Lecture 10
Sept 26

HW 2 posted. Due date: Monday Oct 1
Collect HW 1 from MLC (LSK 301/302, 11am-5pm)

Clicker Q: Find the following limits using the graph of \( f(x) \).

1. \( \lim_{{x \to 0}} f(x) = 1 \) and \( f(0) = \text{undefined} \)
   - A. 0
   - B. 2
   - C. 1
   - D. -1
   - E. Not sure

2. \( \lim_{{x \to 2^+}} f(x) = -2 \)
   - A. -1
   - B. -2
   - C. 1
   - D. 2
   - E. Again not sure

3. \( \lim_{{x \to 2^-}} f(x) = -1 \)
   - A. -1
   - B. -2
   - C. 1
   - D. 2
   - E. Again!!!

What is the full limit at 2 then:

\( \lim_{{x \to 2}} f(x) \) = Does NOT exist, because the one-sided limits are not equal and we don't get a unique value.
Limit Exists or Does Not Exist (DNE)

Exists: If one-sided limits both from left and right exist and they are equal then the full limit exists, and the value of the limit is just the value of one-sided limits ⇒ Clicker Q: 1

\[
\lim_{x \to 0^+} f(x) = 1 = \lim_{x \to 0^-} f(x) = 1 \quad x \to 0
\]

The full limit exists and it is equal to the left & right lim

Does Not Exist

DNE: If one-sided limits are NOT equal to each other then the full limit DNE ⇒ Clicker Q 2 & 3

\[
\lim_{x \to 2^+} f(x) = -2 \quad \lim_{x \to 2^-} f(x) = -1 \quad x \to 2
\]

In math language

If \( \lim_{x \to a} f(x) = L \) \( \text{implies} \) \( \lim_{x \to a^+} f(x) = L \) and vice versa

If \( \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \) then \( \lim_{x \to a} f(x) = L \)

⇒ This is a two-way conclusion:

\[
\lim_{x \to a^+} f(x) = L \quad \iff \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \quad x \to a
\]
So far we could find the limits using the graph of a given function, but what if there's no graph given and we only have the expression for the function? Let's start with two easy examples.

**Clicker Q:** What is \( \lim_{x \to 1} (3) = 3 \)

- A. 0
- B. 1
- C. 2
- D. 3
- E. DNE

\[ f(x) = 3 \]

\[ g(x) = x \]

\[ \lim_{x \to 2} x = 2 \]

\[ \lim_{x \to a} C = C \]

*In general if \( f(x) = C \) where \( C \) is a constant number then \( \lim_{x \to a} C = C \)***

**Clicker Q:** What is \( \lim_{x \to 2} x = 2 \)

Using the above simple limit and the following "limit laws" we can compute limits of "nice" functions.

**Limit Laws:** Suppose \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then

(I) \( \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M \)

(II) \( \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M \)

(III) \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \) where \( M \neq 0 \)

(IV) \( \lim_{x \to a} (f(x) \cdot g(x)) = (\lim_{x \to a} f(x)) \cdot (\lim_{x \to a} g(x)) = L \cdot M \)
\[
\frac{\frac{9 + 6}{-2} + \frac{6}{-2} + \frac{3}{-2}}{-2} = \frac{-6}{-2} = 3
\]

So in this example, we can simplify substituting the given \( x \) value.

Now let's substitute -2 into the function:

\[
\frac{\frac{(x-2)}{x} + \frac{6}{x}}{x-2} = \frac{\frac{6}{x}}{x-2}
\]

so the limit value is -6.

\[
\frac{x+2}{x} + \frac{6}{x}
\]

and

\[
\frac{x-2}{x} + \frac{6}{x}
\]

Example 1.

\[
\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^2 + 6}{x^2 - 3x + 2}
\]

\[
\frac{\frac{9 + 6}{-2} + \frac{6}{-2} + \frac{3}{-2}}{-2} = \frac{-6}{-2} = 3
\]

So in this example, we can simplify substituting the given \( x \) value.

Now let's substitute -2 into the function:

\[
\frac{\frac{(x-2)}{x} + \frac{6}{x}}{x-2} = \frac{\frac{6}{x}}{x-2}
\]

so the limit value is -6.

\[
\frac{x+2}{x} + \frac{6}{x}
\]

and

\[
\frac{x-2}{x} + \frac{6}{x}
\]

Example 1.

\[
\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^2 + 6}{x^2 - 3x + 2}
\]
Example 2: Find \( \lim_{x \to 3} \frac{x^2 + x - 20}{x - 4} \)

Let's start with the boring way:

\[
\lim_{x \to 3} \frac{x^2 + x - 20}{x - 4} = \lim_{x \to 3} x^2 + x - 20 \quad \lim_{x \to 3} x - 4 = \lim_{x \to 3} x - \lim_{x \to 3} 4
\]

\[
= \left( \lim_{x \to 3} x \right)^2 + \lim_{x \to 3} x - \lim_{x \to 3} 20.
\]

\[
= \lim_{x \to 3} x - \lim_{x \to 3} 4
\]

\[
= 3^2 + 3 - 20 = 9 - 20 = -8
\]

Now let's try direct substitution:

\[
\lim_{x \to 3} \frac{x^2 + x - 20}{x - 4} = \frac{(3)^2 + 3 - 20}{3 - 4} = 8 \sqrt{\ }
\]

Note:

The 1st step in finding a limit is Direct Substitution.

If the function is "nice", direct substitution always works.

"Nice" functions include:

- polynomials (constant)
- rational functions of polynomials with no zero in the denominator.
- \( \sin x, \cos x, e^x \)
- \( \sqrt{x} \) and \( \log_b x \) (when \( x > 0 \))
Example 3: Find the following limits

1. \( \lim_{x \to 2} (\ln x) = \ln 2 \)

2. \( \lim_{x \to \pi} (\sqrt{x} + \cos x) = \sqrt{\pi} + \cos \pi = \sqrt{\pi} - 1 \)

Recall:

\[ y = \ln x \]

As long as we are away from 0, everything is nice for \( y = \ln x \).

What if \( f(x) \) is NOT "nice"?

Let's re-do example 2 with limit in another way:

\[ \lim_{x \to 4} \frac{x^2 + x - 20}{x - 4} = \frac{(4)^2 + 4 - 20}{4 - 4} = \frac{0}{0} \]

\( \frac{0}{0} \) is NOT a number, we call \( \frac{0}{0} \) an indeterminate form.

How to remove \( \frac{0}{0} \)? Manipulate the function doing some algebra

\[ \lim_{x \to 4} \frac{x^2 + x - 20}{x - 4} = \lim_{x \to 4} \frac{(x + 5)(x - 4)}{x - 4} = \lim_{x \to 4} \frac{x + 5}{1} = 4 + 5 = 9 \]

\( x \) approaches 4, \( x \neq 4 \)

Problem found: cancel it

\( \frac{0}{0} \) re-substitute.

Question: Write \( f(x) = \frac{x^2 + x - 20}{x - 4} \) as a piece-wise function and plot that.

Recall from Lab 1: \( f(x) = \frac{(x + 5)(x - 4)}{x - 4} = \begin{cases} x + 5 & x \neq 4 \\ \text{undefined} & x = 4 \end{cases} \)

\* Since when \( x \to 4 \) then \( x \neq 4 \) so we can cancel \( x - 4 \) without worrying about undefined case.
Note
Re-visit limit evaluation:

1st step: Always start with direct substitution.

If you get a number

Voila! You have the limit.

If you get an indeterminate form NOT a number: \( \frac{0}{0} \)!!!

Manipulate the function by doing algebra (factoring, common denominator, ...) and remove the term which makes top and bottom 0 substitute again.

Example 4:
Find \( \lim_{x \to -1} \frac{x^2 - 1}{x^2 + x} \)

Start with direct substitution:

\[
\lim_{x \to -1} \frac{x^2 - 1}{x^2 + x} = \frac{(-1)^2 - 1}{(-1)^2 + (-1)} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \text{!!!}
\]

Factor

\[
= \lim_{x \to -1} \frac{(x-1)(x+1)}{x(x+1)}
\]

\[
= \lim_{x \to -1} \frac{x-1}{x} = \frac{-1 - 1}{-1} = 2
\]