Backwardation and Contango in Commodity Futures Markets

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July, 2020 (Draft Copy)

ABSTRACT

This paper uses a competitive stochastic storage model to generate a set of spot and futures prices for corn with a commonly observed seasonal pattern and an endogenous risk premium paid by short hedgers. The main objective is to clarify the hedging and speculation implications of the dual definitions of backwardation and contango, one referring to the difference between the futures price and expected future spot price, and thus a focus on risk premium, and the second referring to the difference between the futures price the current spot price, and thus a focus on the slope of the forward curve. The implications of the two definitions of backwardation and contango are related but due to the presence of storage costs and convenience yield the cause and effect are asymmetric. It is shown that a positive risk premium (normal backwardation) may result in a downward sloping forward curve (backwardation) but a downward sloping forward curve does not directly affect the risk premium. This paper also examines roll yield for institution investors who passively hold long futures contracts. The goal is to clarify an incorrect claim by hedging professionals that negative (positive) roll yields in a contango (backwardated) market result in a financial loss (gain) for investors. The paper concludes with a discussion about pricing inefficiency and the emergence of a contango market in times of unexpected excess inventory.

Key words: backwardation, contango, hedging, speculation, roll yield

JEL codes: G11, G13, Q11, Q14

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1 Introduction

In the analysis of commodity futures markets the concepts of backwardation and contango are important. There currently exists two definitions, and the distinction between these dual defintions is rather subtle. Indeed, the original definition relates to the difference between the futures price the *expected future* spot price, and the subsequent definition relates to the difference between the futures price and the *current* spot price. The expected future spot price is non-observable and so it is not possible to confirm the presence of backwardation and contango with the original definition, whereas it is straight forward to identify backwardation and contango with the subsequent definition. It is likely for this reason that outside of the academic literature (e.g., financial news reports) the subsequent definition of backwardation and contango is used almost exclusively.

The purpose of this paper is to construct a model which clearly shows the economic distinction between the two definitions of backwardation and contango. The analysis begins by noting that the original definition of backwardation is equivalent to Keynesian normal backwardation, and the risk premium that short hedgers are expected to pay to long speculators. Similarly, the original definition of contango implies a risk premium that long hedgers pay to short speculators. The subsequent definition of backwardation and contango pertain to the term structure of futures prices (i.e., the forward curve). Backwardation is said to be present at a particular point in time if the forward curve slopes down. With a negative slope, the spot price is above the price of a short-term futures contract, and and the spreads between futures prices are negative. Contango is said to be present in the opposite case where the forward curve has a positive slope.

Distinguishing between the dual definitions of backwardation and contango requires showing how each definition is linked to the expected profits of hedgers and speculators. The analysis shows that the expected gross profits for a short hedger are positively related to the slope of the forward curve and negatively related to the size of the risk premium.[1]. Thus, expected gross profits for the short hedger may be positive because the market is in contango (subsequent definition) but reduced in size because of the presence of backwardation (original definition). The analysis also shows that the original definition of backwardation and contango are determinants of the net expected profits of speculators whereas there is no direct connection between speculator profits and the subsequent definition of backwardation and contango. This last result appears to contradict the general belief by hedging professionals that passive long investments in commodity futures earn positive (negative) returns due to a positive (negative) roll yield when the slope of the forward curve is negative (positive). The final part of this analysis shows that with the subsequent definition, indeed the roll yield is positive (negative) in a contango (backwardated) market but roll yield determines gross returns and not net returns for the passive long speculator.

The analysis of backwardation and contango takes place within an eight-quarter commodity market model with harvest occurring in Q1 and Q5. Spot prices are determined by consumption and storage decisions, which depend on stock levels, storage costs and convenience yield. The analysis begins after the realization of the Q1 harvest with traders making storage decisions after observing the forecast of the Q5 harvest. In Q2 the forecast of the Q5 harvest is updated, and this update causes the Q2 price and the expected values of the Q3 to Q8 prices to change in response to the change in storage decisions by traders. Traders who own stock use a futures contract which expires in Q3 to hedge against the Q1 to Q2 price change. The short hedge reduces but does not eliminate price risk. The size of the reduction combined with the hedger's level of risk aversion determines the risk premium the hedger is willing to pay and thus the level of normal backwardation. The model is calibrated to the U.S. corn market with the simplifying assumption of zero transportation costs between markets (or, equivalently, a single market location).

The next section provides a brief review of the relevant literature. In Section 3 the model is built and calibrated. In Section 4 the model is used to highlight the dual definitions of backwardation and contango. This includes showing that the shape of the forward curve can lead to incorrect conclusions about expected profits for hedgers and speculators, and how the recent

[1]See Whalen [2018] for a discussion about how the forward curve is used in the marketing of hogs.

surge in institutional investment in commodity futures potentially shifted the long run market outcome from a positive to a negative risk premium. The analysis in Section 5 examines the issue of rollover yield for the passive long investor. Concluding comments are provided in Section 6.

Before proceeding note that in order to avoid use of the rather awkward terms "original" and "subsequent" definitions of backwardation and contango the following conventional is adopted. When referring to the original definition and the presence of a risk premium the terms "normal backwardation" and "normal contango" will be used. When referring to the subsequent definition and the slope of the forward curve, the terms "backwardation" and "contango" will be used.

2 Literature Review

Carter [2012] explains that for a typical agricultural commodity, the spot price is expected to rise over time during the months following harvest, peak in the months leading up to harvest, and then fall throughout the harvest period (see his Figure 3.3). In the no-bias case, where the futures price is a measure of the expected value of the future spot price, it follows that the set of futures prices with different maturities (i.e., the forward curve) will have a similar shape. For example, with an August - September harvest, at a given point in time the following pricing relationship is expected:

$$F_{May} > F_{July} > F_{September} < F_{December} < F_{March}$$

The September to May period is referred to as a normal carrying charge period, and the May to September period is referred to as an inverted market period. There are often deviations from this particular pattern of futures prices, and Carter [2012] provides some explanations for these deviations.

A common way to explain the above pricing pattern is with storage costs and convenience yield [Working, 1948, 1949, Brennan, 1958, Telser, 1958]. The details of this theory are provided in the next section but for now it is sufficient to note that in the absence of a stock out, arbitrage ensures that the price spread between adjacent futures prices (e.g., December and March) is equal to the net marginal carrying charge. This net cost is the difference between the marginal cost of storage and a marginal convenience yield, which is the non-cash benefit that those which handle the physical commodity implicitly obtain from having stocks on hand. As the stocks decline over time the net carrying charge decreases and turns negative just prior to the next harvest. This systematic change in the net carrying charge gives rise to the positive and negative slope portions of the forward curve, as was described above.

Carter [2012] also describes how a shortage of long speculators who are willing to contract with short hedgers may cause the futures price to fall below the expected future spot price. This price gap is an implicit mechanism for short hedgers to pay a risk premium to long speculators. Keynes [1930] was the first to recognize this possibility, and he called the process of short speculators paying risk premium to long speculators normal backwardation. The theory of normal backwardation was worked out in greater detail by Johnson [1960] and Stoll [1979]. Hicks [1939] pointed out that the excess demand for contracts by long hedgers can result in a contango market, in which case the current futures price is above the expected value of the future spot price, and the risk premium flows from long hedgers to short speculators. The combined theories of normal backwardation and contango are sometimes referred to as the hedging pressure theory of commodity futures.

The hedging pressure theory of commodity futures was eventually incorporated into a more comprehensive capital asset pricing model (CAPM) framework. Here the risk premium emerges endogenously because it depends on the degree of systematic risk of the futures price within a well-diversified portfolio (i.e., the beta). There is a large empirical literature both within agricultural economics and general finance on estimating beta values and the corresponding risk premium for commodity futures (e.g., Dusak [1973], Carter, Rausser, and Schmitz [1983], Fama and French [1987], Dewally, Ederington, and Fernando [2013], Hambur and Stenner [2016]). The results are generally highly varied across studies. For example, Dusak [1973] estimated betas for wheat, corn and soybeans in the range of 0.05 to 0.1. By including commodities in the market porfolio and allowing the net position of speculators to vary over the crop year, Carter et al. [1983] show that Dusak's re-estimated betas are in the range of 0.6 to 0.9. A more general conclusion from this literature is that risk premiums are difficult to detect in short-term contracts, and they vary considerably by commodity in long-term contracts.

Despite the lack of evidence of a significant risk premium in agricultural and non-agricultural commodity futures, there has been since the early 2000s strong interest by institutional investors in these instruments. Indeed, Basu and Gavin [2011] noted that investment in commodity index funds grew from approximately $20 billion in 2002 to approximately $250 billion in 2008. No doubt much of this investment was driven by the spectacular growth in commodity prices over the 2004 to 2008 period [Irwin and Sanders, 2011]. With long-only passive investment in commodity index funds the concept of roll yield, and connection between roll yield and the slope of the forward curve is currently emphasized by investment professionals. Indeed, this connection has resulted in the term "super contango" being used to describe a market with a very steep forward curve, something which is particularly relevant for the crude oil futures market. [Brusstar and Norland, 2015, Salzman, 2020].

Gorton and Rouwenhorst [2006] explicitly discuss the dual use of the terms backwardation and contango. For example, they point out that commodities can be in contango (a positive slope for the forward curve) but at the same time have normal backwardation. Agricultural economists generally refer to spreads in futures prices rather than the forward curve, and likely for this reason the terms backwardation and contango (subsequent definition) are seldom used. In the finance literature, the risk premium interpretation of backwardation and contango was in widespread use in the 1970s and 1980s [Grauer and Litzenberger, 1979, Stoll, 1979, Paroush and Wolf, 1989] but in more recent years this interpretation is less common, likely because of the large empirical literature which demonstrates that risk premiums in commodity futures are either non existent or comparatively small. The forward curve interpretation of backwardation and contango likely started in the literature which focused on crude oil futures (e.g., Gabillon [1991]). It appears that the subsequent definitions of backwardation and contango are now routinely used in papers which focus on the term structure of futures prices (i.e., the forward curve) and are not interested in the existence of risk premium [Routledge, Seppi, and Spatt, 2000, Pindyck, 2001].

3 Model

This section is divided into four subsections. The first subsection sets out the assumptions and highlights the theory of storage, as originally developed by Brennan [1958]. Section 2.2 is used to derive the set of equations which govern consumption, storage and prices over time. Of particular importance is the density function which governs the change in the spot price from from Q1 to Q2 in response to the release of an updated forecast of the Q5 harvest. In Section 2.3 futures trading and hedging is incorporated into the model. It is here that hedging pressure is featured in preparation for the analysis of normal backwardation and normal contango. Section 2 concludes by calibrating the model to the U.S. corn market and presenting simulation results.

3.1 Storage Supply and Demand

The single-location market operates for eight quarters, beginning immediately after harvest in the fall quarter of year 1. Demand is constant over time and so the role of speculators in Q1 through Q4 is to estimate the size of the year 2 harvest, which takes place at the beginning of Q5. Updated forecasts about the size of H5 arrive in the market at the beginning of Q2, Q3 and Q4. The updated information results in a revised set of consumption and storage decisions, and this revision causes the current price and the density functions which govern the set of futures prices to change. Initial stocks are assumed to be sufficiently large to ensure that the year 1 market does not stock out in the event of an unusually high forecast for H5. After harvest is realized in Q5, prices remain constant until the market terminates at the end of Q8.

Supply consists of a set of competitive farmers producing a homogeneous commodity and selling this commodity in a competitive cash/spot market. Inverse demand in quarter t is given by $P_t = a - bX_t$ where P_t is the market price and X_t is the level of consumption. Let S_t denote the level of stocks at the end of quarter t. This variable is endogenous in the model except for S_0, which is the exogenous level of stocks which are carried into year 1 and combined with year 1 harvest in Q1, and S_8, which is the level of carry out stocks at the end of Q8. Of particular importance is S_4 because this is a measure of the stocks which are carried out of Q4 in year 1 and combined with year 2 harvest in Q5.

The merchants' marginal cost of storing the commodity from one quarter to the next consists of a physical storage cost and an opportunity cost of the capital that is tied up in the inventory. The combined marginal cost of storage is given by the increasing function $k_t = k_0 + k_1 S_t$.[2] This specification ensures that the marginal storage cost is highest in the fall quarter when stocks are at a maximum, and gradually decline as the marketing year progresses. Merchants also receive a convenience yield from owning the stocks rather than having to purchase stocks on short notice.[3] Let $c_t = c_0 - c_1 S_t$ denote the marginal convenience yield for quarter t. This function decreases with higher stocks because the marginal transaction cost associated with external procurement is assumed to be highest (lowest) when stocks are lowest (highest). Combined storage cost and convenience yield is referred to as the carrying cost. Following [Brennan, 1958] let $m_t = k_t - c_t$ denote the marginal carrying cost for quarter t.

Competition between merchants ensures that the expected compensation for supplying storage, $E\{P_{t+1}\} - P_t$, is equal to the net cost of carry, m_t, provided that stocks are positive (i.e., no stock out). Substituting in the expressions for k_t and c_t, the supply of storage equation can be written as

$$E\{P_{t+1}\} - P_t = m_0 + m_1 S_t \tag{1}$$

where $m_0 = k_0 - c_0$ and $m_1 = k_1 + c_1$. Equation 1 can be interpreted as the intertemporal LOP.

Brennan [1958] defines the demand for storage by first noting that quarter t consumption, X_t, can be written as $X_t = S_{t-1} + H_t - S_t$ where H_t is the level of harvest in quarter t. Inverse demand in quarter t can therefore be expressed as $P_t = a - b(S_{t-1} + H_t - S_t)$, and the demand for storage function is

$$P_{t+1} - P_t = [a - b(S_t + H_{t+1} - S_{t+1})] - [a - b(S_{t-1} + H_t - S_t)] \tag{2}$$

Equation (2) shows that $P_{t+1} - P_t$ is a decreasing function of S_t and thus represents a demand for storage. Brennan [1958] explains that higher stocks carried out of quarter t implies a higher

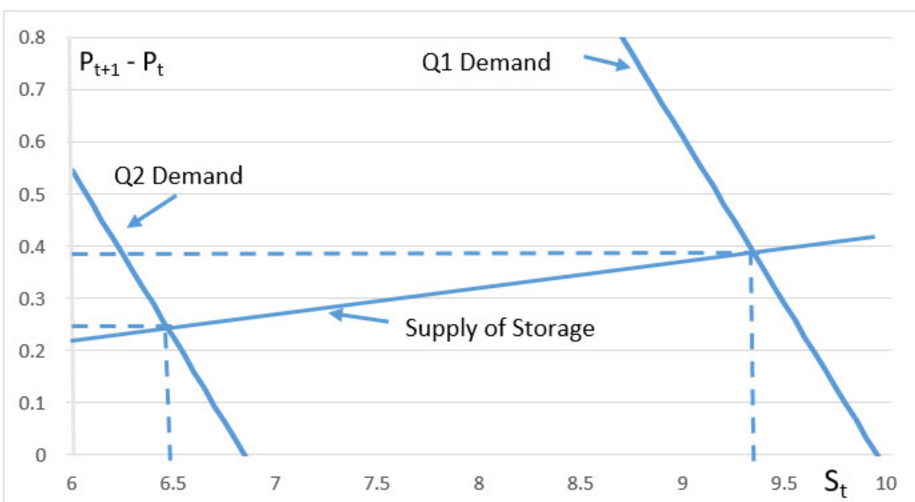
[2]Later in the analysis it is shown that the equilibrium price is a linear function of S_t. Thus, the opportunity cost of capital which is proportional to the commodity's price is assumed to be embedded in the K_0 and K_1 parameters.

[3]A standard explanation of convenience yield is that by having stocks on hand a merchant can fill unexpected sales orders or create sales opportunities that would otherwise not be possible due to the high transaction costs associated with short-notice spot market transactions.

value for P_t since less is available for consumption in quarter t, and lower P_{t+1} since more is available for consumption in quarter $t+1$.

Figure 1 shows the supply and demand for storage for the U.S. corn market. The equations which underlie Figure 1 are derived in the Appendix and the calibration details are provided below. When stocks are high in Q1, the marginal cost of storage is relatively high and the marginal convenience yield is relatively low. The demand for storage is therefore shifted far to the right, and the price increase between Q1 and Q2 is relatively large. The lower level of stocks in Q2 lowers the marginal cost of storage and raises the marginal convenience yield. The resulting inward shift of the demand for storage implies that the price increase between Q2 and Q3 is less than the price increase between Q1 and Q2. For Q3 the marginal convenience yield exceeds the marginal cost of storage, in which case the price change between Q3 and Q4 is negative rather than positive (this is not shown in Figure 1). For Q4 the gap between marginal convenience yield and marginal cost of storage is larger and so the price decrease from Q4 to Q5 is larger than the price decrease from Q3 to Q4. With the arrival of new stocks in Q5, the marginal storage cost surges up, the marginal convenience yield rapidly drops and once again price begins to increase. The full pricing pattern is illustrated in greater detail below.

Figure 1: Q1 and Q2 Simulated Supply and Demand for Storage



3.2 Spot Prices

Rather than using equation (2) explicitly to solve for the set of equilibrium prices, the analysis returns to the core equations. Specifically:

$$P_{t+1} - P_t = m_0 + m_1 S_t \tag{3}$$

$$P_t = a - bX_t \tag{4}$$

$$S_t = S_{t-1} + H_t - X_t \qquad \text{where} \quad H_t = 0 \quad \text{for} \quad t = 2,3,4,6,7,8 \tag{5}$$

Equation (3) is the supply of storage, equation (4) is quarterly demand for the commodity and equation (5) is the equation of motion, which ensures that ending stocks must equal beginning stocks plus harvest minus consumption. Initial stocks are at level S_0, and stocks at the end of Q8 are restricted to equal the predetermined level $\bar{S}_8$.

In the Appendix it shown that the equilibrium Q1 spot price and the expected values of the Q2 and Q3 spot prices as of Q1 can be expressed as linear functions of the known Q1 forecast of the year 2 harvest.[4] The random value for the Q2 spot price given the updated forecast can be expressed as a linear function of the known Q1 forecast and the random Q2 forecast. Similarly, the expected Q3 price given the updated forecast can be expressed as a linear function of the known Q1 forecast and the random Q2 forecast. The corresponding equations can be expressed as

$$\begin{aligned} &P_1^1(H_5^1) = \delta_0^1 + \delta_1^1 H_5^1 \quad \text{and} \quad \bar{P}_2^1(H_5^1) = \delta_0^2 + \delta_1^2 H_5^1 \quad \text{and} \quad \bar{P}_3^1(H_5^1) = \delta_0^3 + \delta_1^3 H_5^1 \\ &\tilde{P}_2^2(H_5^1, \tilde{H}_5^2) = \bar{\delta}_0^2 + \bar{\delta}_2^2 H_5^1 + \bar{\delta}_1^2 \tilde{H}_5^2 \\ &\bar{P}_3^2(H_5^1, \tilde{H}_5^2) = \bar{\delta}_0^3 + \bar{\delta}_2^3 H_5^1 + \bar{\delta}_1^3 \tilde{H}_5^2 \end{aligned} \tag{6}$$

For the P variables in equation (6) the superscript 1 implies that the random Q2 forecast is not yet available and the superscript 2 implies that it is available. Similarly, a tilde ($\sim$) implies that P is random, and a bar (-) implies an expected value. Note that the time-dependent δ parameters are functions of the individual parameters which define the model.

[4] The implicit assumption is the forecasted value of the year 2 harvest is not expected to change over time.

3.3 Futures Prices and Hedging

Assume that in both Q1 and Q2 a futures contract which calls for delivery in Q3 trades in a competitive market (i.e., a Q3 contract). A futures price is said to be unbiased if its current value is equal to the expected value of the spot price when the underlying contract expires. The Q1 price of a Q3 contract can be therefore be expressed as $F_{1,3}(H_5^1) = \bar{P}_3(H_5^1) - \pi_1$ where a zero, positive or negative value for π_1 implies a zero, downward and upward bias, respectively. Later in the analysis a value of π_1 is specified as a function the level of hedging pressure. Immediately after the revised forecast of H_5 is received in Q2 the price of the Q3 futures contract is updated. The futures price that will emerge in Q2 is given by $\tilde{F}_{2,3}(H_5^1, \tilde{H}_5^2) = \bar{P}_3^2(H_5^1, \tilde{H}_5^2) - \pi_2$.

Of interest is the random change in the spot price and futures price given the updated H_5 forecast in Q2. Using the previous equations together with the expressions from equation (6), this pair of random changes can be expressed as

$$\begin{aligned} \tilde{P}_2^2(H_5^1, \tilde{H}_5^2) - P_1^1(H_5^1) &= \bar{\delta}_0^2 - \delta_0^1 + (\bar{\delta}_2^2 - \delta_1^1)H_5^1 + \bar{\delta}_1^2 \tilde{H}_5^2 \\ \tilde{F}_{2,3}(H_5^1, \tilde{H}_5^2) - F_{1,3}(H_5^1) &= \bar{\delta}_0^3 - \delta_0^3 + (\bar{\delta}_2^3 - \delta_1^3)H_5^1 + \bar{\delta}_1^3 \tilde{H}_5^2 \end{aligned} \tag{7}$$

As is shown below, the distributions for this pair of price changes can be plotted if values are assigned to the various parameters and if a density function is specified for the forecast update variable, $\tilde{H}_5^2$.

Merchants who store the commodity between Q1 and Q2 or who combine a Q1 forward sale with a Q2 deferred purchase incur price risk due to the random changes in the spot price. Merchants who store the commodity use a short hedge, which means taking a short futures position in Q1 and offsetting this position in Q2. Merchants who forward sell the commodity in Q1 and purchase the commodity in Q2 to fulfill their delivery obligation use a long hedge. In this case the merchant takes a long futures position in Q1 and offsets this position in Q2. The profits for both the short and long hedge depend on how the basis changes between Q1 and Q2. To see this note that the basis, which is defined as the spot price minus the futures price, can be expressed as $B_1 = P_1^1(H_5^1) - F_{1,3}(H_5^1)$ for Q1, and $\tilde{B}_2 = \tilde{P}_2^1(H_5^1, \tilde{H}_5^2) - \tilde{F}_{2,3}(H_5^1, \tilde{H}_5^2)$ for Q2.

The expression for the change in the basis from Q1 to Q2 can be rearranged and written as the change in the spot price minus the change in the futures price:

$$\tilde{\Delta}_B = \tilde{P}_2^1(H_5^1, \tilde{H}_5^2) - P_1(H_5^1) - [\tilde{F}_{2,3}(\tilde{H}_5^2) - F_{1,3}(H_5^1)] \tag{8}$$

Equation (8) makes it clear that the gross profits of the short hedger (i.e., before subtracting carrying costs) is equal to the change in the basis. With some additional rearranging it can be shown that the gross profits of the long hedger are equal to $-\tilde{\Delta}_B$. Hedging profits for both types of hedgers are stochastic because the quarter when the hedge is lifted (i.e., Q2) is different than the quarter when the futures contract expires (i.e., Q3). Equation (8) is consistent with the standard textbook claim that hedging substitutes basis risk for price risk.

The linearity of the model implies that the variance in the distribution of the change in the spot price between Q1 and Q2 does not depend on the consumption and storage decisions. The same is true for the variance in the distribution of the change in the basis between Q1 and Q2. Consequently, the reduction in price risk that results from a short or long hedge, and the risk premium that merchants are willing to pay to substitute basis risk for price risk via the hedge, take on fixed values.

The net position of short and long hedgers determines the net position of speculators. Regardless of whether these speculators are net long or net short, they require compensation for accepting the risk that is associated with their futures position. Later in the analysis the demand for futures positions by speculators is generalized by accounting for the diversification benefits. With the current assumption, if the number of short hedgers is greater than the number of long hedgers then speculators are net long, and hedging pressure causes the futures price to be bid below the expected spot price by the amount $\pi > 0$. In the opposite case where hedgers are not long the futurs price is bid above the expected spot price by the amount $\pi < 0$.

3.4 Calibration for Simulations

In this section, the pricing model is calibrated to approximately represent the U.S. corn market. Annual USDA data from the Feed Grains Database reveals that the average production and ending stocks for corn for the most recent five years (2015 - 2019) was 14.278 and 2.097 billion bushels, respectively.[5] As well, the average price received by farmers over this time period was \$3.51/bushel. Assume that the size of the year 1 harvest, H_1, and the Q1 forecast of the year 2 harvest, H_5^1, both are equal to 14.278. Initial Q1 stocks, S_0, is set equal to 2.097, and Q8 carry out stocks, S_8, is set equal to 2.0.[6]

The remaining parameters to specify are those which define the marginal cost of storage and the marginal convenience yield. Estimates of the cost of storage exist but there are no reliable estimates of convenience yield. As an alternative, the m_0 and m_1 parameters of the cost of carry function are chosen to match as close as possible the average quarterly price spreads, which are estimated using historical data.

The parameter values are shown in Table 1. This set of values results in a simulated demand elasticity (calculated with average Q1 expected prices and quantities) equal to -0.19, which is similar to the -0.2 estimate that was reported by Moschini, Lapan, and Kim [2017]. These parameter values, together with the various equations in the Appendix, are sufficient to solve for P_1 and the set of expected prices for Q2 through Q8. In Figure 2 a chart of the simulated prices for Q1 through Q4 has been overlain on a chart of the 1980 - 2019 average quarterly prices received by farmers. Each price in the historic data series has been scaled up by a fixed percentage to ensure that the yearly average of the average quarterly prices received by farmers is equal to the yearly average of the simulated prices. Figure 2 demonstrates that the pricing model is well structured because the correspondence between the set of actual and simulated prices is close.

Figure 3 shows the full set of prices assuming both an upward and downward revision of the H_5 forecast at the beginning of Q2. Specifically, the left columns in the triplet are the initial Q1 through Q8 expected prices. The middle (right) columns represent the Q2 through Q8 expected prices assuming that the forecast for the year 2 harvest is decreased (increased) by 5 percent at the beginning of Q2. The relatively large price response to a 5 percent plus or minus change

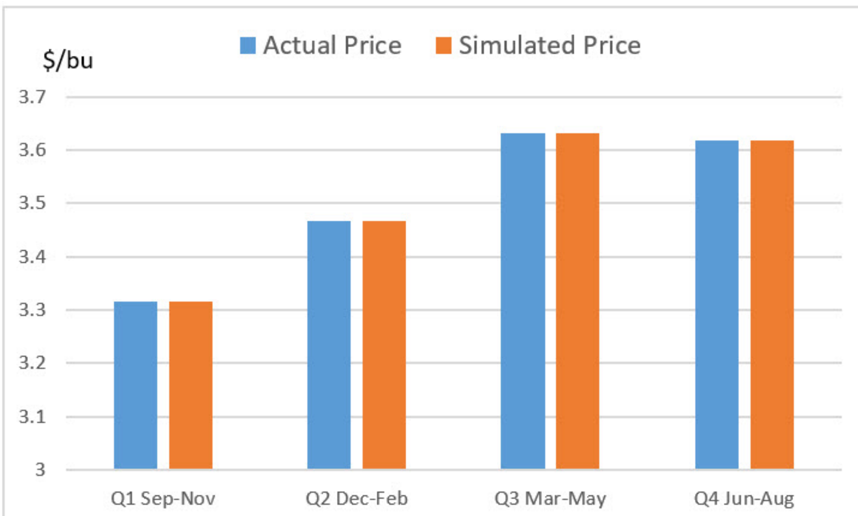
[5]See `https://www.ers.usda.gov/data-products/feed-grains-database/`.

[6]The value chosen for S_8 is not important in the analysis because adjustments in the demand intercept and S_8 have approximately the same impact on the set of prices.

Table 1: Parameters for Base Case Simulation Model

Parameter	**Value**	**Parameter/Elasticity**	**Value**
a (inv. demand intercept)	21.35	H_1 (Year 1 harvest)	14.28
b (inv. demand slope)	5.0	H_5^1 (Q1 forecast of Year 2 harvest)	14.28
m_0 (carry cost intercept)	-0.22	S_0 (Q1 carry in stocks)	2.097
m_1 (carry cost slope)	0.03	S_8 (Q8 carry out stocks)	2.0

Figure 2: Quarterly Corn Prices: Actual (1980-2018 avg.) versus Simulated.

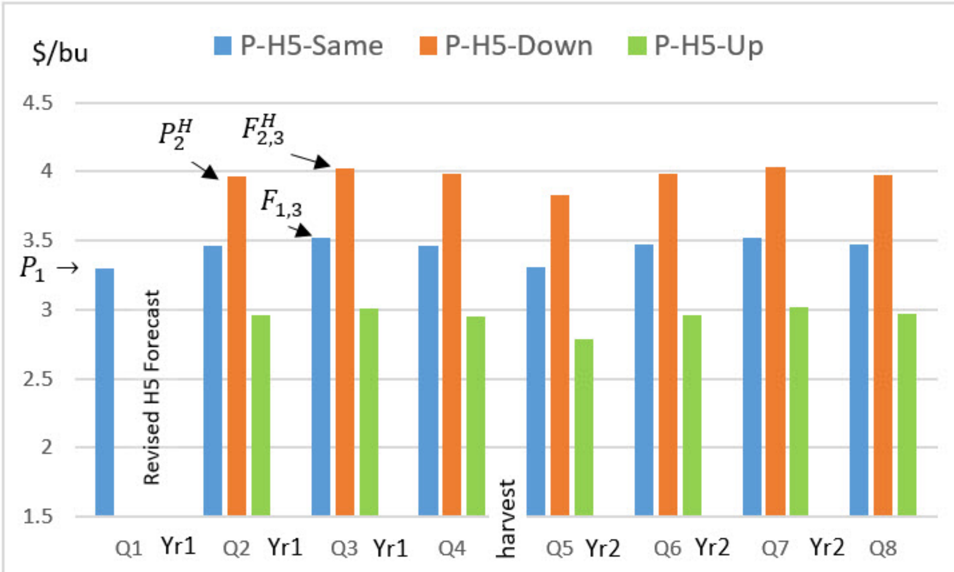


in the expected size of the year 2 harvest can be attributed to the highly inelastic demand.[7] When analyzing Figure 3 keep in mind the theory of storage costs and convenience yield as a determinant of the shape of the pricing pattern over time.

Figure 3 illustrates the risk facing a merchant who stores the commodity between Q1 and Q2, and a speculator who holds a zero bias ($\pi = 0$) futures contract between Q1 and Q2. For the merchant, the cash price of the commodity will increase from P_1 to P_2^H, and for the speculator the futures price will increase from $F_{1,3}$ to $F_{2,3}^H$ if the forecast for H_5 is revised downward. Conversely, both sets of prices will decrease if the forecast is revised upward.

[7]The price response would be much lower if the model incorporated an export market with a relatively elastic demand.

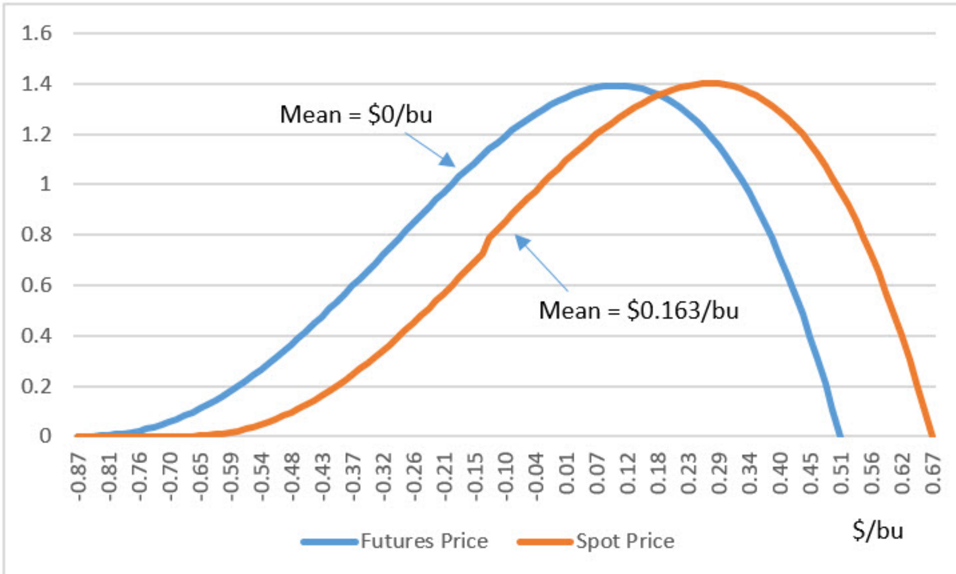
Figure 3: Simulated Corn Prices with Alternative H5 Forecasts.



It is also useful to examine the price risk as reflected by the distributions for the various pricing outcomes. Assume that $\tilde{H}_5^2$ is drawn from a four parameter beta distribution with mean H_5^1 and support $[H_5^{Low}, H_5^{High}]$. For the simulations assume that H_5^{Low} is 5 percent below H_5^1 and H_5^{High} is 8.5 percent above H_5^1. Futher assume that the p and q parameters of the beta distribution are equal to 2 and 3.42, respectively (these values ensure the mean value of the forecast revision is zero).[8] Figure 4 shows the density function for the difference between the Q2 and Q1 spot price and futures price, once again assuming $\pi = 0$. The shape of the distributions are roughly the same but the mean values are different. This is expected because the spot price is expected to increase by an amount equal to the Q1 marginal carrying charge, whereas the futures price is expected to remain constant over time.

[8]The assumed shape of the beta distribution are not important for the general results. The values were chosen to ensure that the corresponding diagrams effectively highlight the key results.

Figure 4: Simulated Density for Q1 to Q2 Change in Spot and Futures Prices



4 Backwardation and Contango

This section is divided into three subsections. In the first subsection the relationship between expected hedging profits and the slope of the forward curve when there is no price bias is made explicit. This relationship connects expected hedging profits with the forward curve measures of backwardation and contango. In the second subsection it is shown that the forward curve in two markets can appear identical, even though there is strong normal backwardation in one market but not in the other market. This is important because it means that the visual properties of the forward curve alone cannot be used to make statements about either expected hedging profits or expected profits for speculators.[9] In the third subsection the potential transition of a market from normal backwardation to normal contango due to strong growth in institutional investment is examined.

[9]A similar claim can be made regarding normal contango. This feature is not emphasized because normal contango is not commonly discussed in the agricultural economics literature.

4.1 Forward Curve and Expected Hedging Profits

Recall from the previous section the gross profits for a short hedger (i.e., profits before deducting carrying costs) are measured by the change in the basis between when the hedge is initiated and when it is lifted. To see the relationship between the change in the basis and the slope of the forward curve, pass the expectations operator through equation (8) to obtain $E_1\{\tilde{\Delta}_B\} = E_1\{\tilde{P}_2^2(H_5^1, \tilde{H}_5^2)\} - P_1(H_5^1) - [E_1\{\tilde{F}_{2,3}(H_5^1, \tilde{H}_5^2)\} - F_{1,3}(H_5^1)]$. The expression in square brackets is equal to the Q1 to Q2 pricing bias, π. Moreover, $E_1\{\tilde{P}_2^1(H_5^1, \tilde{H}_5^2)\}$ is equal the Q1 price of a Q2 futures contract assuming that such a contract exists. If a futures contract also existed in Q1 then arbitrage ensures that the value of the Q1 contract at the point of expiry is equal to P_1^1. This means that the expression for the expected change in basis can be rewritten as

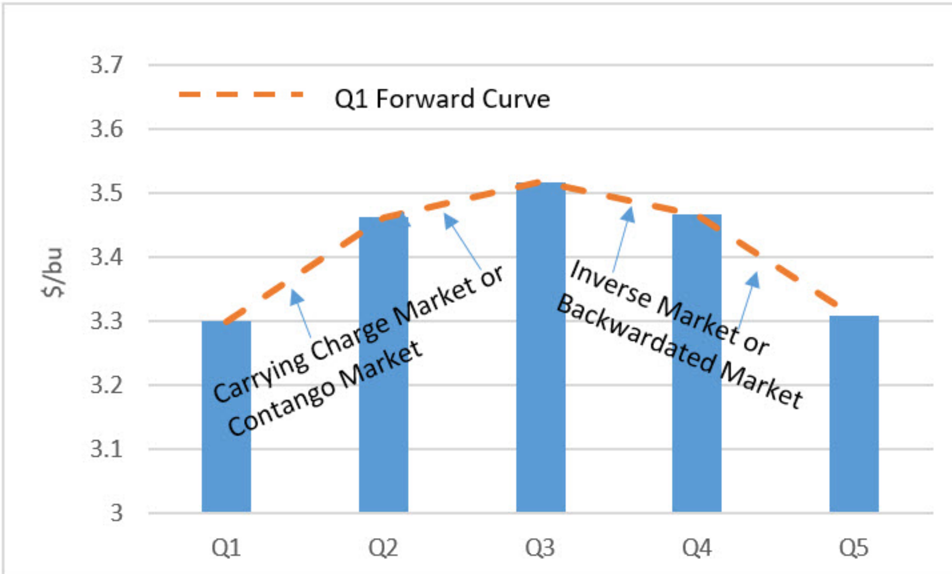
$$E_1\{\tilde{\Delta}_B\} = F_{1,2}(H_5^1) - F_{1,1}(H_5^1) - \pi \tag{9}$$

If there is no price bias from hedging pressure (i.e., $\pi = 0$) then equation (9) shows that expected gross profits for the short hedger is equal to the spread in the Q1 and Q2 futures prices. This spread is a measure of the slope of the forward curve between Q1 and Q2. Thus, in the absence of a price bias, the expected gross profits for the short hedger is equal to the Q1 slope of the forward curve.

The dashed line in Figure 5 is the forward curve for the first five quarters of the simulated corn market with the assumption of zero price bias (i.e., $\pi = 0$) and a futures contract which expires in each of the five quarters. Notice that the first segment of the curve is upward sloping due to relatively large stocks and thus a positive carrying charge. In this contango section of the forward curve, expected gross profits for the short (long) hedger are positive (negative). The second section of the forward curve is downward sloping due to a high convenience yield and thus a negative carrying charge (i.e., the market is inverted). In this backwardated section of the forward curve, expected gross profits for the short (long) hedger are negative (positive). This connection between the slope of the forward curve and expected gross profits for the hedger is seldom emphasized in the agricultural economics literature.

At this point it is worth emphasizing that a positive slope for the forward curve (i.e., contango) or a negative slope of the forward curve (i.e., backwardation) has no impact on the ex-

Figure 5: Forward Curve Definition of Backwardation and Contango



pected profits of the speculator. Figure 5 was constructed with the assumption of zero price bias and as such the expected profits from taking a long or short futures position are equal to zero. This outcome is consistent with Figure 3, which shows that the mean of the distribution of the change in the futures price between Q1 and Q2 is zero. This outcome has important implications for the last section of this paper, which examines the roll yield for institutional investors.

4.2 Normal Backwardation

In this section the implications of normal backwardation are highlighted. To achieve a sharp focus, assume there is no convenience yield and the marginal cost of storage is constant at level m per quarter. This assumption means that the spot price is expected to rise by a constant amount m over time. Normal backwardation implies that short hedging pressure results in a futures price which is biased downward (i.e., $\pi > 0$). It is reasonable to assume that the level of bias increases for contracts with a longer time to expiry because the longer a merchant holds unpriced inventory the greater the variance in the distribution of the eventual selling price for the commodity. For the simulations below assume that $\pi = \beta n^{\gamma}$ where n is the number of quarters until the contract expires and $\gamma > 1$.

Table 2 shows for Q1 through Q5 the futures price minus the current spot price (i.e., the negative of the basis) at different points in time and for contracts with different maturities. The columns of Table 2 contain the data for the forward curves, the first being the most important because it reflects the forward curve for Q1. Notice that the normal backwardation price bias is largest for the Q5 contract and steadily decreases for contracts with a shorter maturity. The rows of Table 2 show how the price spread for a particular contract changes over time. For example, the last row shows how the gap between the futures price and spot price decreases over the five quarters. Table 2 makes it clear that normal backwardation, as measured by $\pi = \beta n^{\gamma}$, affects both the shape of the forward curve and the time path for the price of a particular futures contract.

Table 2: Expected Futures Price Spreads with Normal Backwardation

	Current Quarter				
Expiry	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**
Q1	0	na	na	na	na
Q2	$m - \beta(1)^{\gamma}$	0	na	na	na
Q3	$2m - \beta(2)^{\gamma}$	$m - \beta(1)^{\gamma}$	0	na	na
Q4	$3m - \beta(3)^{\gamma}$	$2m - \beta(2)^{\gamma}$	$m - \beta(1)^{\gamma}$	0	na
Q5	$4m - \beta(4)^{\gamma}$	$3m - \beta(3)^{\gamma}$	$2m - \beta(2)^{\gamma}$	$m - \beta(1)^{\gamma}$	0

It is important to note that Table 2 is consistent with the underlying no arbitrage restrictions. A riskless portfolio can be constructed because arbitrage ensures convergence of the spot and futures prices – for the case at hand this implies $\tilde{F}_{3,3} = \tilde{P}_3$.[10] Suppose the commodity is purchased in Q1 and sold in Q3, and this long position is hedged with a short Q3 futures contract. Zero joint profits for this portfolio, including the cost of storing the commodity, can be expressed as $\tilde{P}_3 - P_1 + F_{1,3} - \tilde{F}_{3,3} - 2m = 0$. Using $\tilde{F}_{3,3} = \tilde{P}_3$, this equation reduces to $F_{1,3} = P_1 + 2m$. It

[10]If these conditions did not hold then the commodity could be purchased in the spot market and immediately delivered against the short futures contract for a profit, or the commodity could be borrowed, immediately sold in the spot market and the loan immediately repaid by taking delivery of the commodity using a long futures contract.

can now be seen that normal backwardation, which results in $F_{1,3} \leq P_1 + 2m$, does not violate the no-arbitrage condition.

Figure 6 shows the simulated expected time paths for the spot price and for the Q2 through Q5 futures contracts, achieved by setting $m_0 = 0.2$, $\beta = 0.1$ and $\gamma = 1.1$. The strong backwardation is revealed by the relatively steep expected pricing paths for the various futures contracts. In each of the first four quarters the slope of the forward curve is given by the vertical distance between the expected pricing paths. The slope of the Q1 forward curve as measured by the price spreads is approximately constant at \$0.10/bu. Notice that the forward curves become slighly steeper as time progresses toward Q4. Figure 6 shows that despite the strong normal backwardation, the market is in a state of contango.

Figure 6: Futures Prices with Contango and Normal Backwardation

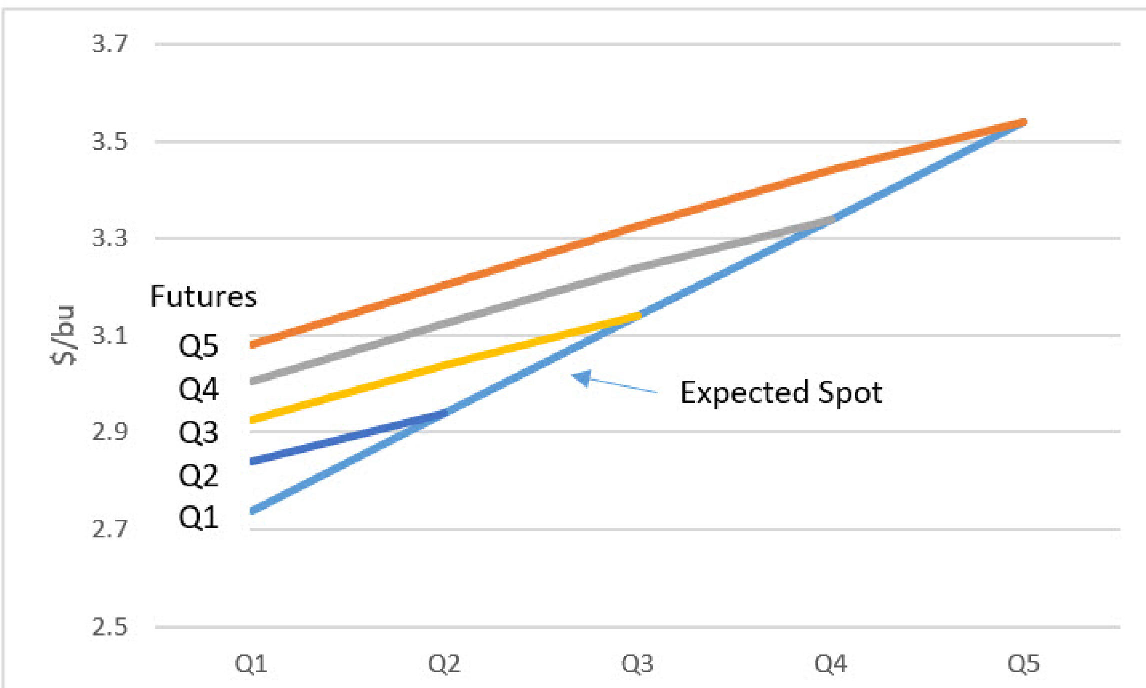
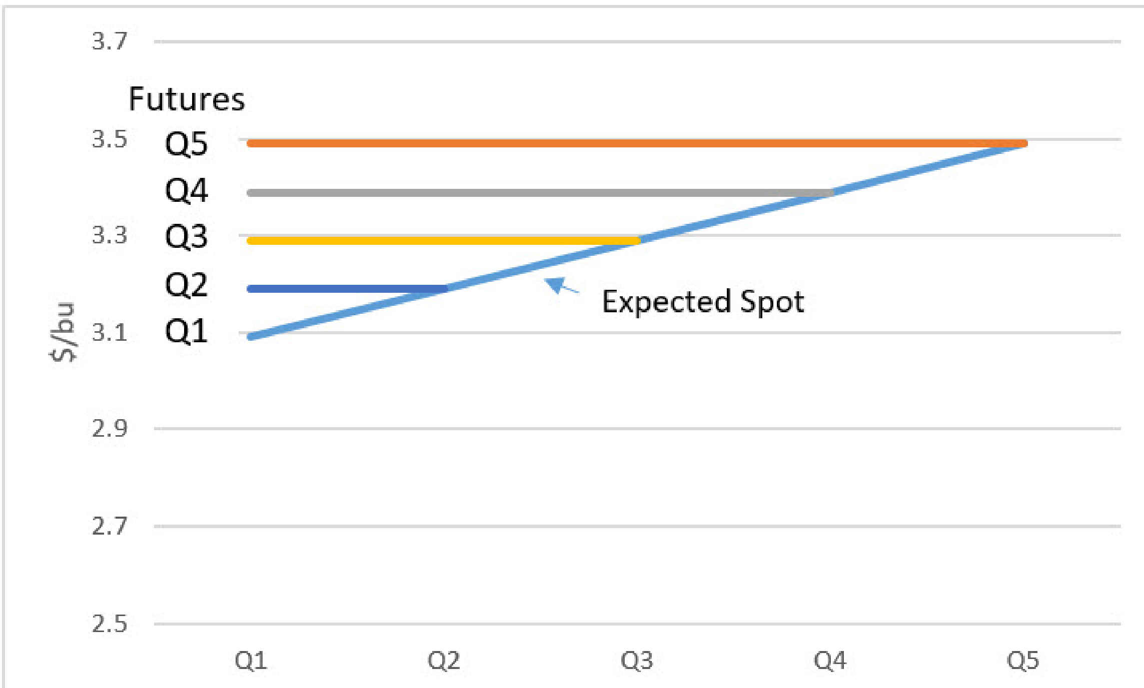


Figure 7 is similar to Figure 6 except in this case there is no backwardation (achieved by setting $\beta = 0$) and the linear storage costs are weaker (achieved by setting $m_0 = 0.1$). The lack of backwardation is revealed by the flat time paths for the expected price of the various futures contracts. Notice that in Q1 the slope of the forward curve as measure by the vertical distance between the pricing paths is constant and equal to 0.1. Thus, the slope of the Q1 forward curve in Figure 7 is very similar to the slope of the Q1 forward curve in Figure 6, despite the fact that there is no normal backwardation in the former and strong normal backwardation in the latter. This comparison reveals that the slope of the forward curve cannot be used to make inference about how the price of a particular futures contract is expected to change over time.

Figure 7: Futures Prices with Linear Storage Costs



The comparison of Figures 6 and 7 leads to an important conclusion. Expected gross hedging profits only depend on the slope of the forward curve and not whether the slope emerges because of a combination of high marginal storage costs and strong backwardation, or low marginal storage costs and no backwardation. In contrast, profits for a speculator are not precisely defined by the slope of the forward curve. Indeed, the slope of the forward curves in Figures 6 and 7 are approximately equal but in the first case the long speculator expects relatively large profits and in the second case zero profits.

4.3 Normal Contango

In the previous section the presence of normal backwardation or normal contango depended on the direction of the hedging pressure (i.e., whether hedgers are net short or net long). The commodity price spikes in the mid 2000s and the simultaneous surge in investment in commodity futures by index and hedge funds induced economists to consider an alternative pathway for the futures price to be bid away from the expected future spot price. Hamilton and Wu [2014, 2015] and Vercammen and Doroudian [2014] constructed models where the passive demand for long futures by institutional investors raised the equilibrium futures price above the expected future spot price (i.e., normal contango). A simplified version of the Hamilton and Wu model is combined with the model used in this paper to show how the recent growth in demand for long agricultural commodity futures by institutional investors potentially changed the long run market outcome from normal backwardation to normal contango.

In the simplest version of the Hamilton and Wu [2014, 2015] model institutional investors have a perfectly inelastic demand for long futures positions at level N_I, and their futures position is continually rolled forward through time as existing contracts expire and new contracts become available. Hamilton and Wu focus on the arbitrageurs who take short positions to facilitate the rolling long positions of the institutional investors. This current analysis differs from Hamilton and Wu in that the net short position of hedgers are added to the demand for short positions by arbitrageurs before determining the market equilibrium.

Recall from the analysis above that short hedgers face a fixed reduction in the variability of profits when converting price risk to basis risk through hedging. Assuming a fixed risk premium at level RP_H, the demand for contracts by short hedgers is perfectly elastic at level RP_H for the $N_H \equiv S_1$ units of the commodity which are hedged in Q1. Prior to the surge in demand for long contracts by institutional investors, assume that $N_I < N_H$, which implies that arbitrageurs have a net long futures position. After the surge in demand, assume that $N_I > N_H$, which implies that arbitrageurs have a net short futures position.

In addition to taking a long or short position in the futures market in Q1, arbitrageurs invest in a well diversified bundle of financial assets which return in Q2 a random cash flow $\tilde{R}$ with mean $\bar{R}$. Let W denote the Q2 return to the arbitrageurs' portfolio as a whole. Equation 7 shows that the change in the futures price between Q1 and Q2 is a linear function of $\tilde{H}_5^2$, which is the random update in the forecast of the year 2 harvest. It follows from Equation 7 that the expected value of the change in the futures price is the price bias variable, π. As well, the variance in the change in the futures price is given by $\sigma_F^2 = (\bar{\delta}_1^3)^2 Var(\tilde{H}_5^2)$. Let $COV(\tilde{H}_5^2, \tilde{R})$ denote the covariance between the forecast update and the returns from the arbitrageur's diversified portfolio. The arbitrageur is assumed to maximize a mean-variance utility function of the form $U = E(W) - \lambda Var(W)$. With the above assumptions, this function can be written as

$$U = N_A \pi + \bar{R} - [\lambda N_A^2 \sigma_F^2 + \sigma_R^2 + 2N_A \bar{\delta}_1^3 COV(\tilde{H}_5^2, \tilde{R})] \tag{10}$$

Within equation (10) the variable N_A takes on a negative (positive) value if the arbitrageur is net long (short).

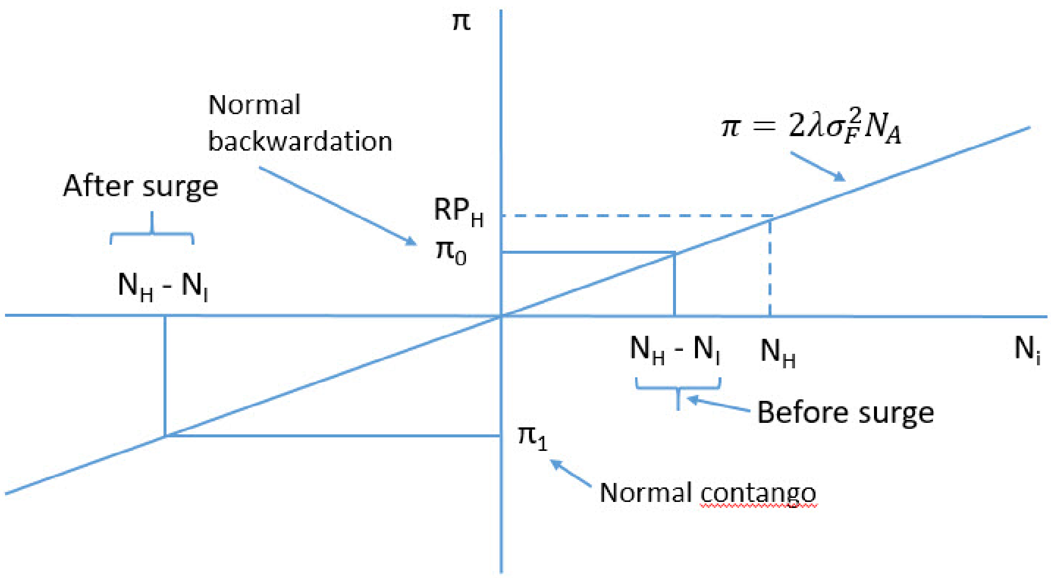
The first order condition for choosing N_A to maximize U can be expressed as

$$\pi = 2\lambda \bar{\delta}_1^3 COV(\tilde{H}_5^2, \tilde{R}) + 2\lambda \sigma_F^2 N_A \tag{11}$$

In the simple case where $COV(\tilde{H}_5^2, \tilde{R}) = 0$ equation (11) shows that the arbitrageur requires the futures price to be below the expected spot price (i.e., $\pi > 0$) in order to take a long futures position (i.e., $N_A > 0$) and opposite for a short futures position. A positive value for $COV(\tilde{H}_5^2, \tilde{R})$ strengthens this result and a negative value potentially reverses it. This is expected because a large negative value for $COV(\tilde{H}_5^2, \tilde{R})$ implies strong diversification benefits from holding the futures contract and thus a willingness to pay rather than requiring compensation to accept the futures price risk. In real world applications it is unlikely that $COV(\tilde{H}_5^2, \tilde{R})$ is large enough to switch the positive relationship between N_A and π in equation (11).

Figure 9 shows how the surge in institutional investors taking passive long positions has potentially converted agricultural markets from normal backwardation to normal contango. The dashed schedule shows the perfectly elastic demand for short contracts by hedgers up to level N_H, which corresponds to the level of inventory to be hedged. The right panel shows that prior to the surge in demand for long contracts, arbitrageurs faced a perfectly inelastic demand for long contracts at level $N_H - N_I$. Using the $\pi = 2\lambda\sigma_F^2 N_A$ supply function, which is the upward sloping schedule in Figure 8, it can be seen that prior to the surge in demand by institutional investors, the equilibrium value of π is positive, and thus the outcome is normal backwardation. After the surge in demand, the combined demand by hedgers and institutional investors has switched from net short to net long (see the left panel in Figure 8). Facing a net demand for short contracts, the risk premium charged by arbitrageurs implies a negative value for π and thus a normal contango outcome.

Figure 8: Transition to Normal Contango with Surge in Institutional Positions



It may seem unrealistic to model the passive long investors as tolerating an expected loss on their hedge fund investment. In real world markets futures prices have occasionally trended up over extended periods of time due to unanticipated growth in demand (e.g., 2004 to 2007). Investors with naive expectations may well be attracted to investing in a commodity hedge or index fund in the presence of normal contango if the past returns have been sizeable.

5 Rollover Yield

Professional market analysts often note that investors who passively roll over long positions in commodity futures should expect negative rollover yield in a contango market and a positive rollover yield in a backwardated market. The following examples support this observation.

"And the contango structure in most commodity futures markets means that long futures positions are being rolled forward at a loss... (In a contango market, short futures positions can be rolled forward at a profit, while long futures positions incur a loss every time they are extended)." (Kemp 2015)

"Why Investors Should Be Wary of Contango...After contango literally ate half of his returns, Alex no longer ignores the shape of the futures curve." (Cardwell, 2017)

"If the oil market structure shifts into contango, an industry term for when spot prices are trading below contracts for future delivery, then an ETF might have to sell its contracts at the lower price, then buy the next month's contract at a higher price just to maintain its holdings." (Sheppard, 2020)

"Contango tends to cause losses for investors in commodity ETFs that use futures contracts, but these losses can be avoided by buying ETFs that hold actual commodities." (Chen, 2020)

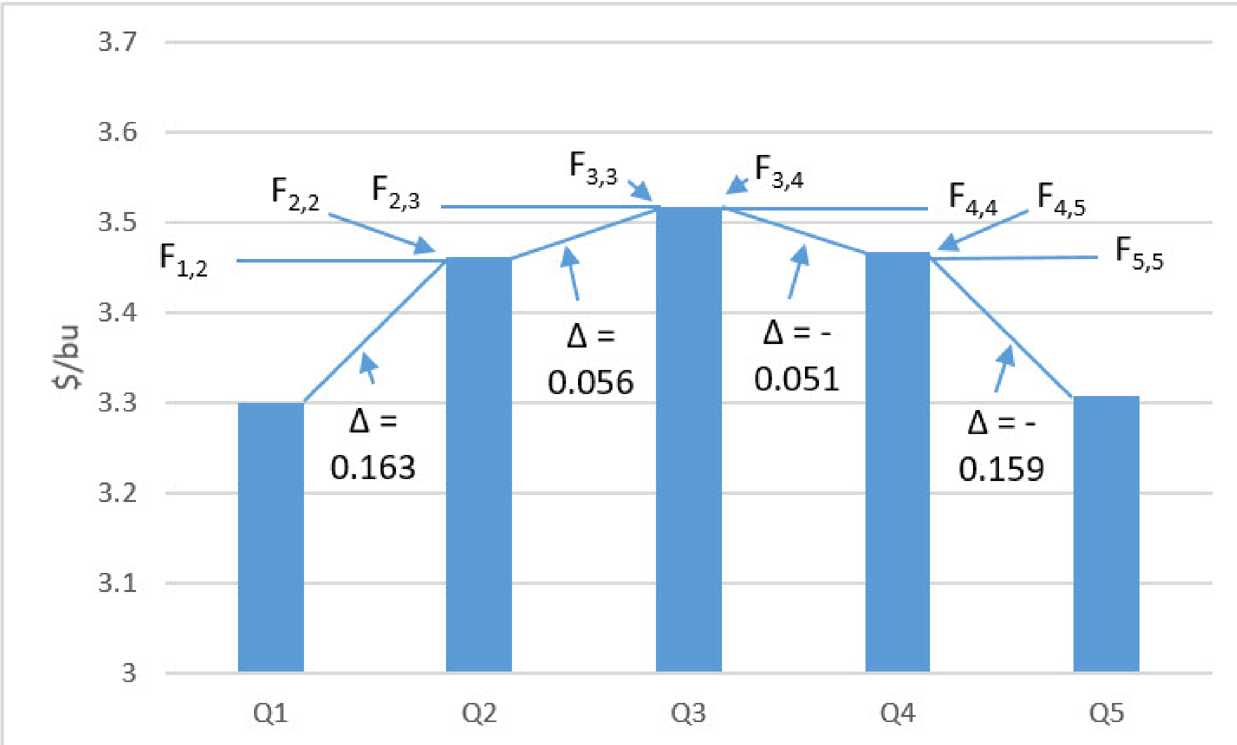
In general, these statements about negative (positive) profits for a passive long speculator in a contango (backwardated) market are not accurate. It is true that the roll yield is negative (positive) in a contango (backwardated) market. However, to make a statement about financial return, the roll yield must be adjusted to account for the cost of carrying the physical commodity, including observable storage costs and non-observable convenience yield. Moreover, the positive or negative roll yield has nothing to do with the fact that expiring contracts are rolled into new contracts, generally at a higher or lower price. This concern about the incorrect interpretation of the roll yield has been acknowledged by Main et al. (2015) and Bessembinder (2018). However, the issue is not likely familiar to most agricultural economists and thus warrants repeating in this paper.

The roll yield is defined as the difference between the profit or loss on a futures contract and the change in the underlying spot price. If an investor could purchase the commodity, store it at zero cost and receive zero convenience yield, then the roll yield would reflect the increase in net profit holding a futures contract rather than investing in the physical commodity. In this case, holding a futures contract in a contango (backwardated) market rather than holding the spot commodity would result in financial loss (gain). Because storage costs and convenience yield are not considered in the calculation of roll yield, this measure should be interpreted as a gross gain or loss. It is generally not possible to convert the roll yield to a net financial return because storage costs and convenience yield cannot be measured.

Figure 9 shows the calculated roll yield for the first five quarters of the simulated market. The assumption of normal price expectations implies that futures prices are expected to remain constant over time and as such expected profits are zero for all futures contracts. Between Q1 and Q2 the market is in contango and the spot price is expected to increase by \$0.163/bu. In this case the calculated roll yield is given by $RY_{1,2} = 0 - 0.163 = -0.163$ dollars per bushel. Similarly, the roll yield is equal to -\$0.056 between Q2 and Q3. The intertemporal LOP implies that the marginal carrying cost is equal to \$0.163/bu between Q1 and Q2, and \$0.056 between Q2 and Q3. Thus, the expected roll yield is equal to the negative of the marginal carrying cost. As previously noted, ignoring this carrying cost when calculating the return on investing in a rolling long futures position gives rise to incorrect conclusions about the profitability of the futures contract. This is particularly true in the crude oil market where oil gluts typically cause steep forward curves due to high storage costs.

The right side of 9 shows why the roll yield is positive in a backwardated market. For example, the roll yield between Q3 and Q4 can be expressed as $RY_{3,4} = 0 - (-0.051) = 0.051$ dollars per bushel. Similarly, the roll yield between Q4 and Q5 is equal to \$0.159/bu. It is easy to see why some investors believe that the positive roll yield in a backwardated market implies positive investment profits from holding a futures contract relative to investing in the physical commodity. Indeed, it seems obvious that breaking even on the futures contract rather than facing a declining spot price is a profitable strategy, at least in relative terms. What these investors are failing to realize is that by holding a futures contract rather than investing in the physical commodity the convenience yield is lost. After this lost convenience yield is accounted

Figure 9: Roll Overs with a Positive Carrying Charge and Normal Price Expectations



for, the net profitability of holding the futures contract and holding the physical commodity are the same, at least for the case at hand.

It is unfortunate that the term roll yield was used to define the performance of a futures contract relative to the change in the spot price. As shown in the quotes above, commodity market professionals sometimes mistakenly attribute the negative roll yield in a contango market to the price difference for the futures contract on the day of the roll over. With reference to Figure 9 the argument goes that closing out the long futures position for the expiring Q2 contract at price $F_{2,2}$ and rolling into a new contract at price $F_{2,3}$ generates a loss because the investor is forced to sell low and buy high. In actual fact, the size and sign of the price spreads within a set of futures contracts do not affect expected profits. All that matters is the difference between the futures price when the rollover position is complete versus initiated.

6 Discussion and Conclusions

The main goal of this paper was to critically examine the dual definitions of backwardation and contango in the context of commodity futures markets. The paper began by noting that by outward appearance the two definitions are very similar. Indeed, (normal) backwardation and (normal) contango, is concerned with the difference between the current futures price and the expected spot price, whereas backwardation and contango is concerned with the difference between the current futures price and the current spot price. Viewed differently, normal backwardation/contango is about the expected time path of the futures price whereas backwardation/contango is about the slope of the forward curve (i.e., term structure) at a given point in time. Normal backwardation (contango) decreases (increases) the slope of the forward curve and in that sense the degree of backwardation/contango depends on the level of normal backwardation/contango. However, the reverse is not true because backwardation/contango is not necessary nor sufficient for the presence of normal backwardation/contango. This asymmetry in cause and effect can be attributed to the marginal carrying cost, which is the net contribution of storage costs and convenience yield.

In the world of professional investing and hedging the concept of normal backwardation/contango is seldom discussed. This is not surprising because the level of normal backwardation/contango cannot be observed, and empirical estimates of the magnitude of the phenomena are generally small and inconsistent across time periods and different commodities. Instead, the degree of backwardation and contango as measured by the slope of the forward curve is commonly identified as a major determinant of roll yield, and roll yield is typically equated with financial return. As was demonstrated in this paper, if normal backwardation/contango is absent, then the slope of the forward curve is a good measure of the expected gross returns to a short hedger and it has no impact on the expected profits for both active speculators and institutional investors. In this paper roll yield was shown to be a measure of the gross return rather than net return from investing in a futures contract rather than holding the physical commodity. By ignoring storage costs and convenience yield, and equating the gross return of holding a futures contract with net financial return, investing professionals are incorrectly critical of the actual financial performance of holding a commodity futures contract.

Due to space constraints, this paper was unable to formally examine backwardation and contango in the context of short term pricing disequilibrium and market inefficiencies. Beck [1994] and others have pointed out that a gap between the futures price and the expected future spot price (i.e., a futures price bias) may be due to either a risk premium caused by normal backwardation/contango or a market inefficiency. Using a cointegration framework to distinguish between the two effects, Beck [1994] concluded that market inefficiency explains futures price biases for most commodities but not all of the time. Similarly, McKenzie and Holt [2002] concluded that the futures markets for the major agricultural commodities are unbiased in the long run but they do exhibit short-run inefficiencies and pricing biases.[11]

Pricing inefficiency has potentially important implications for backwardation and contango through the physical delivery mechanism for settling a commodity futures contract. For example, with the onset of the COVID-19 pandemic, a rapid build up of oil inventories and a severe shortage of oil storage capacity caused expiring May, 2020 crude oil futures to trade at a negative price [Hansen, 2020]. More generally, according to Saefong [2020] there was very little spread between the prices of the May through October futures contracts up until the onset of the pandemic but the spreads rapidly increased and the market experienced steep (super) contango when the pandemic arrived in mid March of 2020. Saefong [2020] attributes the super contango to traders' uncertainty about the short to medium term availability of oil storage capacity.

The idea that information asymmetries and bottlenecks cause pricing inefficiencies is much more appealing than the alternative hypothesis that traders make irrational decisions and/or fail to recognize arbitrage opportunities. Indeed, it is well known from the spatial pricing integration literature that capacity constraints in transportation networks typically reduce the level of spatial pricing integration [Birge, Chan, Pavlin, and Zhu, 2020]. Based on this line of reasoning, constraints in the U.S. grain storage and transportation network is likely to occasionally create a strong contango, similar to that observed in the futures market for crude oil. To see

[11] An anonymous referee identified this literature and indicated its relevance for this current study of backwardation and contango.

how this may work, monthly data on the spread between the USDA spot price for corn at the Gulf Coast and St. Louis was collected for the period January, 2004 to January, 2020.[12] After confirming stationarity, the price spread was regressed on an index of the price of barge services between St. Louis and the Gulf Coast, with the error term specified as a seasonal ARIMA [SARIMA(1,0,1,12)].

The price spread regression revealed a strong association between the spread in spot prices between the U.S. midwest and the U.S. Gulf Coast and the cost of transporting the corn between these two locations. Despite strong seasonality in the price of barge services, the monthly seasonal lag in the error term was also statistically significant. Moreover, the fitted values revealed that the price spread follows the usual seasonal pattern (i.e., relatively large in the months following harvest and relatively small in the months leading up to harvest). This pair of results suggest that barge capacity is being rationed in the post harvest period because the price of barge services is not clearing the market. The near-contract futures price for corn closely follows the St. Louis spot price because St. Louis is a designated futures contract delivery location. Consequently, asymmetric information concerning barge availability in the short to medium term is likely to cause the slope of the forward curve and the corresponding strength of the underlying contango to vary over time and occasionally surge, similar to that which has been described in the market for crude oil.

In the early agricultural economics literature on commodity futures, the goal was to model the cyclical nature of annual crop production, uncertain harvest volumes, seasonal convenience yield and the risk premium for farmers and small scale grain merchants and processors. Beginning in the mid 2000s agricultural economists have largely shifted their focus to the so-called commodification of food, which can be attributed to relatively large inflows of capital from institutional investors in agricultural commodity index funds, and the strengthening of the linkage between energy and food markets due to the large volume of corn currently being used to produce ethanol. As a result of this shift in focus it is likely that the alternative definition of backwardation and contango, which currently enjoys widespread use among investment profes-

[12]The data and empirical results will **eventually** be available in an on-line supplement.

sionals and in general finance, will gradually become mainstream in agricultural economics as well.

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Appendix

A Derivation of Conditional Demand for Storage

Equation (2) is the dynamic demand for storage equation. The goal is to derive a conditional quarterly demand for storage of the form $P_{t+1} - P_t = A_t + B_t S_{t-1} + C_t S_t$ for $t \in \{1, ..., 7\}$ where A_t, B_t and C_t are time dependent groups of parameters, and the implicit assumption is that storage in quarters beyond t is along the equilibrium path. With this specification, $A_t + B_t S_{t-1}$ is the intercept of the demand for storage in quarter t conditioned on stocks brought into quarter t as measured by S_{t-1}, and C_t is the slope. To derive expressions for A_t, B_t and C_t it is useful to first establish the following dynamic variable definitions.

$$\eta_t = \begin{cases} \frac{b}{m_1 + 2b - b\eta_{t+1}} & \text{if } t = 1, ..., 6 \\ 0 & \text{if } t = 7 \end{cases}$$

$$\lambda_t = \begin{cases} \frac{-(m_0 - b\lambda_{t+1}) + bH_t}{m_1 + 2b - b\eta_{t+1}} & \text{if } t = 1, 5 \\ \frac{-(m_0 - b\lambda_{t+1})}{m_1 + 2b - b\eta_{t+1}} & \text{if } t = 2, 3, 6 \end{cases}$$

$$\lambda_t = \begin{cases} \frac{-(m_0 - b\lambda_{t+1}) - bH_t}{m_1 + 2b - b\eta_{t+1}} & \text{if } t = 4 \\ 0 & \text{if } t = 7 \end{cases}$$

In Q7, with zero harvest in Q7 and Q8, and zero stocks carried out of Q8, the demand for storage as given by equation (2) can be expressed as $P_8 - P_7 = a - b(S_7 + 0 - 0) - (a - b(S_6 + 0 - S_7))$. This equation reduces to $P_8 - P_7 = bS_6 - 2bS_7$. Thus, $A_7 = 0$, $B_7 = b$ and $C_7 = -2b$. Now combine the Q7 demand for storage, $P_8 - P_7 = bS_6 - 2bS_7$, with the Q7 supply of storage, $P_8 - P_7 = m_0 + m_1 S_7$ to obtain $S_7 = \lambda_7 + \eta_7 S_6$ where the expressions for λ_7 and η_7 have been defined above. This equation, which shows the optimal level of storage in quarter 7 as a function of the level of stocks which were brought into quarter 7, can now be substituted for S_7 in the Q6 demand for storage equation as given by (2). The resulting equation reduces to $P_7 - P_6 = b\lambda_7 + bS_5 - (2b - b\eta_7)S_6$. Thus, $A_6 = b\lambda_7$, $B_6 = b$ and $C_6 = -(2b - b\eta_7)$. The above procedure can be repeated for $t = 5, 4, 3, 2, 1$ while accounting for the positive levels of harvest in Q1 and Q5. The result is $B_t = b$ for $t = 1, ..., 7$ and the following set of expressions for A_t and B_t:

$$C_t = \begin{cases} -(2b - b\eta_{t+1}) & \text{if } t = 1, ..., 6 \\ -2b & \text{if } t = 7 \end{cases}$$

$$A_t = \begin{cases} b\lambda_{t+1} + bH_t & \text{if } t = 1, 5 \\ b\lambda_{t+1} & \text{if } t = 2, 3, 6 \end{cases}$$

$$A_t = \begin{cases} b\lambda_{t+1} - bH_{t+1} & \text{if } t = 4 \\ 0 & \text{if } t = 7 \end{cases}$$

B Solution for Main Model

This section of the Appendix consists of two subsections. In the first subsection the model is solved conditional on the Q1 forecast of the H_5 variable. The last subsection presents the solution conditional on the Q2 forecast of the H_5 variable.

B.1 Results Conditioned on Q1 Forecast

To simplify the notation let $Z = m_0/b$ and $m = m_1/b$ respectively denote the intercept and slope of the carrying cost function, each normalized by the slope of the inverse demand schedule. Equations (3), (4) and (5) from the text are repeated for convenience.

$$P_{t+1} - P_t = m_0 + m_1 S_t \tag{B.1}$$

$$P_t = a - bX_t \tag{B.2}$$

$$S_t = S_{t-1} + H_t - X_t \qquad \text{with} \quad H_t = 0 \quad \text{for} \quad t = 2, 3, 4, 6, 7, 8 \tag{B.3}$$

Equation (B.1) with $t \in \{1, 2, 3\}$ and equations (B.2) and (B.3) with $t \in \{1, 2, 3, 4\}$ are a system of 11 equations with twelve unknown variables: $X_1...X_4$, $S_1...S_4$ and $P_1...P_4$. If X_1 is treated as known parameter then the system can be solved. Of particular interest is the solution value for consumption, $X_t = \omega_0^t + \omega_1^t X_1$ for $t \in \{2,3,4\}$, and the solution value for stocks, $S_t = \gamma_0^t + \gamma_1^t X_1$ for $t \in \{1,2,3,4\}$. The recursive expressions for the set of ω and γ coefficients are displayed in Table 3 below.

The conditional solution for Q5 through Q8 can be obtained in a similar fashion. Equations (B.1) and (B.3) with $t \in \{5,6,7\}$, and equation (B.2) with $t \in \{5,6,7,8\}$ are a system of 10 equations with twelve unknown variables: $X_5 ... X_8$, $S_5 ... S_8$ and $P_5 ... P_8$. The boundary condition, $S_8 = \bar{S}_8$, provides one additional equation. If X_5 is treated as a known parameter then the system can be solved. Of particular interest is the solution value for consumption, $X_t = \omega_0^t + \omega_1^t X_5 + \omega_2^5 (S_4 + H_5)$ for $t \in \{6,7,8\}$, and the solution value for stocks, $S_t = \gamma_0^t + \gamma_1^t X_5 + \gamma_2^t (S_4 + H_5)$ for $t \in \{5,6,7,8\}$. In this equation the boundary condition, $S_4 + H_5$ is kept explicit because S_4 is a variable.[13]. The recursive expressions for the set of ω and γ coefficients are displayed in Table 3 below.

Table 3: Coefficients for X and S

	Consumption (X_t)			Stocks (S_t)		
	ω_0^t	ω_1^t	ω_2^t	γ_0^t	γ_1^t	γ_2^t
Q1	na	na	na	$S_0 + H_1$	-1	0
Q2	$-Z - m(S_0 + H_1)$	$1 + m$	0	$\gamma_0^1 - \omega_0^2$	$-(1 + \omega_1^2)$	0
Q3	$\omega_0^2 - Z - m\gamma_0^2$	$\omega_1^2 - m\gamma_1^2$	0	$\gamma_0^2 - \omega_0^3$	$\gamma_1^2 - \omega_1^3$	0
Q4	$\omega_0^3 - Z - m\gamma_0^3$	$\omega_1^3 - m\gamma_1^3$	0	$\gamma_0^3 - \omega_0^4$	$\gamma_1^3 - \omega_1^4$	0
Q5	na	na	na	0	-1	1
Q6	$-Z - m\gamma_0^5$	$1 - m\gamma_1^5$	$-m\gamma_2^5$	$\gamma_0^5 - \omega_0^6$	$\gamma_1^5 - \omega_1^6$	$\gamma_2^5 - \omega_2^6$
Q7	$\omega_0^6 - Z - m\gamma_0^6$	$\omega_1^6 - m\gamma_1^6$	$\omega_2^6 - m\gamma_2^6$	$\gamma_0^6 - \omega_0^7$	$\gamma_1^6 - \omega_1^7$	$\gamma_2^6 - \omega_2^7$
Q8	$\omega_0^7 - Z - m\gamma_0^7$	$\omega_1^7 - m\gamma_1^7$	$\omega_2^7 - m\gamma_2^7$	$\gamma_0^7 - \omega_0^8$	$\gamma_1^7 - \omega_1^8$	$\gamma_2^7 - \omega_2^8$

[13]In Q1 the $S_0 + H_1$ boundary condition is suppressed because S_0 and H_1 are both parameters

The next step is to solve for the equilibrium values of X_1 and $E_1(X_5)$. The expectations operator on X_5 indicates that the equilibrium values are relevant for Q1 through Q4. Later in the analysis X_5 is reintroduced as a random variable whose outcome depends on the random H_5 outcome. To solve for the equilibrium values of X_1 and $E_1(X_5)$ substitute $X_4 = \omega_0^4 + \omega_1^4 X_1$ and $S_4 = \gamma_0^4 + \gamma_1^4 X_1$ into the boundary condition, $\gamma_0^8 + \gamma_1^8 E(X_5) + \gamma_2^8(S_4 + H_5) = \bar{S}_8$, and the equilibrium pricing equation which governs the transition from Q4 to Q5: $E_1(P_5) - P_4 = m_0 + m_1 S_4$. Solving for X_1 and $E_1(X_5)$ involves solving for the equilibrium value of S_4. The desired expression is $S_4 = \Gamma_0^4 + \Gamma_1^4 E_1(H_5)$, where:

$$\Gamma_0^4 = \frac{\gamma_1^4[\bar{S}_8 - \gamma_0^8 + \gamma_1^8(Z - \Phi)]}{\gamma_1^8\omega_1^4 - \gamma_1^4(m\gamma_1^8 - \gamma_2^8)} \qquad \Gamma_1^4 = -\frac{\gamma_1^4\gamma_2^8}{\gamma_1^8\omega_1^4 - \gamma_1^4(m\gamma_1^8 - \gamma_2^8))} \tag{B.4}$$

Within equation (B.4) note that $\Phi = (\gamma_1^4\omega_0^4 - \omega_1^4\gamma_0^4)/\gamma_1^4$. It is important to keep in mind that the solutions for Q5 through Q8 depend indirectly on $E_1(H_5)$ because all of the Q5 variables depend on S_5, and S_5 depends on S_4, which itself depends on $E_1(H_5)$.

Equation (B.4) can now be used to specify solution values for X and S as a function of $E_1(H_5)$ for Q1 through Q4, and random H_5 for Q5 through Q8. Specifically, $X_t = \Omega_0^t + \Omega_1^t E_1(H_5)$ and $S_t = \Gamma_0^t + \Gamma_1^t E_1(H_5)$ for $t \in \{1, 2, 3, 4\}$, and $X_t = \Omega_0^t + \Omega_1^t H_5$ and $S_t = \Gamma_0^t + \Gamma_1^t H_5$ for $t \in \{5, 6, 6, 8\}$. The expressions for the Ω and Γ coefficients are in Table 4 below. Finally, quarterly price, $P_t = a - bX_t$, can be expressed as $P_t = \delta_0^t + \delta_1^t E(H_5)$ for $t \in \{1, 2, 3, 4\}$, and $P_t = \delta_0^t + \delta_1^t H_5$ for $t \in \{5, 6, 7, 8\}$, where $\delta_0^t = a - b\Omega_0^t$ and $\delta_1^t = -b\Omega_1^t$. Once again, the Q5 through Q8 prices depend indirectly on $E_1(H_5)$ via the storage functions.

Results Conditioned on Q2 Forecast

In the beginning of Q2 market participants receive an updated forecast of the year 2 harvest, H_5^2. The full set of prices will adjust due to the changes in current and future consumption and storage. After the adjustment $P_2 = \bar{\delta}_0^2 + \bar{\delta}_2^2 H_5^1 + \bar{\delta}_1^2 H_5^2$. What follows is the derivation of the expressions for the three coefficients, $\bar{\delta}_0^2$, $\bar{\delta}_1^2$ and $\bar{\delta}_2^2$.

Table 4: Additional Coefficients for X and S

	Consumption (X_t)		Stocks (S_t)	
	Ω_0^t	Ω_1^t	Γ_0^t	Γ_1^t
Q1	$-\frac{\gamma_0^4}{\gamma_1^4}+\frac{\Gamma_0^4}{\gamma_1^4}$	$\frac{\Gamma_1^4}{\gamma_1^4}$	$S_0+H_1-\Omega_1^1$	$-\Omega_1^1$
Q2	$\omega_0^2+\omega_1^2\Omega_0^1$	$\omega_1^2\Omega_1^1$	$\gamma_0^2+\gamma_1^2\Omega_0^1$	$\gamma_1^2\Omega_1^1$
Q3	$\omega_0^3+\omega_1^3\Omega_0^1$	$\omega_1^3\Omega_1^1$	$\gamma_0^3+\gamma_1^3\Omega_0^1$	$\gamma_1^3\Omega_1^1$
Q4	$\omega_0^4+\omega_1^4\Omega_0^1$	$\omega_1^4\Omega_1^1$	See eqn. (B.4)	See eqn. (B.4)
Q5	$\frac{\bar{S}_8-\gamma_0^8}{\gamma_1^8}-\frac{\gamma_2^8}{\gamma_1^8\left(\Gamma_0^4+\Gamma_1^4E(H_5)\right)}$	$-\frac{\gamma_2^8}{\gamma_1^8}$	$\gamma_0^5+\gamma_1^5\Omega_0^5+\gamma_2^5S_4^*$	$\gamma_1^5\Omega_1^5+\gamma_2^5$
Q6	$\omega_0^6+\omega_1^6\Omega_0^5+\omega_2^6S_4^*$	$\omega_1^6\Omega_1^5+\omega_2^6$	$\gamma_0^6+\gamma_1^6\Omega_0^5+\gamma_2^6S_4^*$	$\gamma_1^6\Omega_1^5+\gamma_2^6$
Q7	$\omega_0^7+\omega_1^7\Omega_0^5+\omega_2^7S_4^*$	$\omega_1^7\Omega_1^5+\omega_2^7$	$\gamma_0^7+\gamma_1^7\Omega_0^5+\gamma_2^7S_4^*$	$\gamma_1^7\Omega_1^5+\gamma_2^7$
Q8	$\omega_0^8+\omega_1^8\Omega_0^5+\omega_2^8S_4^*$	$\omega_1^8\Omega_1^5+\omega_2^8$	$\gamma_0^8+\gamma_1^8\Omega_0^5+\gamma_2^8S_4^*$	$\gamma_1^8\Omega_1^5+\gamma_2^8$

Note that $S_4^*=\Gamma_0^4+\Gamma_1^4E(H_5)$

The system of equations to be solved include: $S_2=S_1^*-X_2$, $S_3=S_2-X_3$, $S_4=S_3-X_4$ $X_3=X_2-Z-mS_2$ and $X_4=X_3-Z-mS_3$ where $S_1^*=\Gamma_0^1+\Gamma_1^1E_1(H_5)$ is the expression for Q1 ending stocks. The solution is given by:

$$X_4=\bar{\omega}_0^4+\bar{\omega}_1^4X_2 \tag{B.5}$$

and

$$S_4=\bar{\gamma}_0^4+\bar{\gamma}_1^4X_2 \tag{B.6}$$

The expressions for the coefficients in this pair of equations are as follows:

$$\begin{aligned}\bar{\omega}_0^4&=-(2+m)Z-m(2+m)S_1^* & \bar{\omega}_1^4&=m^2+3m+1\\ \bar{\gamma}_0^4&=(3+m)Z+(m^2+3m+1)S_1^* & \bar{\gamma}_1^4&=-(m^2+4m+3)\end{aligned} \tag{B.7}$$

Following the steps from the previous section, it can be shown that $S_4=\bar{\Gamma}_0^4+\bar{\Gamma}_1^4E_2(H_5)$ where the expressions for $\bar{\Gamma}_0^4$ and $\bar{\Gamma}_1^4$ are given by equation (B.4) with the coefficients in equation (B.7) substituting for the corresponding coefficients in equation (B.4). Make a similar substitution for the Φ expression, which resides immediately below equation (B.4).

The next step is to combine $S_4 = \bar{\gamma}_0^4 + \bar{\gamma}_1^4 X_2$ and $S_4 = \bar{\Gamma}_0^4 + \bar{\Gamma}_1^4 H_5^2$ with expressions for the four coefficients substituted in. This gives

$$X_2 = \bar{\Omega}_0^2 + \bar{\Omega}_1^2 H_5^2 + \bar{\Omega}_2^2 S_1^* \tag{B.8}$$

The coefficients in this equation have the following expressions:

$$\begin{aligned}\bar{\Omega}_0^2 &= \tfrac{(3+m)Z}{m^2+4m+3} - \tfrac{\bar{\gamma}_1^4(\bar{S}_8-\gamma_0^8)}{K_0} \\ &+ \tfrac{\gamma_1^8}{K_0}\left((m^2+4m+3)(2+m)Z - (m^2+3m+1)((3+m)Z - \bar{\gamma}_1^4 Z)\right)\end{aligned} \tag{B.9}$$

$$\bar{\Omega}_1^2 = \frac{\bar{\gamma}_1^4 \gamma_2^8}{\left(\gamma_1^8 \bar{\omega}_1^4 - \gamma_1^4(m\gamma_1^8 - \gamma_2^8)\right)(m^2+4m+3)} \tag{B.10}$$

and

$$\bar{\Omega}_2^2 = \tfrac{\gamma_1^8}{K_0}\left((m^2+4m+3)(2+m)m - (m^2+3m+1)^2\right) + \tfrac{m^2+3m+1}{m^2+4m+3} \tag{B.11}$$

Within this set of equations $K_0 = (\gamma_1^8 \bar{\omega}_1^4 - \bar{\gamma}_1^4(m\gamma_1^8 - \gamma_2^8))(m^2+4m+3)$.

The finish this section substitute equation (B.8) and $S_1^* = \Gamma_0^1 + \Gamma_1^1 H_5^1$ into $P_t = a - bX_t$ to obtain the following:

$$P_2 = \bar{\delta}_0^2 + \bar{\delta}_2^2 H_5^1 + \bar{\delta}_1^2 H_5^2 \tag{B.12}$$

where: $\bar{\delta}_0^2 = a - b\bar{\Omega}_0^2 - b\bar{\Omega}_2^2\Gamma_0^1$, $\bar{\delta}_1^2 = -b\bar{\Omega}_1^2$ and $-b\bar{\Omega}_2^2\Gamma_1^1$