Math 104 section 108 Quiz #5 Column A Date: Grade:

First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

You must show your work

1. 3 marks A wholesale flour producer wants to sell flour in cubic containers. The cost of making a cubic container with edge x meters is $2x^3 + 98x^{\frac{3}{2}}$ dollars. A market research shows that people are willing to be y cubic meters of flour for $100\sqrt{y}$ dollars. Find the size of the edge of the box which will produce the maximal profit per box.

Answer: $x = \left(\frac{1}{2}\right)^{\frac{2}{3}}$

Solution: The volume of the box is $V(x) = x^3$ and the revenue per box is

$$R(x) = 100\sqrt{V(x)} = 100x^{\frac{3}{2}}.$$

The profit function is

$$P(x) = R(x) - C(x) = 2x^{\frac{3}{2}} - 2x^3.$$

We are interested in values of x in the open interval $(0, \infty)$. We look for critical points

$$P'(x) = 3\sqrt{x} - 6x^2$$

Solving $P'(x) = 0$ for x we get $x^{\frac{3}{2}} = \frac{1}{2}$ so the only critical point of $P(x)$ is at $\left(\frac{1}{2}\right)^{\frac{2}{3}}$.
The second derivative of P is

$$P''(x) = \frac{3}{2\sqrt{x}} - 12x.$$

Which is positive for $0 < x < \left(\frac{3}{24}\right)^{\frac{2}{3}}$ and negative for $x > \left(\frac{3}{24}\right)^{\frac{2}{3}}$. In particular, it follows that P is increasing for $0 < x < \left(\frac{1}{2}\right)^{\frac{2}{3}}$ and decreasing for $x > \left(\frac{1}{2}\right)^{\frac{2}{3}}$ so $x = \left(\frac{1}{2}\right)^{\frac{2}{3}}$ is the absolute maximum of $P(x)$ at $(0, \infty)$. The profit is going to be \$0.5.

2. 3 marks Approximate $\sqrt[3]{15}$ as rational number, using a 2-nd Taylor Polynomial of $f(x) = \sqrt[3]{x}$.

Answer: $2+\frac{7}{12}-\frac{7^2}{288}$ or $3-\frac{12}{27}-\frac{12^2}{2187}$

Solution:

- The closest cubes to 15 are 8 and 27. I solve for both choices but you needed choose only one. Write

$$A(x) = a_0 + a_1(x-8) + a_2(x-8)^2$$
$$B(x) = b_0 + b_1(x-27) + b_2(x-27)^2$$

- 0-term:
$$a_0 = f(8) = 2, \quad b_0 = f(27) = 3.$$
In particular $2 < \sqrt[3]{15} < 3$ (since f is increasing).

- 1-term: $f'(x) = \frac{1}{3x^{\frac{2}{3}}}$
$$a_1 = f'(8) = \frac{1}{3 \cdot 2} = \frac{1}{12}, \quad b_1 = f'(27) = \frac{1}{3 \cdot 9} = \frac{1}{27}$$

- 2-term: $f''(x) = \frac{-2}{9x^{\frac{5}{3}}}$
$$a_2 = \frac{f''(8)}{2} = \frac{-2}{2 \cdot 9 \cdot 2^5} = \frac{-1}{288}, \quad b_2 = \frac{f''(27)}{2!} = \frac{-2}{2 \cdot 9 \cdot 243} = \frac{-1}{2187}$$

- The 2-nd taylor polynomial centered at $x = 8$ is
$$A(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$
$$B(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$$

- The approximations of $\sqrt[3]{15}$ are given by
$$A(15) = 2 + \frac{7}{12} - \frac{7^2}{288}, \quad B(15) = 3 - \frac{12}{27} - \frac{12^2}{2187}.$$

3. 4 marks Sketch the graph of the function $f(x) = \frac{2(x-4)^2}{x^2-4}$. You may use the fact that $f''(x) = 0$ only at $x_0 = \frac{5+\sqrt[3]{3}+\sqrt[3]{3^2}}{2} \approx 5.7$.

Solution:

- The domain of $f(x)$ is $x \neq \pm 2$, i.e. $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.
- Vertical asymptotes:
$$\lim_{x \to 2^+} f(x) = \infty$$
$$\lim_{x \to 2^-} f(x) = -\infty$$
$$\lim_{x \to -2^+} f(x) = -\infty$$
$$\lim_{x \to -2^-} f(x) = \infty$$

Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = 2$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x) = 2$$

- The first derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= 2\frac{2(x-4)(x^2-4)-(x-4)^2 2x}{(x^2-4)^2} \\ &= 2\frac{(x-4)(2(x^2-4)-(x-4)2x)}{(x^2-4)^2} \\ &= 2\frac{(x-4)(2x^2-8-2x^2+8x)}{(x^2-4)^2} \\ &= 2\frac{(x-4)(8x-8)}{(x^2-4)^2} = 16\frac{(x-4)(x-1)}{(x^2-4)^2} = 16\frac{x^2-5x+4}{(x^2-4)^2} \end{aligned}$$

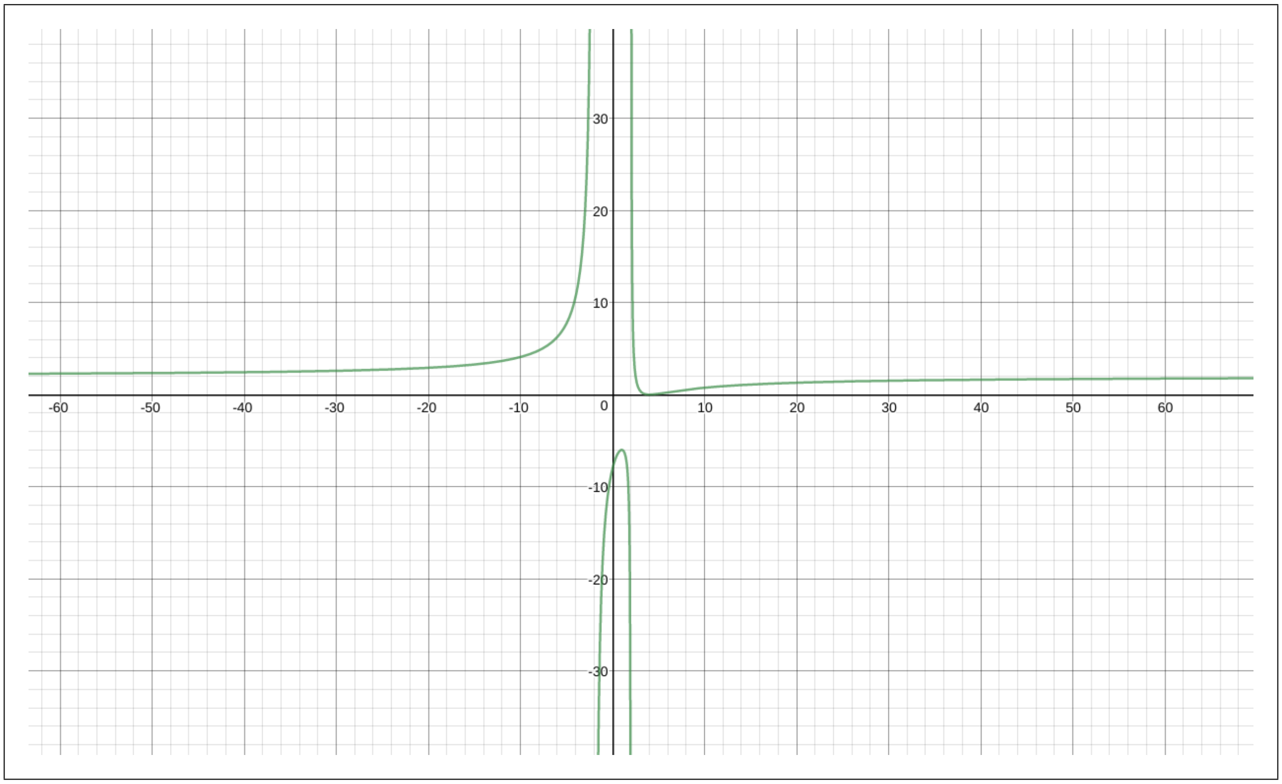
The derivative is defined for any x in the domain (i.e. $x \neq \pm 2$) and so the only critical points are the points c where $f'(c) = 0$, i.e. $c = 1$ and $c = 4$. Also, since the denominator is always positive, the sign of the first derivative is the same as the sign of its numerator. So, $f'(x)$ is positive (and $f(x)$ is increasing) for $x < -2$ and $x > 2$ and it is negative (and $f(x)$ is decreasing) for $-2 < x < 2$. It follows that $c = 1$ is a local maximum and $c = 4$ is a local minimum.

- $f(1) = -6$, $f(4) = 0$

- The second derivative of $f(x)$ is

$$\begin{aligned} f''(x) &= 16\frac{(2x-5)(x^2-4)^2-(x^2-5x+4)2(x^2-4)2x}{(x^2-4)^4} \\ &= 16\frac{(2x-5)(x^2-4)-4x(x^2-5x+4)}{(x^2-4)^3} \\ &= 16\frac{(2x^3-5x^2-4x+20)-(4x^3-20x^2+16x)}{(x^2-4)^3} = 16\frac{-2x^3+15x^2-24x+20}{(x^2-4)^3} \end{aligned}$$

The denominator is positive for $x > 2$ and $x < -2$ and negative for $-2 < x < 2$. The numerator is negative for $x > x_0$ and positive for $x < x_0$. It follows that $f''(x)$ is positive (and $f(x)$ is concave up) for $x < -2$ and $2 < x < x_0$ and negative (and $f(x)$ is concave down) for $-2 < x < 2$ and $x > x_0$. So x_0 is an inflection point.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