MATH 110 : Learning Goals for October Midterm Exam.

Course Objectives

In this course students will learn the basic ideas, tools and techniques of Differential Calculus and will use them to solve problems derived from simple, real-life applications. Specifically, students will learn to

- analyze the behaviour of basic mathematical functions (polynomial, radical, rational, trigonometric, exponential, logarithmic, absolute-valued, composite, and piecewise functions) both graphically and analytically;
- perform differentiation operations and other basic algebraic operations on the above functions fluently;
- recognize when and explain why differentiation operations are required, and use them to solve simple abstract and applied problems.
- use basic calculus concepts to find approximations of functions.???

Students will apply the above knowledge and skills to translate a problem involving real-life applications into a mathematical problem, and solve it by means of Calculus. The applications include science and engineering problems involving velocity/acceleration of moving objects and other rates of change of physical quantities, exponential growth/decay models of populations, optimization problems, curve sketching, approximation techniques.

In general, when solving a problem, students should be able to:

- after reading a problem, correctly state what the problem is asking in mathematical terms, and what information is given that is needed in order to solve the problem;
- after restating the problem, identify which mathematical concepts and techniques are needed to find the solution;
- apply those concepts and techniques and correctly perform the necessary algebraic steps to obtain a solution;
- interpret results within the problem context and determine if they are reasonable.

Students will also learn how to construct simple proofs. They will learn to show that a given mathematical statement is either true or false by constructing a logical explanation (proof) using appropriate Calculus theorems and properties of functions studied in the course. In particular, when applying a theorem, students will recognize the importance of satisfying its hypotheses and drawing logical conclusions.

Prerequisites

As a prerequisite to this course, students are required to have a reasonable mastery of pre-calculus mathematics (e.g., B.C. Principles of Mathematics 10--12), as this material is required throughout the course. In particular, students are expected to be able to perform fluently the tasks listed below.

Note: The skills labels by (R) have been reviewed explicitly in the course.
Prerequisites Topics:

1. Algebra

1.1: When working with expressions containing the following basic mathematical functions: polynomial, radical, rational, trigonometric, exponential, logarithmic, absolute-valued, and compositions of the above, students should be able to perform the following algebraic manipulations:

a) factor a GCF, difference of two squares, difference of two cubes, trinomials, and expressions with exponential and logarithmic functions;
b) apply properties of exponents to simplify expressions (R);
c) apply properties of logarithms to simplify expressions (R);
d) add and subtract and/or simplify complicated rational expressions;
e) rationalize a denominator;
f) convert an expression with a negative exponent to a reciprocal, and vice versa;

1.2: When solving equations and systems of equations, students should be able to do the following:

a) manipulate and solve linear equations
b) manipulate and solve quadratic equations using factoring or the quadratic formula;
c) determine under what conditions a quadratic equation has one, two, or no solutions;
d) manipulate and solve rational equations and simple equations containing radical, absolute-valued, trigonometric (R), exponential (R), logarithmic (R) expressions,
e) solve systems of linear and quadratic equations;
f) interpret the solution(s) of an equation graphically as the x-intercept(s) of a curve (R), similarly interpret the solution(s) of a system of equations as the points of intersection of two or more curves.

1.3: When working with inequalities, students should be able to manipulate and solve linear and quadratic inequalities (R).

2. Geometry

2.1: Euclidean geometry. Students should be able to

a) apply Pythagoras' theorem;
b) determine trigonometric relationships involving the sides and angles of a right triangle,
c) express proportional relations between similar triangles;
d) apply properties of isosceles and equilateral triangles;
e) compute perimeter, area, and volume of basic shapes (e.g. triangle, square, rectangle, trapezoid, rombus, etc.) and solids (e.g. rectangular box, sphere, circular cylinder, cone, etc.);

2.2: Analytic geometry: points of the plane. Students should be able to:

a) represent and identify points on the plane using the Cartesian coordinate system,
b) compute the distance between two points ,
c) find the midpoint of a line segment,
d) compute the slope between two points (R).

2.3: Analytic geometry: lines and other curves. Students should be able to

a) find the slope of a line from a graph or given its equation (R),
b) sketch a line given its equation in any form (R),
c) using the point-slope and point-intercept forms of a line, find the equation of:
   c.1) a line of known slope that goes through a given point (R),
   c.2) a line that goes through two given points (R),
   c.3) a line given its graph (R),
d) find the equation of a vertical/horizontal line that goes through a given point,
e) determine if two lines are parallel or perpendicular, relate the slopes of parallel lines, relate the slopes of perpendicular lines (R),
f) determine whether a point of known coordinates lies on a line/curve of given equation (R)
g) find the x- and y-intercepts of a line/curve, find the intersection point(s) of two curves, including a curve and a line (R),
h) relate, and vice versa (R),
i) sketch the graph of a parabola given its equation, or relate the graph of a parabola to its quadratic equation (R),
j) use the above skills solve basic geometrical problems.

3. Functions

3.1 Basic concepts: Students should be able to

a) give a definition of function,
b) identify independent and dependent variables in a function
c) find function values given any representation of a function (graph, table of values, equation),
d) given a representation of a function, construct a different representation,
e) determine if a graph represents a function.

3.2. Polynomials and rational functions and ratios p(x)/q(x), where p and q are compositions of polynomials and roots: Students should be able to

a) find the domain
b) find the range of simple functions
c) give examples of functions of given domain and range
d) sketch the graph of: linear and quadratic functions, x^3, 1/x,
e) find zeros and y-intercepts (if they exist).

3.3. Piece-wise functions defined by polynomial and rational functions and ratios p(x)/q(x), where p and q are compositions of polynomials and roots: Students should be able to,

a) find the domain
b) find the range of simple functions
c) give examples of piecewise functions of given domain and range
d) sketch the graph of piecewise functions defined by polynomials or simple rational and root functions,
e) find zeros and y-intercepts (if they exist).

3.4 Absolute-valued functions: Students should be able to,

a) find the domain
b) find the range of simple functions
c) write down the equation of an absolute-valued function as a piecewise defined function,
d) sketch the graph given an equation
e) find zeros and y-intercepts (if they exist).

3.5 Inverse functions: Students should be able to,

a) give a definition of a one-to-one function, and recognize whether a given function is one-to-one,
b) give a definition of inverse function
c) given a function, determine if it is invertible, and explain why or why not.
d) Given the graph of a one-to-one function, find domain and range and sketch the graph of its inverse, in particular find domain and range and sketch the graph of root functions.

3.6 Function composition: Students should be able to

a) Use function composition of the above functions to build new functions
b) Identify individual functions in a composite function of given equation

Calculus Topics

1. Limits and Continuity

1. The limit of a function: Students should be able to
   a) Explain in your own words and using a picture the meaning of $\lim_{x \to a} f(x) = L$, as well as $\lim_{x \to a+} f(x) = L$, $\lim_{x \to a-} f(x) = L$, where a is a finite number and L can be either a finite number or (positive or negative) infinity.
   b) given any of the functions listed in Section 3, evaluate the above limits graphically or algebraically, or determine the limit does not exist and explain why it does not exist; if the limit is of the form 0/0, use algebraic manipulations (e.g. factoring) to evaluate it.
   c) Give examples of rational functions whose limits at a given finite number of points do not exist.
   d) Evaluate limits that contain a parameter.
   e) Evaluate limits that arise from the definition of derivative, see later.

Sketch the behaviour of a function at values of x near a point *a* given $\lim_{x \to a} f(x) = L$, or one-sided limits, where *L* can be either a finite number or (positive or negative) infinity.

2. Limits at infinity: Students should be able to
Calculate the limits as x approaches infinity to identify horizontal asymptotes

1.3. Continuity: Students should be able to

a) explain in your words and using a graph what it means for a function to be continuous or discontinuous at a point and on an interval,
b) give a definition of continuous function using formal mathematical notation.
c) determine whether the functions listed in section 1 are continuous on their domain,
d) give examples of continuous and discontinuous functions that model real-life quantities,
e) determine the parameters of a piecewise function that make the function continuous.

1. Properties of continuous functions: The intermediate value theorem: Students should be able to

a) State the Intermediate Value Theorem, and recognize when it is applicable.
b) Give examples of functions that do not satisfy the Intermediate Value Theorem by virtue of their discontinuity either in the interior of an interval or at its endpoints.
c) Use the Intermediate Value Theorem to estimate roots of functions (for functions listed above).

2. Estimating instantaneous velocities and slopes of tangent lines

2.1 The difference quotient: Students should be able to

a) Given the equation or graph of a function *f*, compute or represent graphically the change in function values for a change in the independent variable.
b) Given the equation or graph of a position function, compute the average velocity over a given time interval.
c) Given the equation or graph of a function $f(x)$, compute the average rate of change over a given interval of x values.
d) Given the equation or graph of a curve $y = f(x)$, compute the average slope of the curve between two points.
e) Express any of the average quantities listed in a), b), and c) as a difference quotient, and represent it graphically.

2.2. Velocity and the idea of limit: Students should be able to

a) Using difference quotients, explain how to define the instantaneous velocity of an object, and the instantaneous rate of change of a function, as a limit; in other words, argue why the concept of instantaneous velocity/rate of change requires the notion of limit of a function.
b) Estimate the velocity of an object at a given time instant using appropriate difference quotients.
c) Estimate the rate of change of a function *f* at a given point using appropriate difference quotients; if *f* describes a physical quantity, determine the units of the rate of change of *f*.

2.3 The Tangent line and the idea of limit: Students should be able to

a) Using a graph, explain how to define the slope of a tangent line as a limit of slopes of secant lines (without introducing derivative notation)
b) Estimate the slope of the tangent line to a curve at a point using appropriate secant lines and difference quotients.
c) Given the graph of a curve, draw the tangent line to the curve at a given point.

3. The Derivative

3.1 The definition of derivative: Students should be able to

a) Represent the process of estimating velocities and slopes of tangent lines at a point x by using the notation $[f(x+h)-f(x)]/h$ for some small h and taking its limit as h approaches 0.
b) Define the derivative of a function (at a point or as a function) using limits.
c) Using the limit definition, calculate derivatives of polynomial functions of degree 3 or less and of rational functions $p(\text{x})/q(\text{x})$ where p and q are linear or quadratic functions, and of other (simple) functions in general form (e.g. $cf(x)$ where c is a constant);
d) Given a curve $y = f(x)$, relate the derivative f'(a) to the slope of a tangent line to the curve at (a, f(a)), and compute the slope of a tangent at given points using derivatives.
e) Given the equation of a position function, relate the slope of a tangent line to the value of instantaneous velocity, and vice versa, and compute instantaneous velocities using derivatives.

3.2 Graphing a derivative; Students should be able to,
a) Given the graph of a function f, sketch the graph of f', and vice versa.
b) Sketch the graphs of velocity and acceleration of a particle given a graph of its position.
c) Given graphs of the position, velocity and acceleration of a particle, identify which is which.

3.3. Derivatives and tangent lines: Given the equation of a curve and using the relationship between the derivative and the slope of a tangent line, find the equation of lines tangent to the graph of the curve at specific points, and solve simple geometrical problems involving tangent lines.

3.4 Differentiable functions: students should be able to
a) Define what it means for a function to be differentiable.
b) Explain why differentiable functions are continuous.
c) Demonstrate, using an example, that continuous functions need not be differentiable.
d) Determine the parameters of a piecewise function that make the function differentiable.

4. Differentiation Rules

4.1 Students should be able to compute the derivative of a given function using any of the following rules: the power rule, the constant-multiple rule, the sum/difference rule, the product rule, the quotient rule, the chain rule. In particular, students should be able to

a) recognize when the above rules apply,
b) apply each rule separately or in combination with other differentiation rules,

4.2 Derivatives of trig functions, exponentials, logarithms

4.3 Implicit differentiation