 3.282 | 3.417 | 3.485 | 3.583 | **-0.302** | **-0.166** | **-0.099** |
| Kansas City Basis | -0.499 | -0.363 | -0.296 | -0.197 | **-0.302** | **-0.166** | **-0.099** |
| Expected Gross $\pi$ | | | | | **0.302** | **0.166** | **0.099** |

* Actual price outcomes for Q2, Q3 and Q4. Pre-harvest expected price outcome for Q6.

respectively. These results are expected given the downward sloping segment of the forward curve, as illustrated in Figure 6. In contrast, for the carry over scenario, the results in the bottom half of Table 3 reveal positive gross profits regardless of when the hedge is initiated within the Q2 to Q4 window. Moreover, gross hedging profits are smaller the shorter the hedging time period. Both of these results are expected because in a market with positive carryover the change in the basis is positive over any hedging period but the size of the change is lower with a shorter time horizon and thus a lower carrying cost.

The difference in the Kansas City basis (second row from the bottom) and the carrying cost (second row from the top) is the contribution of the change in the location-specific price discount toward explaining gross hedging profits. This contribution is zero if the hedge is initiated in Q2 because in this case the hedge is initiated and lifted in the same season. If the hedge is initiated in Q3 then the location adjustment contribution is -0.023 - (-0.026) = \$0.003/bu. The analogous value for a Q4 hedge is -0.088 - (-0.041) = -\$0.047/bu. In the Q3 case the contribution of location in explaining hedging profits is negligible whereas this contribtion is relatively large (i.e., close to 50 percent) if the hedge is initiated in Q4. Similar results emerge for the case of a positive carry over, as detailed in the bottom half of Table 3.

Table 4 shows the hedging simulation results for the Gulf Coast. Notice that with one exception, the Gulf Coast basis values are positive, which is opposite the negative basis values which characterize the Kansas City basis. The Gulf Coast basis takes on a negative value in Q2 because in this quarter the carrying cost exceeds the Gulf Coast - Chicago price discount (these two factors have opposite impacts on the basis). The change in the Gulf Coast basis over the hedging period, and thus the expected gross profits for a Gulf Coast short hedger, are similar to the values reported for Kansas City with the important exception that the Gulf Coast basis values do not include the seasonal changes in $\tau(n, S_t^R)$. This difference is important enough to cause sign differences in the expected profits from hedging at the two locations. For example, in the stockout scenario that corresponds to the top half of Tables 3 and 4, expected gross profits from the short hedge are positive in the Gulf Coast market and negative in the Kansas City market. Overall, gross expected hedging profits are larger in the Gulf Coast market than the Kansas City market for both the stockout and carry over scenarios.

Table 4: Simulated Expected Hedging Profits in the Gulf Coast Market

Gulf Coast Export Market

| Low L (Scenario 1): Year 1 Market Stocks Out | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Q2 | Q3 | Q4 | Q6 | Q2 - Q6 | Q3 - Q6 | Q4 - Q6 |
| Gulf Coast Spot | 3.666 | 3.751 | 3.818 | 3.777 | **-0.111** | **-0.026** | **0.041** |
| Carry Cost to Q7 | 0.194 | 0.109 | 0.042 | 0.083 | **0.111** | **0.026** | **-0.041** |
| $\delta(3, S_7^R)$ | 0.358 | 0.358 | 0.358 | 0.358 | **0** | **0** | **0** |
| $F_t^7$ (Futures) | 3.502 | 3.502 | 3.502 | 3.502 | **0** | **0** | **0** |
| Gulf Coast Basis | 0.164 | 0.249 | 0.316 | 0.275 | **-0.111** | **-0.026** | **0.041** |
| Expected Gross $\pi$ | | | | | **0.111** | **0.026** | **-0.041** |

| High L (Scenario 2): Year 1 Carry Over Stocks | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Q2 | Q3 | Q4 | Q6 | Q2 - Q6 | Q2 - Q6 | Q2 - Q6 |
| Gulf Coast Spot | 3.747 | 3.833 | 3.903 | 4.048 | **-0.302** | **-0.215** | **-0.146** |
| Carry Cost to Q7 | 0.379 | 0.293 | 0.223 | 0.078 | **0.302** | **0.215** | **0.146** |
| $\delta(3, S_7^N)$ | 0.345 | 0.345 | 0.345 | 0.345 | **0** | **0** | **0** |
| $F_7$ (Futures) | 3.781 | 3.781 | 3.781 | 3.781 | **0** | **0** | **0** |
| Gulf Coast Basis | -0.034 | 0.053 | 0.122 | 0.268 | **-0.302** | **-0.215** | **-0.146** |
| Expected Gross $\pi$ | | | | | **0.302** | **0.215** | **0.146** |

* Actual price outcomes for Q2, Q3 and Q4. Pre-harvest expected price outcome for Q6.

# 5 Hedging with Backwardation and Contango

In this final section the dual interpretations of backwardation and contango are examined. Recall from the Introduction that the first interpretation of backwardation and contango is that these concepts refer to a gap between the futures price and the expected future spot price due to hedging pressure. The classic example is Keynesian normal backwardation, where an excess demand for short contracts by short hedgers bids the futures price below the expected future spot price of the commodity. With normal backwardation the futures price is expected to gradually rise over time as the hedging pressure diminishes with the approach of the contract's expiry date. This expected rise is equivalent to a forward curve that slopes down because contracts which mature later are most affected by the backwardation. In the opposite case where an excess demand for long contracts bids the futures price above the expected spot price, the forward curve slopes up due to the underlying contango.

The second interpretation of backwardation and contango is that these concepts refer to a situation where a temporary shortage or surplus of the commodity has caused the price to rise above or fall below the equilibrium price, and traders anticipate the price gradually regaining equilibrium. In a backwardated (contango) market, the forward curve slopes down (up) because with the gradual elimination of the commodity shortage (surplus), the spot and futures price are expected to gradually decrease (increase) over time. With this interpretation of backwardation and contango traders have normal price expectations because the spot and futures price are expected to move in the same direction in response to the temporary shortfall or surplus. In summary, the two interpretations of backwardation and contango both give rise to a sloped forward curve but the reasons for the slope are very different.

The hedging pressure interpretation of backwardation is illustrated in Figure 7. The Gulf Coast merchant is assumed to be placing a short hedge in Q4 using a F7 futures contract and then lifting the hedge when the futures contract expires in Q7. The height of the last column on the right side of the graph is a measure of the market price of the F7 contract when trading is taking place in Q4. The backwardation gap measures the extent that this futures price is below the expected Q7 Chicago spot price, which is given by the second column in from the right side of Figure 7. If futures contracts were allowed to trade in Q5 and Q6 then from the perspective of a merchant in Q4 the backwardation gap for these two contracts would be smaller than the backwardation gap for the F7 contract because the time until contract maturity is shorter with the Q5 and Q6 contracts. A comparison of Figures 5 and 7 reveals that the backwardation has flattened the forward curve and has even created a negative spread in the F4 and F5 pair of futures contracts.

Figure 7: Gulf Coast Basis with Carryover and Backwardation

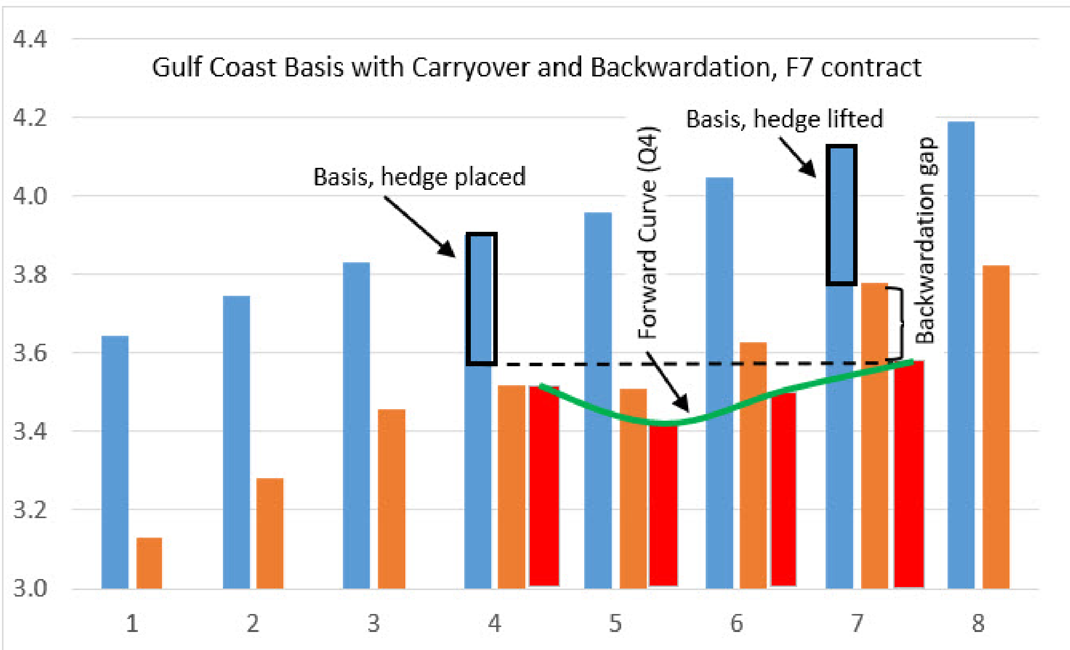



Figure 7 can be used to identify the impact of backwardation on the expected profits of the short hedger in the Gulf Coast market. When the hedge is initiated in Q4 the F7 futures price that includes the backwardation gap is used to calculate the basis. This gap has vanished by Q7 and thus the basis is normal when the hedge is lifted in Q7. The fact that the basis is larger than normal when the hedge is initiated in Q4 and normal when the hedge is lifted in Q7 implies that the increase in the size of the basis is smaller with versus without backwardation. The smaller increase (and in some cases a decrease) implies that expected gross profits for the short hedger are lower with backwardation than without.

The excess shortage interpretation of backwardation can be modeled as follows. Begin by changing the values of the parameters which define the cost of carry schedule. Specifically, replace $m_0 = 0.05$ with $m_0 = -0.04$ and replace $m_1 = 0.005$ with $m_1 = 0.008$. This change implies a larger convenience yield for merchants who hold stock. Assuming that the market stocks out, the equilibrium Gulf Coast spot prices for the first four quarters are as follows: $P_1 = 3.664$, $P_2 = 3.706$, $P_3 = 3.720$ and $P_4 = 3.707$. Notice that the price increases from Q1 through Q3 and then decreases from Q3 to Q4. This occurs because as stocks diminish toward the end of the marketing year the carrying cost switches from a positive to a negative value due to the rising convenience yield.

In the previous example, the forward curves slopes up for Q1 through Q3 and slopes down for Q3 to Q4. Those who use the second interpretation of backwardation such as Routledge, Seppi, and Spatt [2000] believe that the downward sloping forward curve when moving from Q3 to Q4 is evidence of backwardation. Specifically, traders will understand that the shortage of commodity in Q4 is temporarily driving up convenience yield, which in turn is driving up Q4 spot price of the commodity. When new stocks arrive and the convenience yield declines, the spot price will also decline. Thus, the forward curve slopes down even while maintaining the assumption of normal price expectations for traders. When modeling commodity prices with normal price expectations in the first part of this paper, a high convenience yield which causes the forward curve to slope down would be described as an inverted market and not a backwardated market.

# 6 Conclusions

This paper provides an integrated framework for examining the expected profits of a short hedger with a specific focus on midwest corn that is being exported through the U.S. Gulf Coast. The framework ensures that storage decisions result in a set of spot prices which satisfy the equilibrium conditions for intertemporal pricing. Under the assumption of normal price expectations the price of a commodity futures contract is derived and this allows for the calculation of the local basis. The dynamic properties of the basis is the focus of the analysis because of the strong connection between how the basis changes over time and expected hedging profits.

The main result of this analysis is that expected profits from hedging are impacted by two types of seasonality. First, yearly production and storage gives rise to seasonal variations in the carrying cost of the commodity due to the relationship between storage costs, convenience yield and the level of stocks. During the fall harvesting season the relatively high marginal cost of storage and low marginal convenience yield implies that a weak basis that will strengthen over time. Gross expected profits for a short hedge that is initiated shortly after harvest is therefore expected to be positive. If the spot price is expected to decline with a new harvest due to a market stockout then the basis will switch from positive to negative. This weakening of the basis will reduce expected profits from the short hedge, possibly making them negative.

The second type of seasonality that affects profits from hedging is in the spot price discounts that arise due to transportation costs and the spatial separation of the spot markets. For example, the spot price of corn in Kansas City is consistently below the spot price at the Gulf Coast. The size of the discount is largest during harvest and it gradually decreases over the course of the marketing year. The spatial seasonality adds to the seasonality in the carrying cost, thus giving rise to an overall higher degree of seasonality in the expected profits from a short hedge. The seasonality in the basis that can be attributed to the spatial configuration of the spot markets is smaller in the Kansas City market than in the Gulf Coast market. This outcome emerges because of the relatively strong correlation between the Gulf Coast - Chicago price discount and the Gulf Coast - Kansas City price discount.

This analysis also clarifies the dual interpretation of backwardation and contango as determinants of hedging profits. The distinction between the two interpretations is particular important in this analysis because of the seasonality of corn production. It is not surprising that dual interpretations have emerged because with both versions backwardation causes the forward curve to slope down and contango causes the forward curve to slope up. The big difference is that in the first interpretation which emphasises the role of hedging pressure, the slope of the forward curve is impacted because of wedge that is driven between the price of a particular futures contract and what traders expect that contract to be trading at when the contract expires. This bias in price expectations does not appear in the second interpretation, which emphasizes temporary shortages and surpluses as the cause of the sloped forward curve.

An important limitation of this analysis is that the cost of hedging does not have sufficient structure to draw meaningful conclusions about net profits from hedging. For example, Figure 5 shows a strengthening basis at the Gulf Coast for the case where there are carryover stocks from year 1 to year 2. This outcome implies positive gross profits with a short hedge but no conclusions can be drawn about net profits due to lack of structure on the hedging cost function. The reason why the cost function is not specified at a more detailed level is because empirically little is known about convenience yield at the level of an individual merchant. If a merchant places a high value on maintaining stocks over the hedging period rather than selling the stocks as a substitute for the short hedge, then the convenience yield of this merchant is expected to be higher than the convenience yield for the market as a whole, in which case the merchant's expected profits from the short hedge are positive.

A second limitation of this analysis is that it fails to go the next step and use the results to identify the optimal timing of the short hedge. The reason why this topic is not pursued is that doing so would require either making the strong assumption that the hedger is risk neutral or actually modeling the hedger's risk aversion when choosing the timing of the hedge. The simulation results reveal that if the year 1 market stocks out then expected gross profits from the short hedge are negative if the hedge is initiated prior to harvest and lifted after harvest. This result does not imply that a farmer or merchant should never use a short hedge that spans the harvesting period if a market inversion due to a stockout is imminent. Additional assumptions about the level of risk aversion of the merchant or farmer is required before drawing conclusions about whether or not a short hedge should be pursued under these conditions.

# References

Bektemirova, Janat 2014. Basis trading strategies and returns to storage. Phd dissertation, University of Arkansas.

Brennan, Michael J. 1958. The supply of storage. *The American Economic Review 48*(1), 50–72.

Brusstar, Dan and Erik Norland 2015, August 6. Super-contango and the bottom in oil prices. Report, Chicago Mercantile Exchange (CME).

Denicoff, Marina R., Marvin E. Prater, and Pierre Bahizi 2014. Corn transportation profile. `https://www.ams.usda.gov/sites/default/files/media/Corn%20Transportation%20Profile.pdf`. [Online; accessed 29-May-2019].

Dewally, Michael, Louis H. Ederington, and Chitru S. Fernando 2013, 01. Determinants of Trader Profits in Commodity Futures Markets. *The Review of Financial Studies 26*(10), 2648–2683.

Fama, Eugene F. and Kenneth R. French 1987. Commodity futures prices: Some evidence on forecast power, premiums, and the theory of storage. *The Journal of Business 60*(1), 55–73.

Geman, Helyette 2015. *Agricultural Finance: From Crops to Land, Water and Infrastructure*. John Wiley and Sons, Inc.

Gorton, Gary B., Fumio Hayashi, and K. Geert Rouwenhorst 2012, 08. The Fundamentals of Commodity Futures Returns. *Review of Finance 17*(1), 35–105.

Hambur, Jonathan and Nick Stenner 2016. The term structure of commodity risk premiums and the role of hedging. Available at `https://www.rba.gov.au/publications/bulletin/2016/mar/7.html` (2019/06/20).

Hirshleifer, David 1988. Residual risk, trading costs, and commodity futures risk premia. *The Review of Financial Studies 1*(2), 173–193.

Hirshleifer, David 1990. Hedging pressure and futures price movements in a general equilibrium model. *Econometrica 58*(2), 411–428.

Hofstrand, Don 2009, December. Corn and Soybean Price Basis. *Ag Decision Maker File A2-40*.

Johnson, Leland L. 1960. The theory of hedging and speculation in commodity futures. *The Review of Economic Studies 27*(3), 139–151.

Jovanovic, Slobodan 2014. *Hedging Commodities: A practical guide to hedging strategies with futures and options*. Belgrade: Harriman House.

Kluis, Alan 2000, March. Understanding Basis Signals in the Grain Markets. *Corn and Soybean Digest March 31*.

Leuthold, Raymond M., Joan C. Junkus, and Jean E. Cordier 1989. *The theory and practice of futures markets*. Lexington, Mass.: Lexington Books.

Mitchell, John B. 2010. Soybean futures crush spread arbitrage: Trading strategies and market efficiency. *Journal of Risk and Financial Management 3*(1), 63–96.

Routledge, Bryan R., Duane J. Seppi, and Chester S. Spatt 2000. Equilibrium forward curves for commodities. *The Journal of Finance 55*(3), 1297–1338.

Stoll, Hans R. 1979. Commodity futures and spot price determination and hedging in capital market equilibrium. *The Journal of Financial and Quantitative Analysis 14*(4), 873–894.

Telser, Lester G. 1958. Futures trading and the storage of cotton and wheat. *Journal of Political Economy 66*(3), 233–255.

Whalen, Chip 2018, April. Understanding the Cash Market and Futures Curve. *National Hog Farmer April 30*.

Williams, Jeffrey C. 1986. *The Economic Function of Futures Markets*. Cambridge University Press.

Working, Holbrook 1948. Theory of the inverse carrying charge in futures markets. *Journal of Farm Economics 30*(1), 1–28.

Working, Holbrook 1953, 11. Hedging Reconsidered. *American Journal of Agricultural Economics 35*(4), 544–561.

Working, Holbrook 1962. New concepts concerning futures markets and prices. *The American Economic Review 52*(3), 431–459.

Yoon, Byung Sam and Wade B. Brorsen 2002. Market inversion in commodity futures prices. *Journal of Agricultural and Applied Economics 34*, 459 – 476.

# Appendix

## A Derivation of Equilibrium Conditions

To simplify the notation in the analysis below let $Z = m_0/b$ and $m = m_1/b$. That is, $Z$ and $m$ are the intercept and slope, respectively, of the carrying cost function, each normalized by the slope of the inverse demand schedule. As well, in some parts of the analysis it is useful to substitute $S_0$ for $H$ in year 1 because by assumption the level of stocks that are carried out of quarter 0 and into quarter 1 is equal to the year 1 harvest. Finally, let $H_2 \in \{H, H - L\}$ denote the two alternative values for the level of harvest in year 2.

### Solution for Quarters 1 through 4

In the first four quarters uncertainty has yet to be resolved and so there is just one equilibrium value for price ($P_t$), stocks ($S_t$) and consumption ($X_t$). First quarter consumption, $X_1$, is initially treated as a parameter rather than a variable. This allows the endogenous variables in equations (3) through (5) from Section 2 to be solved as functions of time, $t$, year 1 harvest, $H$, and first quarter consumption, $X_1$. Specifically, substitute $P_t = a - bX_t$ and $P_{t+1} = a - bX_{t+1}$ from equation (4) into equation (3) to obtain:

$$X_{t+1} = X_t - Z - mS_t \tag{A.1}$$

After substituting in equation (5) for $S_t$ in equation (A.1), the following expression for $X_t$ emerges:

$$X_t = -mS_{t-2} + (1 + m)X_{t-1} - Z \qquad t = 2, 3, 4 \tag{A.2}$$

Equations (5) and (A.2) can now be solved iteratively to obtain a complete solution. Begin with

$$S_1 = S_0 - X_1 \tag{A.3}$$

Equation (A.2) with $t - 2$ gives

$$X_2 = -mS_0 + (1 + m)X_1 - Z \tag{A.4}$$

From equation (5) it follows that $S_2 = S_1 - X_2$. Substituting in equations (A.3) and (A.4) gives

$$S_2 = (1+m)S_0 - (2+m)X_1 + Z \tag{A.5}$$

Equation (A.2) with $t = 3$ gives $X_3 = -mS_1 + (1+m)X_2 - Z$. Substituting in equations (A.3) and (A.4) gives

$$X_3 = -m(2+m)S_0 + (m + (1+m)^2)X_1 - (2+m)Z \tag{A.6}$$

From equation (5) it follows that $S_3 = S_2 - X_3$. Substituting in equations (A.5) and (A.6) gives

$$S_3 = (1 + 3m + m^2)S_0 - (3 + 4m + m^2)X_1 + (3+m)Z \tag{A.7}$$

Equation (A.2) with $t = 4$ gives $X_4 = -mS_2 + (1+m)X_3 - Z$. Substituting in equations (A.5) and (A.6) gives

$$X_4 = -m(1+m)(3+m)S_0 + (1 + 6m + 5m^2 + m^3)X_1 - (3 + 4m + m^2)Z \tag{A.8}$$

From equation (5) it follows that $S_4 = S_3 - X_4$. Substituting in equations (A.7) and (A.8) gives

$$S_4 = (1 + 6m + 5m^2 + m^3)S_0 - (4 + 10m + 6m^2 + m^3)X_1 + (3+m)(2+m)Z \tag{A.9}$$

The next step is to derive an expression for the equilibrium value of $X_1$ which up until now has been treated as a parameter. Begin by defining $\hat{R} \in \{0, R\}$ as an indicator variable which defines the level of carryover from year 1 to year 2. For $L \leq L^*$ the year 1 market stocks out and $\hat{R} = 0$. Conversely, for $L > L^*$ the market does not stock out and $\hat{R} = R$ units are carried over (the equilibrium value for $R$ is derived below). This specification implies $S_4 = \hat{R}$ (i.e., stocks which are not consumed in the summer of year 1 are carried over to year 2). The solution value for $X_1$ as a function of $\hat{R}$ can now be obtained by setting equation (A.9) equal to $\hat{R}$ and then solving the resulting equation for $X_1$. The desired expression is

$$X_1^* = \frac{\gamma_1 H + \theta_0 Z - \hat{R}}{\theta_1} \tag{A.10}$$

where:

$$\gamma_1 = 1 + 6m + 5m^2 + m^3 \tag{A.11}$$

$$\theta_0 = (3+m)(2+m) \tag{A.12}$$

$$\theta_1 = 4 + 10m + 6m^2 + m^3 \tag{A.13}$$

To finalize the solution for the first four quarters note that expressions for $P_1$ through $P_4$ can be derived by substituting the above expressions for $X_1$ through $X_4$ into the demand schedule, $P_t = a - bX_t$.

## Solution for Final 4 Quarters

There exists a critical value for $L$, call it $L^*$, which determines whether stock will be carried over from year 1 to year 2. Specifically, for $L \leq L^*$ the price appreciation that results from the reduced year 2 supply is insufficient to cover the marginal carrying cost, and so zero carry over (i.e., a year 1 stockout) is optimal. A positive carry over is optimal if the year 2 production shortfall is sufficiently large (i.e., $L > L^*$). The binary nature of the problem implies four possibilities: (1) a year 1 stockout and a normal year 2 harvest; (2) a year 1 stockout and a low year 2 harvest; (3) positive carry over and a normal year 2 harvest; and (4) a positive carry over and a low year 2 harvest.

The previous set of equations with some minor adjustments can be used solve the problem for the final 4 quarters. Year 2 always stocks out and so in this case $\hat{R} = 0$. If year 1 stocks out due to $L \leq L^*$ and if year 2 harvest is normal (i.e., $H_2 = H$) then the conditional solution for the final 4 quarters, $P_t^N$, $S_t^N$ and $X_t^N$, will be identical to the solution for the first four quarters.[17] If year 1 stocks out due to $L \leq L^*$ and if year 2 harvest is low (i.e., $H_2 = H - L$) then the conditional solution for the final 4 quarters, $P_t^L$, $S_t^L$ and $X_t^L$, can be determined by using the previous set of equations with $H_2 = H - L$ substituting for $S_0 \equiv H$. If year 1 does not stock out due to $L > L^*$ then the solution for the last four quarters can be determined by using the previous set of equations except now $H + R$ substitutes for $S_0$ when year 2 harvest is normal, and $R + H - L$ substitutes for $S_0$ when year 2 harvest is low. If these adjustments are substituted

[17] For the final 4 quarters a superscripted "N" on a variable indicates that year 2 harvest was normal (i.e., $H_2 = H$) and a superscripted "L" indicates that the year 2 harvest was low (i.e., $H_2 = H - L$).

into equation (A.10) then expressions for Q5 consumption conditioned on the outcome for $R$ can be written as[18]

$$X_{1,2}^{N}=\frac{\gamma_1(R+H)+\theta_0 Z}{\theta_1} \qquad \text{and} \qquad X_{1,2}^{L}=\frac{\gamma_1(R+H-L)+\theta_0 Z}{\theta_1} \tag{A.14}$$

What remains is solving for the equilibrium value of $R$, which is the level of carry over when $L>L^*$, and solving for $L^*$, which is the threshold loss that triggers carry over. To ensure there is no arbitrage when there is positive carry over it must the case that $E\{P_{1,2}\}=P_{4,1}+m_0+m_1S_{4,1}$ where $E$ is the expectations operator. In other words, the supply of storage must hold across years as well as within a year when stocks are carried over across years, and it must also hold with respect to the expected pricing outcome as well as the actual pricing outcome.

To make this no arbitrage condition more explicit substitute $R$ for $S_{4,1}$ because both variables describe the level of carryover. Next note that the linearity of the demand equation and the equal probability for the two alternative harvest outcomes imply that the expected price in the first quarter of year 2 can be expressed as $E(P_{1,2})=a-\frac{b}{2}\left(X_{1,2}^{N}+X_{1,2}^{L}\right)$. Using equation (A.14) this expression can be rewritten as

$$E(P_{1,2})=a-\frac{b}{\theta_1}\left(\gamma_1 H+\theta_0 Z-0.5\gamma_1 L+\gamma_1 R\right) \tag{A.15}$$

To construct an expression for the price in the fourth quarter of year 1, $P_{4,1}$, it is useful to rewrite the inverse demand equation as $P_{4,1}=a-b(X_{4,1}-\gamma_1 X_{4,1}^*)+b\gamma_1 X_{4,1}^*$. It follows from equation (A.8) that

$$X_{4,1}-\gamma_1 X_{1,1}^*=\rho_0 S_0+\rho_1 \tag{A.16}$$

where

$$\rho_0=-m(1+m)(3+m) \qquad \text{and} \qquad \rho_1=-(3+4m+m^2)Z \tag{A.17}$$

Also note that the $X_{1,1}^*$ term in equation (A.16) is given by equation (A.10). Thus, an alternative expression for $P_{4,1}$ can be written as

$$P_{4,1}=a-b(\rho_0 S_0+\rho_1)+b\gamma_1 X_{1,1}^* \tag{A.18}$$

[18]To distinguish the final 4 quarters from the first four quarters let the first subscript on a variable denote the quarter and the second subscript denote the year. For example, $X_{3,2}$ refers to consumption in the third quarter of the second year.

To complete the no arbitrage condition substitute the expression for $E(P_{1,2})$ from equation (A.15 and the expression for $P_{4,1}$ from equation (A.18) into the no arbitrage equation, $E\{P_{1,2}\} = P_{4,1} + m_0 + m_1 S_{4,1}$. Under the assumption that $L > L^*$ so that some carry over is optimal, the no-arbitrage equation can be solved for the equilibrium carry over:

$$R^* = \frac{1}{m_1}\left[b(\rho_0 S_0 + \rho_1) + \frac{b\gamma_1}{2\theta_1}L - m_0 - b(\gamma_1 + 1)\left(\frac{\gamma_1 H + \theta_0 Z}{\theta_1}\right)\right] \tag{A.19}$$

The equilibrium value for $L^*$ can now be obtained as the value of $L$ which makes equation (A.19) vanish since this ensures a corner solution outcome where the market is indifferent between zero and positive carry over.