Math 104 section 108 Quiz #2 Column C   Date: Grade:

First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Short answer questions — you must show your work

1. [6 marks] Each part is worth 2 marks.

   (a) Prove that the equation $\ln x = \frac{1}{x^2}$ has a solution.

   Solution: We define $f(x) = \ln(x) - \frac{1}{x^2}$ for $x > 0$. Since

   1. $f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$ and
   2. $f(e) = \ln e - \frac{1}{e^2} = 1 - \frac{1}{e^2} > 0$ and
   3. f is continuous on $[1, e]$

   then IVT implies the existence of $1 < c < e$ such that $f(c) = 0$.

   (b) Differentiate the function $y = \frac{e^x - x^4}{\ln x^2}$. You do not need to simplify your answer. You may use any method taught in class.

   Answer: $y' = \frac{(e^x - 4x^3)\ln x^2 - (e^x - x^4)\frac{2}{x}}{\ln^2 x^2}$

   Solution:

$$
\begin{aligned}
y' &= \frac{(e^x - x^4)' \ln x^2 - (e^x - x^4) \cdot (\ln x^2)'}{\ln^2 x^2} \\
&= \frac{(e^x - 4x^3) \ln x^2 - (e^x - x^4)\frac{1}{x^2}2x}{\ln^2 x^2} \\
&= \frac{(e^x - 4x^3) \ln x^2 - (e^x - x^4)\frac{2}{x}}{\ln^2 x^2}
\end{aligned}
$$

   (c) Write the line equation for the tangent of $f(x) = a\ln(x) - a$ at the point $x = 1$. Find a value a for which this tangent passes through the point $(0, 0)$

   Answer: $l(x) = a(x - 1)$, $a = 0$

   Solution: We have $f(1) = 0$. The tangent formula is $l(x) = f'(1)(x - 1)$. We have $f'(x) = \frac{a}{x}$ and hence $f'(1) = a$. Hence $l(x) = a(x - 1)$.
   For the second part we plug in $x = 0$ and $l = 0$ to get $0 = a(0 - 1)$ and hence $a = 0$.

Long answer question — you must show your work

2. [4 marks] Compute the derivative of $f(x) = \sqrt{x^2 + 3}$ at $x = 1$ and write the line equation of the tangent line. You may only use the definition of the derivative.

Answer: $f'(1) = \frac{1}{2}$, $l(x) = \frac{1}{2}(x - 1) + 2$

Solution:

$$\begin{aligned}
f(1) &= \sqrt{1^2 + 3} = \sqrt{4} = 2 \\
f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x\to 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1} \\
&= \lim_{x\to 1} \frac{(\sqrt{x^2 + 3} - 2)(\sqrt{x^2 + 3} + 2)}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
&= \lim_{x\to 1} \frac{(x^2 + 3) - 2^2}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
&= \lim_{x\to 1} \frac{x^2 - 1}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
&= \lim_{x\to 1} \frac{(x - 1)(x + 1)}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
&= \lim_{x\to 1} \frac{x + 1}{\sqrt{x^2 + 3} + 2} = \frac{1 + 1}{\sqrt{1^2 + 3} + 2} = \frac{2}{4} = \frac{1}{2} \\
l(x) &= \frac{1}{2}(x - 1) + 2
\end{aligned}$$